§120. Flow at the velocity of sound

The simplified form of Chaplygin's equation (i.e., the Euler-Tricomi equation) must, in principle, be used to investigate the basic qualitative properties of steady two-dimensional flow past bodies, resulting from the existence of transonic regions. These include, in the first place, problems concerning the formation of shock waves. In the transonic region, the shock wave is weak, and this is the reason why the Euler-Tricomi equation is applicable under these conditions. We have seen in §§86 and 114 that in a weak shock the changes in the entropy and vorticity are higher-order small quantities; in the first approximation, therefore, we can assume isentropic potential flow behind the discontinuity also.

We shall discuss here a problem of theoretical importance, that of the nature of steady two-dimensional flow past a body when the velocity of the incident stream is exactly equal to the velocity of sound. We shall see, in particular, that a shock wave must extend from the surface of the body to infinity. From this we can draw the important conclusion that the shock wave must first appear for a Mach number M_∞ which is certainly less than unity.

For, let us consider two-dimensional flow past a body ("wing") with infinite span and arbitrary (not necessarily symmetrical) cross-section. Here we are interested in the flow pattern at distances from the body which are large compared with its dimension. For convenience we shall first describe the results in a qualitative manner, and afterwards give a quantitative calculation.

In Fig. 122, *AB* and *A'B'* are **sonic lines**, so that the subsonic region lies to the left of them (upstream); the arrow shows the direction of the main stream, which we shall take as the *x*-axis, with the origin anywhere near the body. At a certain distance from the sonic line we have shock waves leaving the body (*EF* and *E'F'* in Fig. 122). It is found that the characteristics leaving the body (between the sonic line and the shock wave) can be divided into two groups. The characteristics in the first group meet the sonic line and end there (that is to say, they are reflected from it as characteristics which reach the body; Fig. 122 shows one such characteristic). The characteristics in the second group end at the shock wave. The two groups are separated by limiting characteristics, the only ones which go to infinity and meet neither the sonic line nor the shock wave (*CD* and *C'D'* in Fig. 122). Since disturbances (caused, for instance, by a change in the shape of the body) which are propagated from the body along characteristics of the first group reach the boundary of the subsonic region, it is clear that the part of the supersonic region which lies between the sonic line and the limiting characteristic affects the subsonic region, but the flow to the right of the limiting characteristics has no effect on the flow to the left: the flow to the left is not affected by a disturbance of the flow to the right (such as a change in the profile to the right of *C* or *C''*). The flow behind the shock wave has, as we know, no effect on the flow in front of it. Thus the whole flow can be divided into three parts (to the left of *DCC'D'*, between *DCC'D'* and *FEE'F'*, and to the right of *FEE'F'*), such that the flow in the second part has no effect on that in the first, and the flow in the third part has no effect on that in the second.

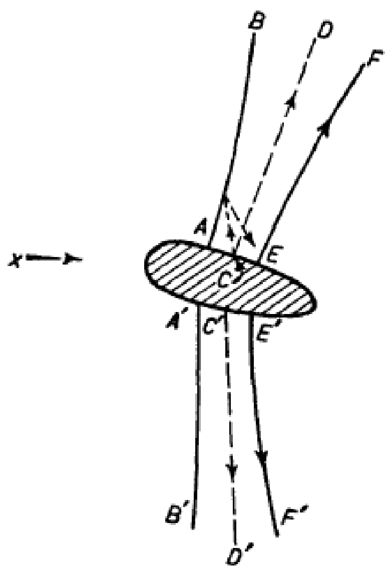

FIG. 122

We shall now give a quantitative account (and verification) of the flow pattern just described.

The origin in the hodograph plane ($\theta = \eta = 0$) corresponds to an infinitely distant region of the physical plane, and the hodograph characteristics leaving the origin correspond to the limiting characteristics *CD* and *C'D'*. Figure 123 shows the neighbourhood of the origin, the

letters corresponding to those in Fig. 122. The shock wave corresponds not to one line but to two lines in the hodograph plane (corresponding to the gas flow on the two sides of the discontinuity); the regions between these lines (shaded in Fig. 123) do not correspond to any part of the physical plane.

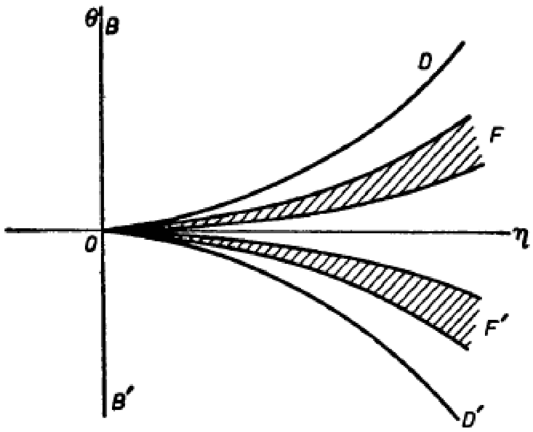

FIG. 123

We must ascertain, first of all, which of the general integrals Φ_k corresponds to this case. If $\Phi(\theta,\eta)$ is of degree *k*, then the functions $x = \frac{\partial \Phi}{\partial \eta}$ and $y = \frac{\partial \Phi}{\partial \theta}$ are homogeneous and of degree $k - \frac{1}{3}$ and $k - \frac{1}{2}$, respectively. As θ and η tend to zero we must, in general, reach infinity in the physical plane (*x* and *y* tend to infinity). It is evident that, for this to be so, we must have $k < \frac{1}{3}$. The limiting characteristics in the physical plane, however, need not lie entirely at infinity, i.e., $y = \pm\infty$ need not hold everywhere on the curve $9\theta^2 = 4\eta^3$. In that case (for $2k + \frac{1}{6} < \frac{5}{6}$), the second term in the brackets in (118.6) must be zero. Thus the function $\Phi(\theta,\eta)$ must be given by the first term of (118.6):

$$\Phi = A\theta^{2k} F\left(-k, -k + \frac{1}{2}; -2k + \frac{5}{6}; 1 - \frac{4\eta^3}{9\theta^2}\right). \qquad (120.1)$$

The function $y(\theta,\eta)$ (which also satisfies the Euler-Tricomi equation) has the same form, but with $k - \frac{1}{2}$ instead of *k*.

If the expression (120.1) is valid near (e.g.) the upper characteristic ($\theta = +\frac{2}{3}\eta^{3/2}$), however, it will not be valid near the lower characteristic also ($\theta = -\frac{2}{3}\eta^{3/2}$) for an arbitrary $k < \frac{1}{3}$. We must therefore require also that the form (120.1) of the function $\Phi(\theta,\eta)$ is maintained on going round the origin in the hodograph plane from one characteristic to the other through the half-plane $\eta < 0$ (the path *A*'*B*' in Fig. 119). This path corresponds to a passage in the physical plane from distant points on one of the limiting characteristics to distant points on the other, along a path which passes through the subsonic region and therefore nowhere intersects the shock wave, at which the flow is discontinuous. The transformation of the hypergeometric function in (120.1) in going along such a path is given

by the first formula (118.13), and we must require that the coefficient of F_2 in this formula be zero. This condition is fulfilled (for $k<\frac{1}{3}$) when $k=\frac{1}{6}-\frac{1}{2}n$ (n = 0,1, 2,...).Of these values, only one can be taken, namely

$$k=-\frac{1}{3}; \tag{120.2}$$

it can be shown that all values of k with $n > 1$ give a mapping of the hodograph plane into the physical plane which is not one-to-one (in going once round the former we go more than once round the latter), and so the physical flow is many-valued, which is of course impossible. The value $k=\frac{1}{6}$, on the other hand, gives a solution in which we do not go to infinity in every direction in the physical plane when θ and η tend to zero; such a solution is, evidently, likewise physically impossible.

For $k=-\frac{1}{3}$ the coefficient of F_1 on the right-hand side of formula (118.13) is unity, i.e., the function Φ is unchanged when we go from one characteristic to the other. This means that Φ is an even function of θ, and the coordinate $y=\frac{\partial\Phi}{\partial\theta}$ is therefore an odd function. Physically, this means that, in the first approximation here considered, the flow pattern at large distances from the body is symmetrical about the plane $y = 0$, whatever the shape of the body, and in particular whether there is a lift force or not.

Thus we have determined the nature of the singularity of $\Phi(\theta,\eta)$ at the point $\eta=\theta=0$. From this we can at once deduce the form of the sonic line, the limiting characteristics and the shock wave at great distances from the body. Each of these lines must correspond to a definite value of the ratio θ^2/η^3 and, since Φ has the form $\theta^{-2/3}f(\eta^3/\theta^2)$, we find from formulae (118.4) that $x\propto\theta^{-4/3}$, $y\propto\theta^{-5/3}$. Hence these lines are given by equations having the form

$$x=\text{constant}\times \text{y}^{4/5}, \tag{120.3}$$

with various values of the constant. Along these lines, θ and η decrease according to

$$\theta=\text{constant}\times y^{-3/5},\quad \eta=\text{constant}\times y^{-2/5}. \tag{120.4}$$

These results are due to F. I. Frankl' (1947) and K. G. Guderley (1948).[1]

In what follows we shall, for definiteness, write the formulae with the signs appropriate to the upper half-plane ($y > 0$).

We shall show how the coefficients in these formulae may be calculated. The value $k=-\frac{1}{3}$ is one of those for which the Φ_k reduce to algebraic functions (see § 119). The particular integral which determines Φ in the present case can be written as $\Phi=\frac{1}{2}a_1\frac{\partial f}{\partial\theta}$, where a_1 is an arbitrary positive constant, and f is that root of the cubic equation

$$f^3-3\eta f+3\theta=0 \tag{120.5}$$

which is the one real root for $9\theta^2-4\eta^3>0$. Hence

$$\Phi=\frac{1}{2}a_1\frac{\partial f}{\partial\theta}=-\frac{1}{2}\frac{a_1}{f^2-\eta}, \tag{120.6}$$

and we have for the coordinates

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[1] Similar results can be obtained for axially symmetrical flow (with $M_\infty=1$).
In cylindrical polar coordinates x, r, the form of the sonic surface, the limiting characteristic and the shock wave, and the velocity variations, are given (far from the body) by $x=\text{constant}\times r^{4/7}$, $v_x\propto r^{-6/7}$, $v_r\propto r^{-9/7}$. See K. G. Guderley, The Theory of Transonic Flow, Oxford 1962; S. V. Fal'kovich and I. A. Chernov, *Journal of Applied Mathematics and Mechanics* 28, 342, 1965.

$$\left.\begin{aligned} x &= \frac{\partial \Phi}{\partial \eta} = \frac{1}{2}\frac{a_1(f^2+\eta)}{(f^2-\eta)^3} \\ y &= \frac{\partial \Phi}{\partial \theta} = -\frac{a_1 f}{(f^2-\eta)^3} \end{aligned}\right\} \qquad (120.7)$$

These formulae can be put in a convenient parametric form by using as a parameter $s = \dfrac{f^2}{f^2 - \eta}$. Then

$$\left.\begin{aligned} \frac{x}{y^{4/5}} &= \frac{a_1^{1/5}(2s-1)}{2s^{2/5}} \\ \eta y^{2/5} &= a_1^{2/5} s^{1/5}(s-1) \\ \theta y^{3/5} &= \frac{1}{3} a_1^{3/5} s^{4/5}(3-2s) \end{aligned}\right\} \qquad (120.8)$$

which give, in parametric form, η and θ as functions of the coordinates. The parameter s takes positive values from zero upwards (s = 0 corresponding to $x = -\infty$, i.e., to the stream incident from infinity). In particular, the value s = 1/2 corresponds to x = 0, i.e., it gives the velocity distribution for large y in a plane perpendicular to the x-axis and passing near the body. The value s = 1 corresponds to the sonic line ($\eta = 0$), and s = 4/3 as is easily seen, to the limiting characteristic. The value of the constant a_1 depends on the actual shape of the body, and can be determined only from an exact solution of the problem in all space.

Formulae (120.8) relate only to the region in front of the shock wave. The necessity for the shock to appear can be seen as follows. A simple calculation from formula (118.5) gives for the Jacobian Δ the expression $\dfrac{a_1^2(4f^2-\eta)}{(f^2-\eta)^3}$. It is easy to see that $\Delta > 0$ (and does not vanish) on the characteristics and everywhere to the left of them, corresponding to the region upstream of the limiting characteristics in the physical plane. To the right of the characteristics, however, Δ becomes zero, and so a shock wave must appear in this region.

The boundary conditions at the shock wave which must be satisfied by the solution of the Euler-Tricomi equation are as follows. Let θ_1, η_1 and θ_2, η_2 be the values of θ and η on the two sides of the discontinuity. First of all, they must correspond to the same curve in the physical plane, i.e.,

$$x(\theta_1, \eta_1) = x(\theta_2, \eta_2), \quad y(\theta_1, \eta_1) = y(\theta_2, \eta_2) \qquad (120.9)$$

Next, the condition that the velocity component tangential to the discontinuity be continuous (i.e., that the derivative of the potential ϕ along the discontinuity be continuous) is equivalent to the condition that the potential itself be continuous:

$$\phi(\theta_1, \eta_1) = \phi(\theta_2, \eta_2); \qquad (120.10)$$

the potential ϕ is determined from the function Φ by (119.3). Finally, another condition can be obtained from the limiting form of the equation (92.6) of the shock polar, which gives a relation between the velocity components on the two sides of the discontinuity. Replacing the angle χ in (92.6) by $\theta_2 - \theta_1$, and introducing η_1, η_2 in place of v_1, v_2, we obtain the relation

$$2(\theta_2 - \theta_1)^2 = (\eta_2 - \eta_1)^2(\eta_2 + \eta_1). \qquad (120.11)$$

In the present case, the solution of the Euler-Tricomi equation behind the shock wave (the region between OF and OF' in the hodograph plane, Fig. 123) has the same form (120.5), (120.6), but of course with a different constant coefficient (which we call $-a_2$) in place of a_1. The four simultaneous equations (120.9) - (120.11) determine the ratio a_2/a_1 and relate the quantities $\eta_1, \theta_1, \eta_2, \theta_2$. The solution of these equations is fairly complicated; it gives the following results. The shock wave corresponds to the value $s = \dfrac{5\sqrt{3}+8}{6} = 2.78$ of the

parameter s in formulae (120.8), which give the form of the shock and the velocity distribution on the forward side of the discontinuity. In the region behind (downstream of) the shock, the coefficient $-a_2$ is negative, and $\dfrac{f^2}{f^2-\eta}$ takes negative values. Using as the parameter the positive quantity $s=\dfrac{f^2}{\eta-f^2}$, we have instead of (120.8) the formulae

$$\left.\begin{aligned} \frac{x}{y^{4/5}} &= \frac{a_2{}^{1/5}(2s+1)}{2s^{2/5}} \\ \eta y^{2/5} &= a_2{}^{2/5}s^{1/5}(s+1) \\ \theta y^{3/5} &= -\frac{1}{3}a_2{}^{3/5}s^{4/5}(2s+3) \end{aligned}\right\} \qquad (120.12)$$

where

$$\frac{a_2}{a_1}=\frac{9\sqrt{3}+1}{9\sqrt{3}-1}=1.14,$$

and s takes values from $\dfrac{5\sqrt{3}-8}{6}=0.11$ on the shock wave to zero at an infinite distance downstream.

Figure 124 shows graphs of $\eta y^{2/5}$ and $\theta y^{3/5}$ as functions of $xy^{-4/5}$, calculated from formulae (120.8) and (120.12) (the constant a_1 being arbitrarily taken as unity).

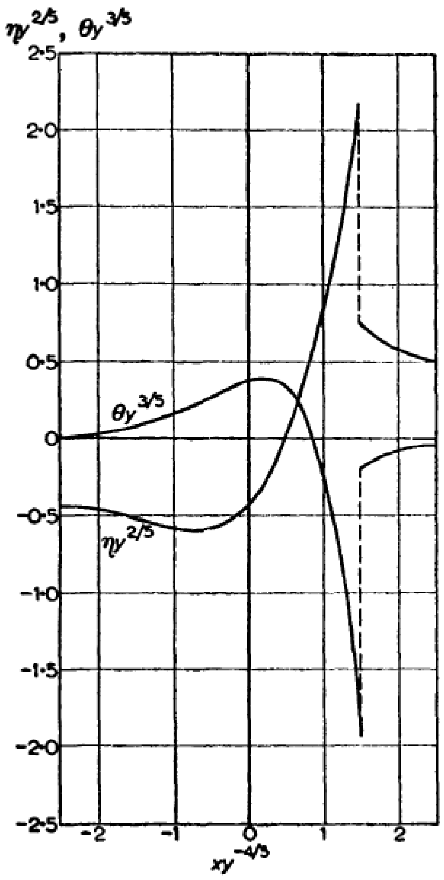

FIG. 124

§121. The reflection of a weak discontinuity from the sonic line

Let us consider by means of the Euler-Tricomi equation the reflection of a weak discontinuity from the sonic line.

We shall assume that the weak discontinuity incident on the sonic line (reaching the point of intersection) is of the ordinary type, formed (say) by flow past an acute angle, i.e., the first spatial derivatives of the velocity are discontinuous in it. It is reflected from the sonic line as another discontinuity, the nature of which, however, is unknown *a priori* and must be determined by investigating the flow near the point of intersection. We take this point as the origin in the *xy*-plane, and the *x*-axis in the direction of the gas velocity there, so that it corresponds to the origin in the hodograph plane also.

Weak discontinuities coincide with characteristics, as we know. Let the characteristic *Oa* in the hodograph plane (Fig. 125a) correspond to the incident discontinuity. Since the coordinates *x*, *y* are continuous at the discontinuity, the first derivatives Φ_η, Φ_θ must be continuous also. The second derivatives of Φ, on the other hand, can be expressed in terms of the first spatial derivatives of the velocity, and therefore must be discontinuous. Denoting the discontinuities of quantities by placing them in brackets, we therefore have

$$\text{on } Oa \quad [\Phi_\eta]=[\Phi_\theta]=0\,;\ [\Phi_{\theta\theta}]\neq 0,\ [\Phi_{\theta\eta}]\neq 0,\ [\Phi_{\eta\eta}]\neq 0. \tag{121.1}$$

The functions Φ themselves in the regions 1 and 2 on each side of the characteristic *Oa* must not have singularities on the characteristic. Such a solution can be constructed from the second term in (118.6) with k = 11/12, which is proportional to the square of the difference $1-\dfrac{4\eta^3}{9\theta^2}$ (the other independent solution $\Phi_{11/12}$ has a singularity on the characteristic; see below). The first derivatives of this function vanish on the characteristic, and the second derivatives are finite. Furthermore, Φ can include those particular solutions of the Euler-Tricomi equation which do not give singularities of the flow in the physical plane. The solution of this kind which is of the lowest degree in θ and η is $\eta\theta$ (§119). Thus we seek Φ near the characteristic *Oa* and on either side of it in the forms:

$$\left.\begin{aligned}\Phi_{a1}&=-A\eta\theta-B\xi^2\theta^{11/6}F\left(\frac{13}{12},\frac{19}{12};3;\xi\right)\\ \Phi_{a2}&=-A\eta\theta-C\xi^2\theta^{11/6}F\left(\frac{13}{12},\frac{19}{12};3;\xi\right)\end{aligned}\right\} \tag{121.2}$$

where the suffixes *a1* and *a2* denote regions 1 and 2 near the characteristic and on each side of it; *A*, *B*, *C* are constants, and $\xi\equiv 1-\dfrac{4\eta^3}{9\theta^2}$; on the characteristics, $\xi=0$.

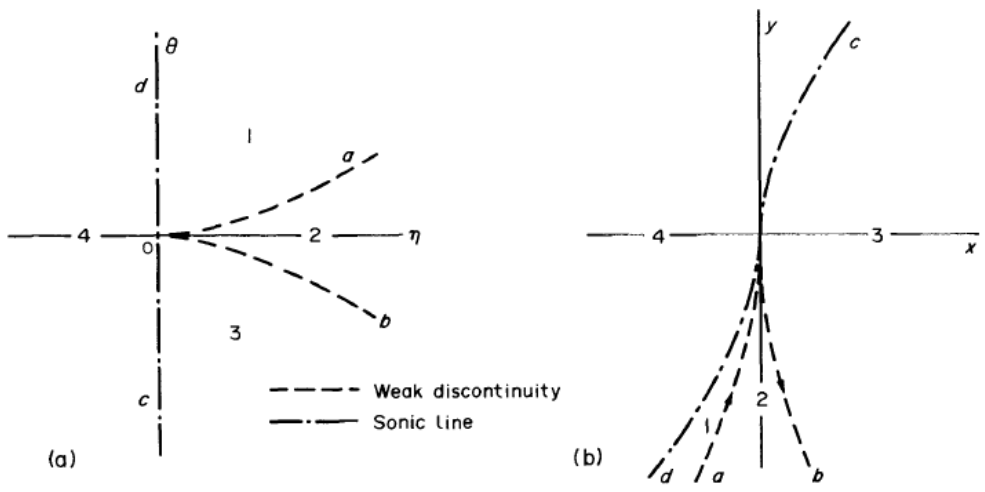

FIG. 125

We shall see that there are two cases, depending on the sign of the product *AB*, the weak discontinuity being reflected as either a logarithmic weak discontinuity or a weak shock wave.

Reflection as a weak discontinuity

Let us take the first case (L. D. Landau and E. M. Lifshitz 1954). A second characteristic in the hodograph plane (*Ob* in Fig. 125a) corresponds to the weak discontinuity reflected from the sonic line. The form of the function Φ near this characteristic is obtained by analytical continuation of the functions (121.2), using (118.11) - (118.13). For k = 11/12, however, the function F_1 is meaningless, and therefore we cannot use these formulae directly. Instead, we must first put $k=(11/12)+\varepsilon$, and then let ε tend to zero. Logarithmic terms then appear, in accordance with the general theory of hypergeometric functions.

The calculation with (118.13) gives the following expression for the function Φ in region 3 near the characteristic *Ob* (we retain terms up to the second order in ξ):

$$\Phi_{b3}=-A\theta\eta+\frac{B}{\pi}(-\theta)^{11/6}\left\{\xi^2\log|\xi|+c_0+c_1\xi+c_2\xi^2\right\}, \qquad (121.3)$$

where c_0, c_1, c_2 are numerical constants.[2]

A similar transformation (using formula (118.11)) of the function Φ_{a2} from the neighbourhood of the characteristic *Oa* to that of *Ob* gives an expression for Φ_{b2} similar to (121.3), with $\frac{1}{2}C$ in place of *B*. The coordinates *x* and *y* of points on the characteristic in the physical plane are calculated as the derivatives (118.4) for $\xi=0$. Starting from (121.3), we find

$$\left.\begin{aligned} x&=-A\theta-\frac{12^{1/3}Bc_1}{\pi}(-\theta)^{7/6}\\ y&=-A\left(-\frac{3}{2}\theta\right)^{2/3}-\frac{B}{\pi}\left(\frac{11}{6}c_0+2c_1\right)(-\theta)^{5/6}\end{aligned}\right\} \qquad \text{‘121.4)}$$

and differentiation of Φ_{b2} gives the same with $\frac{1}{2}C$ instead of *B*. The condition that the coordinates *x* and *y* be continuous at the characteristic *Ob* therefore gives

$$C=2B\,. \qquad (121.5)$$

Next, for this type of reflection to occur, there must be no limiting lines in the hodograph plane (and therefore no non-physical regions in that plane), i.e., the Jacobian Δ must not vanish anywhere. Near the characteristic *Oa*, Δ can be calculated from the functions (121.2), and is seen to be positive; the leading term in Δ is $\cong A^2$. Near the characteristic *Ob*, a calculation using (121.3) gives

$$\Delta\cong A-16\left(\frac{3}{2}\right)^{1/6}AB\eta^{1/4}\log|\xi|\,. \qquad (121.6)$$

As we approach the characteristic, the logarithm tends to $-\infty$, and the second term is the leading one. The condition $\Delta>0$ therefore gives *AB* > 0, i.e., *A* and *B* must have the same sign.

Finally, to determine the form of the sonic line, we need an expression for Φ near the axis $\eta=0$. An expression valid near the upper half is obtained by simply transforming the hypergeometric function in Φ (121.2) into hypergeometric functions of argument $1-\xi=\dfrac{4\eta^3}{9\theta^2}$, which vanishes for $\eta=0$.[3] On retaining only terms of the lowest degrees in η, we obtain

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[2] Their values are $c_0=-2^9\cdot 3^4/385=-108$, $c_1=288/7=41.1$, $c_2=4.86$.

[3] The transformation is given, for example, in *QM*, Mathematical Appendices, formula (e.7).

$$\Phi_d = -A\eta\theta - \frac{2\Gamma(1/3)}{\Gamma(23/12)\Gamma(17/12)} B\theta^{11/6} = -A\eta\theta - 6.25B\theta^{11/6}. \qquad (121.7)$$

An analytical continuation into the region near the lower half of the axis gives

$$\Phi_c = -A\eta\theta - 6.25\sqrt{3}B\theta^{11/6}; \qquad (121.8)$$

the calculations are similar to those used in deriving the transformation formulae (118.13).

We can now determine the form of all the lines under consideration. On the characteristics we have, omitting terms of higher order, $x = -A\theta$, $y = -A\eta$. We arbitrarily suppose that the upper characteristic ($\theta > 0$) corresponds to the weak discontinuity reaching the intersection. Since the gas velocity is in the positive x-direction, this discontinuity is the one which reaches the intersection if it lies in the half-plane $x < 0$. Hence it follows that the constant A, and therefore the constant B also, must be positive. The equation of the line of discontinuity in the physical plane is

$$-y = \left(\frac{3}{2}\right)^{2/3} A^{1/3}(-x)^{2/3} = 1.31A^{1/3}(-x)^{2/3}. \qquad (121.9)$$

The reflected discontinuity, which corresponds to the lower characteristic, is given by the equation[4]

$$-y = 1.31A^{1/3}x^{2/3} \qquad (121.10)$$

(Fig. 125b, in which the lines and regions are marked in correspondence with those in Fig. 125a).

The equation of the sonic line is obtained from the functions (121.7) and (121.8). Effecting the differentiation with respect to η and θ, and then putting $\eta = 0$, we obtain from (121.7) the equation of the part for which $\theta > 0$: $x = -A\theta$, $y = -\frac{11}{6}\cdot 6.25B\theta^{5/6}$, whence

$$y = -11.4BA^{-5/6}(-x)^{5/6}. \qquad (121.11)$$

This is the lower part of the sonic line in Fig. 125b. Similarly, we obtain from (121.8) the equation of the upper part of this line:

$$y = 11.4\sqrt{3}BA^{-5/6}x^{5/6}. \qquad (121.12)$$

Thus both discontinuities and both branches of the sonic line have a common tangent (the y-axis) at the point of intersection O. Near this point the two branches of the sonic line are on opposite sides of the y-axis.

On the discontinuity which reaches O, the spatial derivatives of the velocity are discontinuous; as a characteristic quantity we may consider the discontinuity of the derivative $\left(\frac{\partial\eta}{\partial x}\right)_y$. Using the fact that

$$\left(\frac{\partial\eta}{\partial x}\right)_y = \frac{\partial(\eta, y)}{\partial(x, y)} = \frac{\dfrac{\partial(\eta, y)}{\partial(\eta,\theta)}}{\dfrac{\partial(x, y)}{\partial(\eta,\theta)}} = -\frac{1}{\Delta}\frac{\partial^2\Phi}{\partial\theta^2}$$

and formulae (121.2), (121.5), we obtain

$$\left[\left(\frac{\partial\eta}{\partial x}\right)_y\right]_1^2 = 8\left(\frac{3}{2}\right)^{1/6}\frac{B}{A^2}\eta^{-1/4} = 8.56BA^{-7/4}(-y)^{-1/4}. \qquad (121.13)$$

Thus this discontinuity increases as $(-y)^{-1/4}$ as we approach the point of intersection.

On the reflected weak discontinuity, the derivatives of the velocity are not discontinuous,

[4] When the first correction terms (the second terms in (121.4)) are taken into account, the equation of the reflected discontinuity is

$$-y = 1.31A^{1/3}x^{2/3} - 10.5BA^{-5/6}x^{5/6} \qquad (121.10a)$$

but the velocity distribution has a curious logarithmic singularity. Calculating the coordinates x and y as functions of η, θ from (121.3) (keeping only the first term in the braces), we can put the dependence of η on x for given y near the reflected discontinuity in the parametric form

$$\left.\begin{aligned} \eta &= \frac{|y|}{A} + \frac{x-x_0}{2\sqrt{A|y|}} - \frac{1}{6A}|y|\zeta \\ x - x_0 &= \frac{1}{3\sqrt{A}}|y|^{3/2}\zeta - 5.7\frac{B|y|^{7/4}}{\pi A^{7/4}}\zeta\log|\zeta| \end{aligned}\right\} \tag{121.14}$$

where ζ is the parameter and $x_0 = x_0(y)$ is the equation of the discontinuity in the physical plane.

Reflection as a shock wave

Let us now consider the other case, that of reflection of a weak discontinuity from the sonic line as a shock wave (L. P. Gor'kov and L. P. Pitaevskii 1962).[5]

This case occurs if $AB < 0$. From (121.6), we see that here there are two limiting lines which are exponentially close to the characteristic Ob: the Jacobian Δ is zero for

$$|\zeta| \cong \frac{2}{|\theta|}\left|\theta + \frac{2}{3}\eta^{3/2}\right| e^{-\Theta}, \quad \Theta = \frac{A\pi(2/3)^{1/6}}{16|B|\eta^{1/4}}. \tag{121.15}$$

It is evident from the start that the boundaries of the non-physical region in the hodograph plane (Ob_2 and Ob_3 in Fig. 126a) will also be exponentially close to the characteristic, and the shock wave is therefore exponentially weak.

Neglecting the exponentially small values of ξ on Ob_2 and Ob_3, we find for the coordinates x and y on them the same expressions as on either side of the characteristic Ob in the previous case. The continuity condition for the coordinates at the shock wave therefore always gives the previous relation (121.5). Accordingly, we have the same expression (121.13) for the change in the velocity derivative at the incident discontinuity. Again assuming that the latter corresponds to the upper characteristic Oa in the hodograph plane, we have as before $A > 0$, so that now $B < 0$. It is seen from (121.13), therefore, that the physical criterion for the two cases of weak-discontinuity reflection is the sign of the change in the velocity derivative at the incident discontinuity.

The equations (121.9) and (121.10) for the incident weak discontinuity and reflected shock wave lines remain the same, if exponentially small corrections are neglected. However, since the sign of B is different, the configuration of these lines in the physical plane is changed, as shown in Fig. 126b.

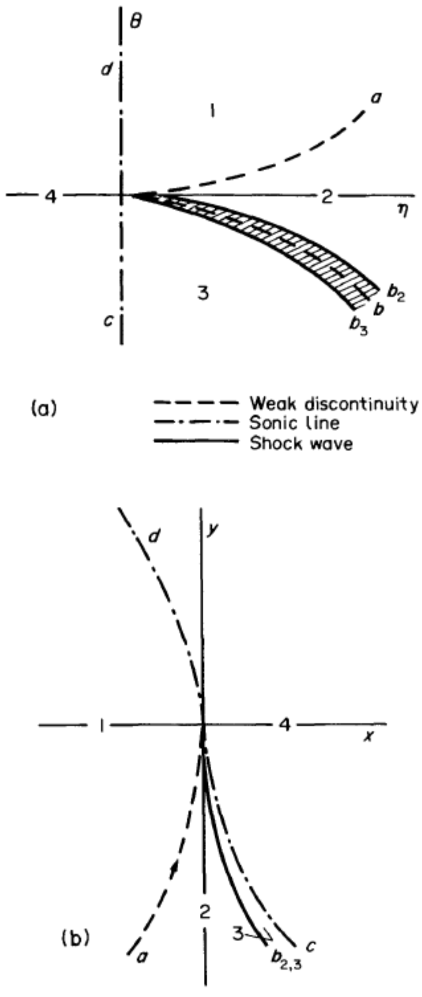

FIG. 126

[5] The possibility in principle of such reflection was noted by K. G. Guderley (1948).

To determine the strength of the shock wave, i.e., the changes $\delta\theta$ and $\delta\eta$ there, we have to use the complete boundary conditions to be satisfied at the shock by the solution of the Euler-Tricomi equation. These have already been formulated as (120.9) - (120.11). The last of these, the equation of the shock polar, becomes $(\delta\theta)^2 = \eta(\delta\eta)^2$, where $\delta\theta = \theta_{b2} - \theta_{b3}$, $\delta\eta = \eta_{b2} - \eta_{b3}$ are exponentially small discontinuities at the shock; the suffixes *b2* and *b3* relate to the lines Ob_2 and Ob_3 in the hodograph plane, that is, to the front and back of the shock in the physical plane respectively. Hence

$$\delta\theta = \sqrt{\eta}\delta\eta\,; \tag{121.16}$$

the choice of the sign of the square root is determined by the fact that, when the gas velocity decreases on passing through the shock, the streamlines must approach the surface of discontinuity.

In accordance with (121.15), we seek the equations of Ob_2 and Ob_3 in the hodograph plane as

$$\theta + \frac{2}{3}\eta^{3/2} = a_{b2}\,|\,\theta\,|\,e^{-\Theta}, \quad \theta + \frac{2}{3}\eta^{3/2} = -a_{b3}\,|\,\theta\,|\,e^{-\Theta},$$

where a_{b2} and a_{b3} are positive. According to (121.16), $\delta\left(gq + \frac{2}{3}\eta^{3/2}\right) = \delta\theta + \sqrt{\eta}d\eta = 2\delta\theta$. The required discontinuities $\delta\theta$ and $\delta\eta$ are therefore

$$\left.\begin{aligned} \delta\theta &= a\frac{x}{A}e^{-\Theta} \\ \delta\eta &= a\left(\frac{2}{3}\right)^{1/3}\left(\frac{x}{A}\right)^{2/3} e^{-\Theta} \\ \Theta &= \frac{A\pi(2/3)^{1/3}}{16\,|\,B\,|}\left(\frac{A}{x}\right)^{1/6} = 0.17\frac{A^{7/6}}{|\,B\,|\,x^{1/6}} \end{aligned}\right\} \quad (121.17$$

where $a = \frac{1}{2}(a_{b2} + a_{b3})$; the variables η and θ are expressed in terms of the coordinates in the physical plane by $x \cong -A\theta$, $y = -A\eta$. The determination of *a* has to take into account also all the remaining boundary conditions, with terms both linear and quadratic in the exponentially small quantity $e^{-\Theta}$.We shall not give these quite lengthy calculations, but simply the result: $a_{b2} = a_{b3} = a = 5.2$.