CHAPTER XIII
FLOW PAST FINITE BODIES

§122. The formation of shock waves in supersonic flow past bodies

Simple arguments show that, in supersonic flow past an arbitrary body, a shock wave must be formed in front of the body. For the disturbances in the supersonic flow caused by the presence of the body are propagated only downstream. Hence a uniform supersonic stream incident on the body would be unperturbed as far as the leading end of the body. The normal component of the gas velocity would then be non-zero at the surface there, in contradiction to the necessary boundary condition. The resolution of this difficulty can only be the occurrence of a shock wave, as a result of which the gas flow between it and the leading end of the body becomes subsonic.

Thus a shock wave is formed in front of the body when the incident flow is supersonic; it is called the ***bow wave***. When the leading end of the body is blunt, the bow wave does not touch the body. In front of the shock wave, the flow is uniform; behind it, the flow is modified and bends round the body (Fig. 127a). The surface of the shock wave extends to infinity, and at great distances from the body, where the shock is weak, it intersects the incident streamlines at an angle approaching the Mach angle. A characteristic feature of flow past a blunt-ended body is the existence of a subsonic flow region behind the shock wave at the most forward part of its surface; this region extends to the body itself, and thus lies between the discontinuity surface, the body, and a lateral sonic surface (the broken curves in Fig. 127a).

The shock wave can touch the body only when the leading end of the latter is pointed. The surface of discontinuity then has a point at the same place (Fig. 127b); in asymmetric flow, part of this surface may be a weak discontinuity.

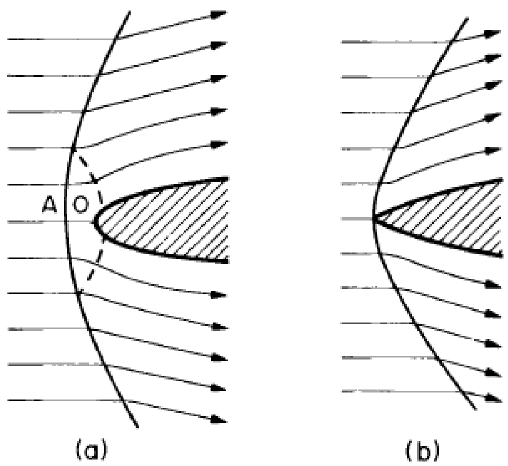

FIG. 127

For a body with a given shape, however, this type of flow pattern is possible only for velocities exceeding a certain limit; at lower velocities, the shock wave is detached from the leading end of the body (see §§113), even if the latter is pointed.

Let us consider **axially symmetrical** supersonic flow past a solid of revolution and determine the gas pressure at the rounded leading end of the body (the stagnation point O in Fig. 127a). It is evident from symmetry that the streamline which terminates at O intersects the shock wave at right angles, so that the velocity component at A normal to the surface of discontinuity is the same as the total velocity. The values of quantities in the incident stream will be denoted, as usual, by the suffix 1, and the values behind the shock wave at the point A by the suffix 2. The latter are determined from formulae (89.6) and (89.7):

$$p_2 = p_1 \frac{2\gamma M_1^{\,2} - (\gamma - 1)}{\gamma + 1},$$

$$v_2 = c_1 \frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1},$$

$$\rho_2 = \rho_1 \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}.$$

The pressure p_0 at the point *O* (where the gas velocity *v* = 0) can now be obtained by means of the formulae which give the variation of quantities along a streamline. We have (see §83, Problem)

$$p_0 = p_2 \left[1+\frac{\gamma-1}{2}\frac{v_2^2}{c_2^2}\right]^{\gamma/(\gamma-1)},$$

and a simple calculation gives

$$p_0 = p_1 \left(\frac{\gamma+1}{2}\right)^{(\gamma+1)/(\gamma-1)} \frac{M_1^2}{\left[\gamma-\frac{\gamma-1}{2}M_1^2\right]^{1/(\gamma-1)}}. \tag{122.1}$$

This determines the pressure at the leading end for a supersonic incident flow ($M_1 > 1$).

For comparison, we give the formula for the pressure at the stagnation point obtained for a continuous adiabatic retardation of the gas, with no shock wave (as would be true for a subsonic incident flow):

$$p_0 = p_1 \left[1+\frac{\gamma-1}{2}M_1^2\right]^{\gamma/(\gamma-1)}. \tag{122.2}$$

For $M_1 = 1$, the two formulae give the same value of p_0, but for $M_1 > 1$ the pressure given by formula (122.2) is always greater than the true pressure p_0 given by formula (122.1).[1]

In the limit of very large velocities ($M_1 >> 1$), formula (122.1) gives

$$p_0 = p_1 \left(\frac{\gamma+1}{2}\right)^{(\gamma+1)/(\gamma-1)} \gamma^{-1/(\gamma-1)} M_1^2, \tag{122.3}$$

i.e., the pressure p_0 is proportional to the square of the incident velocity. From this result we can conclude that the total drag force on the body at velocities large compared with that of sound is proportional to the square of the velocity. It should be noticed that this is the same as the law governing the drag force at velocities small compared with that of sound but yet so large that the Reynolds number is large (see §45).

Besides the fact that shock waves must be formed, we can also say that in supersonic flow past a finite body there must be two successive shock waves at large distances from the body (L. Landau 1945). For the disturbances caused by the body at large distances are small, and can therefore be regarded as a cylindrical sound wave outgoing from the *x*-axis (which passes through the body parallel to the direction of flow); considering the flow, as usual, in a coordinate system where the body is at rest, we have a wave in which the time is represented by x/v_1, and the rate of propagation by $\dfrac{v_1}{\sqrt{M_1^2-1}}$ (see §123). We can therefore apply immediately the results obtained in §102 for a cylindrical wave at large distances from the

[1] This statement is true generally, and does not depend on the assumption of a poly tropic or even a perfect gas in (122.1), (122.2). For, when a shock wave is present, the entropy s_0 of the gas at *O* is greater than s_1, whereas if the shock wave were absent s_0 would be equal to s_1. The heat function is in either case $w_0 = w_1 + (1/2)v_1^2$, since the quantity $w + (1/2)v^2$ is unchanged when a streamline intersects a normal compression discontinuity. From the thermodynamic identity $dw = Tds + dp/\rho$, it follows that the derivative $\left(\frac{\partial p}{\partial s}\right)_w = -\rho T < 0$, i.e., an increase in entropy when *w* remains constant involves a decrease in pressure, whence the result follows.

source. We thus arrive at the following pattern of shock waves far from the body: in the first shock, the pressure increases discontinuously, so that behind it there is a condensation; then follows a region where the pressure gradually decreases into a rarefaction, after which the pressure again increases discontinuously in the second shock. The intensity of the leading shock decreases as $r^{-3/4}$ with increasing distance from the *x*-axis, and the distance between the two shocks increases as $r^{1/4}$.[2]

Let us now examine the appearance and development of the shock waves as the number M_1 gradually increases. A supersonic region first appears for some value of M_1 less than unity, as a region adjoining the surface of the body. At least one shock wave occurs in this region, usually at the edge of the supersonic region.

As M_1 increases, the supersonic region expands, and the length of the shock wave increases. This is the shock wave whose existence for $M_1 = 1$ has been demonstrated (for the two-dimensional case) in § 120; it follows also that the shock wave must first appear for $M_1 < 1$.

As soon as M_1 exceeds unity, another shock wave appears, the **bow wave**, which intersects the whole of the infinitely wide incident stream of gas. For M_1 exactly unity, the flow in front of the body is entirely subsonic. For $M_1 > 1$ but arbitrarily close to unity, therefore, the supersonic part of the incident stream, and consequently the bow wave, are arbitrarily far in front of the body. As M_1 increases further, the bow wave gradually approaches the body.

The shock wave in the local supersonic region must intersect the sonic line in some way. We shall discuss the two-dimensional case. The nature of this intersection is not yet fully understood. If the shock terminates at the intersection, its strength falls to zero there, and the flow is transonic everywhere in the plane near the intersection point. The flow pattern in such a case must be given by the appropriate solution of the Euler-Tricomi equation. In addition to the usual conditions that the solution be single-valued in the physical plane and the boundary conditions at the shock wave, the following conditions must also be satisfied: (1) if the flow is supersonic on both sides of the shock (as when only the shock terminates at the intersection, being "supported" by the sonic line), then the shock wave must be one that reaches the intersection; (2) characteristics in the supersonic region which reach the intersection cannot have any flow singularities, since these could arise only from the intersection itself and would therefore have to be carried away from the intersection point. The existence of a solution of

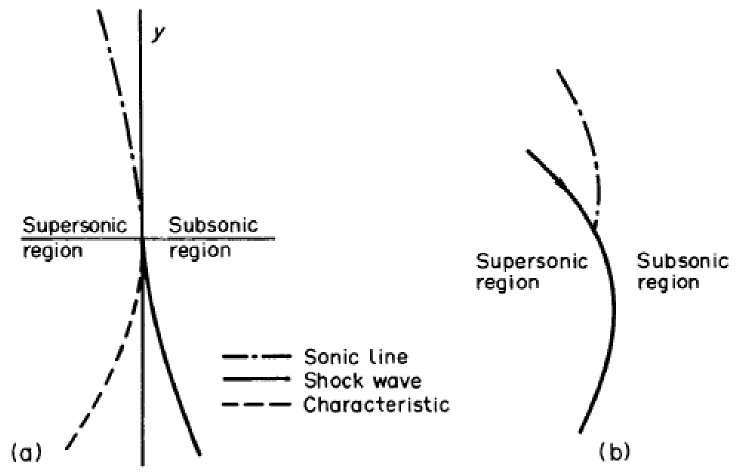

FIG. 128

[2] For shock waves formed in axially symmetrical flow past narrow pointed bodies, the quantitative coefficients in these relationships can be determined; see the second footnote to §12

the Euler-Tricomi equation satisfying these requirements seems to be not yet proved.[3]

Another possible configuration of the shock wave and the sonic line in the local supersonic region is for only the sonic line to terminate at the intersection (Fig. 128b); the shock wave strength need not be zero there, and so the flow near it is transonic only on one side of the shock. The shock wave itself may have one end "supported" by a solid surface, and the other end (or both ends) starting within the supersonic stream (cf. the end of § 115).

§123. Supersonic flow past a pointed body

The shape which a body must have in order to be streamlined in supersonic flow, i.e., to be subject to as small a drag force as possible, is quite different from the corresponding shape for subsonic flow. We may recall that, in the subsonic case, streamlined bodies are those which are elongated, rounded in front, and pointed behind. In supersonic flow past such a body, however, a strong shock wave would be formed in front of it, leading to a considerable increase in the drag. In the supersonic case, therefore, a long streamlined body must be pointed at both ends, and the angle of the point must be small; if the body is inclined to the direction of flow, the angle between them (angle of attack) must also be small.

In steady supersonic flow past a body of this shape, the gas velocity is nowhere very different in magnitude or direction from the incident velocity, even near the body, and the shock waves formed are weak; the intensity of the bow wave decreases with the angle at the front of the body. Far from the body, the gas flow consists of outgoing sound waves. The main part of the drag can be regarded as due to the conversion of kinetic energy of the moving body into the energy of the sound waves which it emits. This drag, which occurs only in supersonic flow, is called ***wave drag***;[4] it can be calculated in a general form valid for any cross-section of the body (T. von Karman and N. B. Moore 1932).

The nature of the flow just described makes it possible to use the linearized equation (114.4) for the potential:

$$\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\beta^2\frac{\partial^2\phi}{\partial x^2}=0\,, \qquad (123.1)$$

where we have introduced for brevity the positive constant

$$\beta^2=\frac{v_1^{\,2}-c_1^{\,2}}{c}\,; \qquad (123.2)$$

the x-axis is in the direction of the flow, the suffix 1 denotes quantities pertaining to the incident stream, and $1/\beta$ is just the tangent of the Mach angle.

Equation (123.1) is formally identical with the two-dimensional wave equation with x/v_1 representing the time and v_1/β the velocity of the waves. This is no accident; the physical significance is that the gas flow far from the body consists, as already mentioned, of outgoing sound waves emitted by the body. If the gas at infinity is regarded as being at rest, and the body as being in motion, the cross-section of the body at a given point in space will vary with time, and the distance to which a disturbance is propagated at time t (i.e., the distance to the Mach cone) will increase as $v_1 t/\beta$. Thus we shall have a two-dimensional emission of sound (propagated with velocity v_1/β) by the variable profile.

Using this "**sonic analogy**" as a guide, we can immediately write down the required expression for the velocity potential of the gas, using formula (74.15) for the potential of cylindrical sound waves emitted from a source (at distances large compared with the dimension of the source) and replacing ct by x/β.

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[3] Germain has found several types of solution of the Euler-Tricomi equation which might represent the intersection of a shock wave with the sonic line, but they have not been at all fully investigated. Some of them do not satisfy condition (1) above. Figure 128a shows a case which might correspond to the termination of a shock wave forming the boundary of the local supersonic region: at the point of intersection, the shock wave and the sonic line both terminate and have a common tangent, and lie on opposite sides of it (the gas moves from left to right). The fulfilment of condition (2) has not been tested, however. Only the possible range of k has been determined ($3/4<k<11/12$), but it is not known whether one can satisfy the condition for the coordinates to be continuous at the shock wave in the physical plane. See P. Germain, *Progress in Aeronautical Sciences* 5, 143, 1964.

[4] The total drag is obtained by adding to the wave drag the forces due to friction and to separation at the trailing end of the body.

Let $S(x)$ be the area of the cross-section of the body in a plane perpendicular to the direction of flow (the x-axis), and l the length of the body in that direction; we take the origin at the leading end of the body. Then

$$\phi(x,r) = -\frac{v_1}{2\pi}\int_0^{x-\beta r}\frac{S'(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}}\,; \qquad (123.3)$$

the lower limit is taken as zero, since $S(x) = 0$ for $x < 0$ (and for $x > l$).

Thus we have completely determined the gas flow at distances r from the axis which are large compared with the thickness of the body.[5] Disturbances leaving the body in a supersonic flow are, of course, propagated only into the region behind the cone $x-\beta r=0$, whose vertex is at the leading end of the body; in front of this cone we have simply $\phi=0$ (uniform flow). Between the cones $x-\beta r=0$ and $x-\beta r=l$, the potential is determined by formula (123.3); behind the latter cone (whose vertex is at the trailing end of the body) the upper limit of the integral in (123.3) is the constant l. Both these cones are weak discontinuities, in the approximation considered; in reality, they are weak shock waves.

The drag force acting on the body is just the x-component of the momentum carried away by the sound waves per unit time. We take a cylindrical surface with large radius r and axis along the x-axis. The x-component of the **momentum flux density** through this surface is

$$\Pi_{xr} = \rho v_r(v_x+v_1) \cong \rho_1\frac{\partial\phi}{\partial r}\left(v_1+\frac{\partial\phi}{\partial x}\right).$$

On integration over the whole surface, the first term gives zero, since the integral of ρv_r is the total mass flux through the surface, which is zero. Thus

$$F_x = -2\pi r\int_{-\infty}^{\infty}\Pi_{xr}dx = -2\pi r\rho_1\int_{-\infty}^{\infty}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial x}dx\,. \qquad (123.4)$$

At large distances (in the wave region), the derivatives of the potential can be calculated as in § 74 (see formula (74.17)), and we have

$$\frac{\partial\phi}{\partial r} = -\beta\frac{\partial\phi}{\partial x} = \frac{v_1}{2\pi}\sqrt{\frac{\beta}{2r}}\int_{-\infty}^{\infty}\frac{S''(\xi)d\xi}{\sqrt{x-\xi-\beta r}}\,.$$

This expression is substituted in (123.4), and the squared integral is written as a double integral; putting for brevity $x-\beta r = X$, we obtain

$$F_x = \frac{\rho_1 v_1^2}{4\pi}\int_{-\infty}^{\infty}\int_0^X\int_0^X\frac{S''(\xi_1)S''(\xi_2)d\xi_1 d\xi_2 dX}{\sqrt{(X-\xi_1)(X-\xi_2)}}\,.$$

The integration over X can be effected; after changing the order of integration, the integral is from the greater of ξ_1 and ξ_2 to infinity. We first take as the upper limit a large but finite quantity L, which later tends to infinity. Thus

$$F_x = -\frac{\rho_1 v_1^2}{2\pi}\int_0^l\int_0^{\xi_2}S''(\xi_1)S''(\xi_2)\left[\log(\xi_2-\xi_1)-\log 4L\right]d\xi_1 d\xi_2\,.$$

The integral of the term containing the constant factor $\log 4L$ is zero, since not only the area $S(x)$ but also its derivative $S'(x)$ vanishes at the pointed ends of the body. We therefore have

$$F_x = -\frac{\rho_1 v_1^2}{2\pi}\int_0^l\int_0^{\xi_2}S''(\xi_1)S''(\xi_2)\log(\xi_2-\xi_1)d\xi_1 d\xi_2\,,$$

or

[5] For axially symmetrical flow past a solid of revolution, (123.3) is valid for all r up to the surface of the body. In particular, it again gives (113.6) for flow past a narrow cone. On the other hand, if this linear-approximation solution is considered at a great distance from the body, a correction can be applied to it for the non-linear distortion of the profile, as was done in § 102 for a cylindrical sound wave. This gives the strength of the shock wave at large distances from a narrow pointed solid of revolution, including the dependence on M_1, i.e., the coefficient in the law of damping ($\propto r^{-3/4}$) described in § 122. See G. B. Whitham, *Linear and Nonlinear Waves*, New York 1974, §9.3.

$$F_x = -\frac{\rho_1 v_1^2}{2\pi} \int_0^l \int_0^l S''(\xi_1) S''(\xi_2) \log|\xi_2 - \xi_1| \, d\xi_1 d\xi_2 . \qquad (123.5)$$

This is the required formula for the **wave drag** on a narrow pointed body.[6] The order of magnitude of the integral is $\left(\frac{S}{l^2}\right)^2 l^2$, where S is some mean cross-sectional area of the body. Hence $F_x \sim \frac{\rho_1 v_1^2 S^2}{l^2}$. The drag coefficient for an elongated body may be conventionally defined, in terms of the square of the length, as

$$C_x = \frac{F_x}{\frac{1}{2}\rho_1 v_1^2 l^2} . \qquad (123.6)$$

Then, in this case,

$$C_x \sim \frac{S^2}{l^4} ; \qquad (123.7)$$

it is proportional to the square of the cross-sectional area.

We may point out the formal analogy between formula (123.5) and formula (47.4) for the induced drag on a thin wing; the function $\Gamma(z)$ in (47.4) is here replaced by $v_1 S'(x)$. On account of this analogy we can use, to calculate the integral in (123.5), the method described at the end of §47.

It should also be noticed that the wave drag given by formula (123.5) is unchanged if the direction of flow is reversed: the integral is independent of the direction in which the body extends. This property of the drag force is characteristic of the linearized theory.[7]

Finally, let us briefly discuss the range of applicability of this formula. This subject may be approached as follows. The amplitude of oscillation of the gas particles in the sound waves emitted by the body is of the order of magnitude of the thickness of the body, which we denote by δ. The velocity of the oscillations is accordingly of the order of the ratio $\delta : (l/v)$ of the amplitude δ to the period l/v_1 of the wave. The linear approximation for the propagation of sound waves (i.e., the linearized equation for the potential), however, always requires that the gas velocity be small compared with the velocity of sound, i.e., we must have $v_1/\beta >> v_1\delta/l$, or, what is in practice the same,

$$M_1 << \frac{l}{\delta} . \qquad (123.8)$$

Thus the theory given above becomes inapplicable for values of M_1 comparable with the ratio of length to thickness of the body.

It is also inapplicable, of course, in the opposite limiting case where M_1 is close to unity and the linearization of the equations is invalid.

PROBLEM

Determine the form of the elongated solid of revolution which experiences the smallest drag for a given volume *V* and length *l*.

Solution. On account of the analogy mentioned in the text, we introduce a variable θ such that $x = \frac{l}{2}(1 - \cos\theta)$ ($0 \le \theta \le \pi$; the origin of *x* is at the leading end of the body); and write

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[6] The lift (for a body not axially symmetrical or a non-zero angle of attack) is zero in the approximation here considered.

[7] It also holds in the theory of the wave drag on thin wings given in §125.

the function $f(x) = S'(x)$ as

$$f = -l\sum_{n=2}^{\infty} A_n \sin n\theta ;$$

the condition $S = 0$ for $x = 0$ and l means that only terms with $n \geq 2$ can appear in the sum. The drag coefficient is then

$$C_x = \frac{\pi}{4}\sum_{n=2}^{\infty} n A_n{}^2 .$$

The area $S(x)$ and the total volume V of the body are calculated from the function $f(x)$ as

$$S = \int_0^x f(x)dx , \quad V = \int_0^l S(x)dx .$$

A simple calculation gives $V = \frac{\pi l^3 A_2}{16}$, i.e., the volume is determined by the coefficient A_2 alone. The minimum F_x is therefore reached if $A_n = 0$ for $n \geq 3$. The result is

$$C_{x,\min} = \frac{128}{\pi}\left(\frac{V}{l^3}\right)^2 = \frac{9\pi}{2}\left(\frac{S_{\max}}{l^2}\right)^2 .$$

The cross-sectional area of the body is $S = \frac{1}{3} l^2 A_2 \sin^3 \theta$, and the radius as a function of x is therefore

$$R(x) = \sqrt{2}\,\frac{8}{\pi}\left(\frac{V}{3l^4}\right)^{1/2} [x(l-x)]^{3/4} .$$

The body is symmetrical about the plane $x = l/2$.[8]

§124. Subsonic flow past a thin wing

Let us consider subsonic flow of a gas past a thin streamlined wing. As for an incompressible fluid, a wing which is streamlined for subsonic flow must be thin, pointed at the trailing edge, and rounded at the leading edge, and the angle of attack must be small. We take the direction of flow as the x-axis and the direction of the span as the z-axis.

The gas velocity nowhere[9] differs greatly from the velocity v_1 of the incident stream, so that we can use the linearized equation (114.4) for the potential:

$$(1 - M_1{}^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 . \qquad (1241)$$

At the surface of the wing (which we call C), the velocity must be tangential; introducing a unit vector $\boldsymbol{n}$ along the normal to the surface, we can write this condition as

$$\left(v_1 + \frac{\partial \phi}{\partial x}\right) n_x + \frac{\partial \phi}{\partial y} n_y + \frac{\partial \phi}{\partial z} n_z = 0 .$$

Since the wing is flattened and the angle of attack is small, the normal $\boldsymbol{n}$ is almost parallel to the y-axis, so that $|n_y|$ is almost unity, while n_x and n_z are small. We can therefore neglect the second-order terms $n_x \frac{\partial \phi}{\partial x}$ and $n_z \frac{\partial \phi}{\partial z}$, and replace n_y by ± 1 (+ 1 on the upper surface of the wing and – 1 on the lower surface). Thus the boundary condition on equation (124.1) is

$$v_1 n_x \pm \frac{\partial \phi}{\partial y} = 0 . \qquad (124.2)$$

[8] Although $R(x)$ vanishes at the ends of the body, $R'(x)$ becomes infinite, i.e., the body is not pointed; the approximation underlying the method is therefore not strictly applicable near the ends.

[9] Except in a small region near the leading edge of the wing, where there is a stagnation line.

Since the wing is assumed thin, $\frac{\partial \phi}{\partial y}$ on its surface can be taken as the limiting value for $y \to 0$.

The solution of equation (124.1) with the condition (124.2) can easily be reduced to the solution of a problem of incompressible flow. To do so, we use instead of the coordinates *x*, *y*, *z* the variables

$$\left.\begin{aligned} x' &= x \\ y' &= y\sqrt{1-M_1{}^2} \\ z' &= z\sqrt{1-M_1{}^2} \end{aligned}\right\}. \qquad (124.3)$$

In these variables, equation (124.1) becomes

$$\frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial^2 \phi}{\partial y'^2} + \frac{\partial^2 \phi}{\partial z'^2} = 0\,, \qquad (124.4)$$

i.e., Laplace's equation. The surface of the body is replaced by another, *C'*, obtained by leaving unchanged the profiles of cross-sections by planes parallel to the *xy*-plane, but reducing in the ratio $\sqrt{1-M_1{}^2}$ all dimensions in the direction of the span (the *z*-direction).

The boundary condition (124.2) then becomes

$$v_1 n_x \pm \frac{\partial \phi}{\partial y'}\sqrt{1-M_1{}^2} = 0\,,$$

and it can be reduced to the previous form by introducing in place of ϕ a new potential ϕ':

$$\phi' = \phi\sqrt{1-M_1{}^2}\,. \qquad (124.5)$$

We then have for ϕ' Laplace's equation with the boundary condition

$$v_1 n_x \pm \frac{\partial \phi'}{\partial y'} = 0\,, \qquad (124.6)$$

which must be satisfied for $y' = 0$.

Equation (124.4) with the boundary condition (124.6), is, however, the equation which must be satisfied by the velocity potential of an incompressible fluid flowing past the surface *C'*. Thus the problem of determining the velocity distribution in compressible flow past a wing with surface *C* is equivalent to that of finding the velocity distribution in incompressible flow past a wing with surface *C'*.

Next, let us consider the lift force F_y acting on the wing. First of all, we note that the derivation of Zhukovskii's formula (38.4) given in §38 is entirely valid for a compressible fluid, since the variable density ρ of the fluid can be replaced in that approximation by a constant ρ_1. Thus

$$F_y = -\rho_1 v_1 \int \Gamma dz\,, \qquad (124.7)$$

where the integration is taken along the span l_z of the wing. From the relation (124.5) and the equality of the transverse profiles of the wings *C* and *C'* it follows that the velocity circulation Γ in compressible flow past the wing *C* is related to the circulation Γ' in incompressible flow past the wing *C'* by

$$\Gamma' = \Gamma\sqrt{1-M_1{}^2}\,. \qquad (124.8)$$

Substituting this in (124.7) and changing to an integration over *z'*, we obtain

$$F_y = -\rho_1 v_1 \int \frac{\Gamma' dz'}{1-M_1{}^2}\,.$$

The numerator is the lift force on the wing *C'* in an incompressible fluid. Denoting it by F'_y, we have

$$F_y = \frac{F'_y}{1-M_1^2} \tag{124.9}$$

Introducing the lift coefficients

$$C_y = \frac{F_y}{\frac{1}{2}\rho_1 v_1^2 l_x l_z}, \quad C'_y = \frac{F'_y}{\frac{1}{2}\rho_1 v_1^2 l_x l_z'}$$

(where l_x, l_z and l_x, $l_z' = l_z\sqrt{1-M_1^2}$ are the lengths of the wings *C* and *C'* in the *x* and *z* directions), we can rewrite this equation as

$$C_y = \frac{C'_y}{\sqrt{1-M_1^2}}. \tag{124.10}$$

For wings with large span (and constant profile), the lift coefficient in an incompressible fluid is proportional to the angle of attack, and does not depend on the length or width of the wing:

$$C'_y = \text{constant} \times \alpha, \tag{124.11}$$

where the constant depends only on the shape of the profile (see §46). In this case, therefore, (124.10) can be replaced by

$$C'_y = \frac{C_y^{(0)}}{\sqrt{1-M_1^2}}, \tag{124.12}$$

where C_y and $C_y^{(0)}$ are the lift coefficients for the same wing in compressible and incompressible fluids, respectively. Thus we have the rule that the lift force acting on a long wing in a compressible fluid is $\frac{1}{\sqrt{1-M_1^2}}$ times that on the same wing (at the same angle of attack) in an incompressible fluid (L. Prandtl 1922, H. Glauert 1928).

Similar relations can be obtained for the drag force. Together with Zhukovskii's formula for the lift force, formula (47.4) for the induced drag on a wing is also entirely applicable to compressible flow. Effecting the same transformations (124.3) and (124.8), we obtain

$$F_x = \frac{F'_x}{1-M_1^2}, \tag{124.13}$$

where F'_x is the drag on the wing *C'* in an incompressible fluid. When the span increases, the induced drag tends to a constant limit (§47). For sufficiently long wings we can therefore replace F'_x by $F_x^{(0)}$ (the drag in an incompressible fluid for the wing *C*). Then the drag coefficient is

$$C_x = \frac{C_x^{(0)}}{1-M_1^2}. \tag{124.14}$$

Comparing this with (124.12), we see that the ratio $\frac{C_y^2}{C_x}$ is the same for compressible and incompressible fluids.

All the results given here are, of course, invalid for values of M_1 close to unity, since the linearized theory then becomes inapplicable.

§125. Supersonic flow past a wing

If a wing is streamlined in a supersonic stream, it must be pointed at both ends, like the thin bodies discussed in §123.

Here we shall consider only the flow past a thin wing with very large span, the profile being constant along the span. Regarding the span as infinite, we have a two-dimensional gas

flow (in the xy-plane). Instead of equation (123.1), we now have for the potential the equation

$$\frac{\partial^2\phi}{\partial y^2} - \beta^2 \frac{\partial^2\phi}{\partial x^2} = 0, \tag{125.1}$$

with the boundary condition

$$\left[\frac{\partial\phi}{\partial y}\right]_{y\to\pm 0} = \mp v_1 n_x, \tag{125.2}$$

where the signs $\mp$ on the right relate to the upper and lower surfaces of the wing, respectively. Equation (125.1) is a one-dimensional wave equation, and its general solution has the form $\phi = f_1(x-\beta y) + f_2(x+\beta y)$. The fact that disturbances which affect the flow start from the body means that above the wing ($y > 0$) we must have $f_2 \equiv 0$, so that $\phi = f_1(x-\beta y)$, and below the wing ($y < 0$) $\phi = f_2(x+\beta y)$. For definiteness, we shall consider the region above the wing, where $\phi = f(x-\beta y)$. The function f is determined from the condition (125.2) by putting $n_x \cong -\zeta_2'(x)$, where $y = \zeta_2(x)$ is the equation of the upper part of the wing profile (Fig. 129a). We have $\left[\frac{\partial\phi}{\partial y}\right]_{y\to +0} = -\beta f'(x) = v_1\zeta_2'(x)$, whence $f = -\frac{v_1\zeta_2(x)}{\beta}$. Thus the velocity distribution for $y > 0$ is given by the potential

$$\phi(x, y) = -\frac{v_1}{\beta}\zeta_2(x-\beta y). \tag{125.3}$$

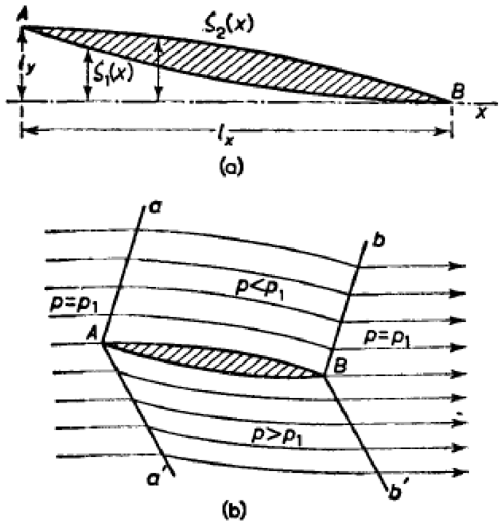

FIG. 129

Similarly we obtain, for $y < 0$, $\phi(x, y) = \frac{v_1}{\beta}\zeta_1(x+\beta y)$, where $y = \zeta_1(x)$ is the equation of the lower part of the profile. It should be noticed that the potential, and therefore the other quantities, are constant along the straight lines $x \pm \beta y = \text{constant}$ (the characteristics), in accordance with the results of § 115, of which the solution just obtained is a particular case.

The flow pattern is qualitatively as follows. Weak discontinuities (aAa' and bBb' in Fig. 129b) leave the pointed leading and trailing edges.[10] In the regions in front of the

[10] This statement is valid only in the approximation used here. In reality we have not weak discontinuities but weak shock waves or narrow centred rarefaction waves, depending on the direction in which the velocity is turned by them.

discontinuity aAa' and behind bBb' the flow is uniform, but between them it is turned so as to go round the surface of the wing; the flow here is a simple wave, and in the present linearized approximation the characteristics are all parallel and inclined at the Mach angle of the incident stream.

The pressure distribution is given by the formula $p - p_1 = -\rho_1 v_1 \dfrac{\partial \phi}{\partial x}$; the term in v_y^2 in the general formula (114.5) can here be omitted, since v_x and v_y are of the same order of magnitude. Substituting (125.3) and introducing the ***pressure coefficient*** C_p, we obtain in the upper half-plane $C_p = \dfrac{p - p_1}{\frac{1}{2}\rho_1 v_1^2} = \dfrac{2\zeta_2'(x - \beta y)}{\beta}$. In particular, the pressure coefficient on the upper surface of the wing is

$$C_{p2} = \frac{2\zeta_2'(x)}{\beta}. \qquad (125.4)$$

Similarly, we find for the lower surface

$$C_{p1} = -\frac{2\zeta_1'(x)}{\beta}. \qquad (125.5)$$

It should be noted that the pressure at any point on the wing profile depends only on the slope of the profile contour at that point.

Since the angle between the profile contour and the x-axis is always small, the vertical component of the pressure force can be taken, with sufficient accuracy, as the pressure itself. The resultant lift force on the wing is equal to the difference of the pressures on the lower and upper surfaces. The lift coefficient is therefore

$$C_y = \frac{1}{l_x}\int_0^{l_x})C_{p1} - C_{p2})dx = \frac{4l_y}{\beta l_x};$$

see Fig. 129a for the definition of l_x, l_y. We define the angle of attack α as the angle between the chord AB through the ends of the profile (Fig. 129a) and the x-axis: $\alpha \cong \dfrac{l_y}{l_x}$, and obtain the following simple formula:

$$C_y = \frac{4\alpha}{\sqrt{M_1^2 - 1}} \qquad (125.6)$$

(J. Ackeret 1925). We see that the lift force is determined by the angle of attack, and does not depend on the form of the wing cross-section, unlike what happens for subsonic flow (see formula (48.7)).

Let us next determine the drag force on the wing (i.e., the wave drag, which is of the same nature as that on thin bodies; see §123). To do so, we must take the x-component of the pressure force and integrate over the profile contour. The drag coefficient is then found to be

$$C_x = \frac{2}{\beta l_x}\int_0^{l_x}\left(\zeta_1'^2 + \zeta_2'^2\right)dx. \qquad (125.7)$$

We put $\zeta_1' = \theta_1 - \alpha$, $\zeta12' = \theta_2 - \alpha$, where $\theta_1(x)$ and $\theta_2(x)$ are the angles between the upper and lower parts of the contour and the chord AB. The integrals of θ_1 and θ_2 are evidently zero, and the result is therefore

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For the profile shown in Fig. 129b, for example, Aa and Bb' are rarefaction waves, while Aa' and Bb' are shock waves. The streamline leaving the trailing edge (B in Fig. 129b) is actually a tangential discontinuity of the velocity (which in practice becomes a narrow turbulent wake).

$$C_x = \frac{4\alpha^2 + 2(\overline{\theta_1^2} + \overline{\theta_2^2})}{\sqrt{M_1^2 - 1}}; \tag{125.8}$$

the bar denotes an average with respect to *x*. For a given angle of attack, the drag coefficient is seen to be least for a wing in the form of a flat plate (for which $\theta_1 = \theta_2 = 0$). In this case $C_x = \alpha C_y$. If we apply formula (125.8) to a rough surface, we find that the roughness may result in a considerable increase in the drag, even if the height of the irregularities is small.[11] For the drag is independent of the height of the irregularities if the mean slope of the surface, i.e., the mean ratio of the height of the irregularities to the distance between them, remains constant.

Finally, we may make the following remark. Here, as everywhere, when we speak of a wing we imply that its edges are perpendicular to the flow. The generalization to the case of any angle γ between the direction of flow and the edge (the ***angle of yaw***) is quite obvious. It is clear that the forces on an infinite wing with constant cross-section depend only on the component of the incident velocity normal to its edges; in an ideal fluid, the velocity component parallel to the edges does not result in a force. The forces acting on a wing at an angle of yaw other than $\pi/2$ in a stream with Mach number M_1 are therefore the same as those on the same wing for $\gamma = \pi/2$ in a stream with Mach number $M_1 \sin\gamma$. In particular, if $M_1 > 1$ but $M_1 \sin\gamma < 1$, the wave drag, which is peculiar to supersonic flow, will not occur.

[11] But nevertheless greater than the thickness of the boundary layer.