§126. The law of transonic similarity

The theory of supersonic and subsonic flow past thin bodies developed in §§ 123-125 is not applicable to transonic flow, when the linearized equation for the potential becomes invalid. In this case the flow pattern in all space is given by the non-linear equation (114.10)

$$2\alpha_* \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \tag{126.1}$$

(or, for two-dimensional flow, by the equivalent Euler-Tricomi equation). The solution of these equations for particular cases is very difficult, however. The similarity rules which can be established for such flows, without finding any particular solution, are therefore of great interest.

Let us first consider two-dimensional flow, and let

$$Y = \delta f(x/l) \tag{126.2}$$

be the equation which gives the shape of the thin contour past which the flow takes place, l being its length (in the direction of flow) and δ some characteristic thickness ($\delta << l$). By varying the two parameters l and δ, we obtain a family of similar contours.

The equation of motion is

$$2\alpha_* \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y^2}, \tag{126.3}$$

with the following boundary conditions. At infinity, the velocity equal the velocity v_1 of the undisturbed stream, i.e.,

$$\frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial \phi}{\partial x} = M_{1*} - 1 = \frac{M_1 - 1}{\alpha_*}; \tag{126.4}$$

see the definition of the potential ϕ, (114.9). On the profile, the velocity must be tangential:

$$\frac{v_y}{v_x} \cong \frac{\partial \phi}{\partial y} = \frac{dY}{dX} = \frac{\delta}{l} f'(x/l); \tag{126.5}$$

since the profile is thin, this condition can be imposed at $y = 0$.

We introduce dimensionless variables thus:

$$x = l\bar{x}, \quad y = \frac{l}{(\theta\alpha_*)^{1/3}}\bar{y}, \quad \phi = \frac{l\theta^{2/3}}{\alpha_*^{1/3}}\bar{\phi}(\bar{x}, \bar{y}); \tag{126.6}$$

here $\theta = \delta / l$ gives the angular thickness of the wing or angle of attack. Then

$$2\frac{\partial \bar{\phi}}{\partial \bar{x}}\frac{\partial^2 \bar{\phi}}{\partial \bar{x}^2} = \frac{\partial^2 \bar{\phi}}{\partial \bar{y}^2},$$

with the following boundary conditions:

$$\frac{\partial \bar{\phi}}{\partial \bar{x}} = K, \quad \frac{\partial \bar{\phi}}{\partial \bar{y}} = 0 \quad \text{at infinity,}$$

$$\frac{\partial \bar{\phi}}{\partial \bar{y}} = f'(\bar{x}), \qquad \text{at } \bar{y} = 0,$$

where

$$K = \frac{M_1 - 1}{(\alpha_*\theta)^{2/3}}. \tag{126.7}$$

These conditions contain only one parameter, K. Thus we have obtained the required **similarity law**: two-dimensional transonic flows with the same value of K are similar, as is shown by formulae (126.6) (S. V. Fal'kovich 1947).

It should be noticed that the expression (126.7) involves only a single parameter α_* which characterizes the properties of the gas itself. The similarity law therefore determines also the similarity with respect to a change in the gas.

In the approximation here considered, the pressure is given by the formula $p - p_1 \cong -\rho_1 v_1 (v_x - v_1)$. A calculation using the expressions (126.6) shows that the pressure

coefficient on the profile has the form

$$C_p = \frac{p - p_1}{\frac{1}{2}\rho_1 v_1{}^2} = \frac{\theta^{2/3}}{\alpha_*{}^{1/3}} P\left(K, \frac{x}{l}\right).$$

The drag and lift coefficients are given by integrals along the contour of the profile:

$$C_x = \frac{1}{l}\oint C_p \frac{dY}{dX} dx \,,$$

$$C_y = \frac{1}{l}\oint C_p dx \,,$$

and therefore have the form[1]

$$\left.\begin{aligned} C_x &= \frac{\theta^{5/3}}{\alpha_*{}^{1/3}} f_x(K) \\ C_y &= \frac{\theta^{2/3}}{\alpha_*{}^{1/3}} f_y(K) \end{aligned}\right\}. \qquad (126.8)$$

In an entirely similar manner, we can obtain the similarity law for a three-dimensional thin body whose shape is given by equations having the form

$$Y = \delta f_1(x/l)\,, \quad Z = \delta f_2(x/l)\,, \qquad (126.9)$$

with the two parameters δ and l ($\delta << l$). There is an important difference from the two-dimensional case, because the potential has a logarithmic singularity for $y \to 0$, $z \to 0$ (see, for instance, the formulae for flow past a narrow cone in §113). Hence the boundary condition at the *x*-axis must determine, not the derivatives $\frac{\partial \phi}{\partial y}$, $\frac{\partial \phi}{\partial z}$ themselves, but the products $y\frac{\partial \phi}{\partial y} = Y\frac{dY}{dx}$, $z\frac{\partial \phi}{\partial z} = Z\frac{dZ}{dx}$, which remain finite. It is easy to see that in this case the similarity transformation is (again with $\theta = \delta / l$)

$$x = l\bar{x}\,, \quad y = \frac{l}{\theta \alpha_*{}^{1/2}}\bar{y}\,, \quad z = \frac{l}{\theta \alpha_*{}^{1/2}}\bar{z}\,, \quad \phi = l\theta^2\bar{\phi}\,, \qquad (126.10)$$

the similarity parameter being

$$K = \frac{M_1 - 1}{\theta^2 \alpha_*} \qquad (126.11)$$

(T. von Karman 1947). The pressure coefficient at the surface of the body is found to have the form $C_p = \theta^2 P(K, x/l)$, and the drag coefficient is accordingly[2]

$$C_x = \theta^4 f(K)\,. \qquad (126.12)$$

All these formulae hold, of course, for both small positive and small negative values of $M_1 - 1$. If $M_1 = 1$ exactly, the similarity parameter K = 0, and the functions in formulae (126.8) and (126.12) reduce to constants, so that these formulae completely determine C_x and C_y as functions of θ and α_*, which represents the properties of the gas.

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[1] The range of validity of these formulae is given by the condition $|M_1 - 1| << 1$. The linearized theory, however, corresponds to large *K*, i.e., $|M_1 - 1| >> \theta^{2/3}$. In the range $1 >> M_1 - 1 >> \theta^{2/3}$, formulae (126.8) must therefore become the formulae (125.6) - (125.8) given by the linearized theory. This means that, for large *K*, the functions f_x and f_y must be proportional to $K^{-1/2}$.

[2] In the range $1 >> M_1 - 1 >> \theta^2$, we must obtain the formula (123.7) given by the linearized theory, according to which $C_x \propto \theta^4$; this means that the function *f*(*K*) tends to a constant as *K* increases.

§127. The law of hypersonic similarity

The linearized theory is invalid for supersonic flow past thin pointed bodies for very large values of the Mach number M_1 (***hypersonic flow***), as has already been mentioned at the end of §114. A simple similarity rule which can be established for this case is therefore of interest.

The shock waves formed in such flow are at a small angle to the direction of flow, of the order of the ratio $\theta = \delta / l$ of thickness to length of the body. These shocks are in general curved and also strong; the velocity discontinuity in them is relatively small, but the pressure discontinuity (and therefore the entropy discontinuity) is large. The gas flow is therefore not in general potential flow.

We shall assume that the Mach number M_1 is of the order of $1/\theta\,0$ or greater. A shock wave reduces the local value of *M*, but the latter always remains of the order of $1/\theta$ (see §112, Problem 2), so that *M* is large everywhere.

We use the "**sonic analogy**" mentioned in §123: a three-dimensional problem of steady flow past a thin body with variable cross-section *S* (*x*) is equivalent to a two-dimensional problem of non-steady emission of sound waves by a contour whose area varies with time according to the law $S(v_1 t)$; the velocity of sound is represented by $\frac{v_1}{\sqrt{M_1^2 - 1}}$, or, for large M_1, by c_1 simply. It should be emphasized that the only condition necessary for the two problems to be equivalent is that the ratio δ / l be small; this enables us to regard small annular regions of the surface of the body as cylindrical. For large M_1, however, the rate of propagation of the emitted waves is comparable with the velocity of the gas particles in the waves (cf. the end of § 123), and the problem therefore has to be solved on the basis of the exact (non-linearized) equations.

The velocity perturbation is small (in comparison with the velocity v_1 of the incident stream) in any supersonic flow past a narrow pointed body. In hypersonic flow, the perturbation of the longitudinal velocity is also small in comparison with the transverse velocities which occur:

$$v_y \sim v_x \sim v_1\theta\,, \quad v_x - v_1 \sim v_1\theta^2\,. \tag{127.1}$$

The pressure and density changes, however, are not small:

$$\frac{p - p_1}{p_1} = M_1^{\,2}\theta^2\,, \quad \frac{\rho_2 - \rho_1}{\rho_1} \sim 1\,, \tag{127.2}$$

and the pressure change can even be indefinitely large when $M_1\theta >> 1$; cf. § 112, Problem 2.

The sonic analogy applies, however, only to the two-dimensional problem of flow in the *yz*-plane, which is perpendicular to the incident stream. In this two-dimensional problem, the linear velocity of the source is of the order of $v_1\theta$; the only other independent parameters of the problem are the velocity of sound c_1, the dimension δ of the source, and the density ρ_1.[3] From these we can form only one dimensionless combination,

$$K = M_1\theta\,, \tag{127.3}$$

which is the **similarity parameter**.[4] The scales of length for the coordinates *y*, *z* and of time must be taken to have the appropriate dimensions, and be formed from the same parameters, e.g., δ and $\frac{\delta}{v_1\theta} = \frac{l}{v_1}$. The natural parameter for the coordinate *x* is the length *l* of the body.

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[3] We are considering, of course, not only the equations of motion of the gas, but also the boundary conditions on them at the surface of the body and the conditions which must be satisfied at the shock waves which are formed. We take the case of a polytropic gas, so that the gas-dynamic properties depend only on the dimensionless parameter γ; the similarity rule obtained below, however, does not determine the dependence of the flow on this parameter.
In flow with $M_1 >> 1$, there is considerable heating of the gas, and there may be a considerable consequent change in its thermodynamics properties. The quantitative significance of the formulae for a polytropic gas (the specific heat being assumed constant) is therefore in practice limited, at hypersonic velocities.

[4] If M_1 is not supposed large, we obtain a similarity rule with parameter $K = \theta\sqrt{M_1^{\,2} - 1}$. This is of no interest, however, since for small M_1 the linearized theory determines all quantities as functions of this parameter.

We can then say that

$$v_y = v_1 \theta v'_y, \quad v_z = v_1 \theta v'_z, \quad p = \rho_1 {v_1}^2 \theta^2 p', \quad \rho = \rho_1 \rho', \qquad (127.4)$$

where v'_y, v'_z, p' and ρ' are functions of the dimensionless variables x/l, y/δ, z/δ and the parameter K; from (127.1) and (127.2), these functions are of the order of unity.[5]

The drag force F_x is calculated as the integral

$$F_x = \oint p dydz,$$

taken over the whole surface of the body; according to the boundary condition $v_n = 0$, the term $v_x(\boldsymbol{v} \cdot \boldsymbol{n})$ in the momentum flux density is zero at the surface of the body, $\boldsymbol{n}$ being the normal to the surface. Changing to dimensionless variables in accordance with (127.4), we get the drag coefficient C_x, defined by (123.6), as

$$C_x = 2\theta^4 \oint p' dy' dz'.$$

The remaining integral is a function of the dimensionless parameter K. Thus

$$C_x = \theta^4 f(K). \qquad (127.5)$$

The same similarity law obviously occurs in the two-dimensional case of flow past a thin wing with infinite span. The drag and lift coefficients then have the form

$$C_x = \theta^3 f_x(K), \quad C_y = \theta^3 f_y(K). \qquad (127.6)$$

In applying the relations (127.5) and (127.6), it must be remembered that the similarity of the flows implies that the shape, size and orientation of the bodies relative to the incident stream are obtained from one another by merely changing the scale δ along the y and z axes and l along the x-axis. This means, in particular, that if the angle of attack α is not zero the ratio α/θ must be the same for similar configurations.

As $K \to \infty$, the functions of this parameter in (127.5) and (127.6) tend to constant limits. This is a consequence of the existence of a limiting flow regime as $M_1 \to \infty$, whose properties are independent of M_1 over a considerable region (S. V. Vallander 1947; K. Oswatitsch 1951). By this we mean a region between the forward strongest part of the bow wave and the surface of the body in the flow not too far from its forward part; it is this region, where the pressure is greatest, which determines the forces acting on the body. If the flow is described by the "reduced" velocity v/v_1, pressure $p/\rho_1 {v_1}^2$ and density ρ/ρ_1 as functions of the dimensionless coordinates, the pattern of flow past a body having a given shape is in this region independent of M_1 in the limit. The reason is that, when expressed in terms of these variables, not only the equations of fluid dynamics and the boundary conditions on the surface of the body but also all conditions at the shock wave surface are independent of M_1. The term "considerable region" is used because the quantities neglected in the latter conditions are of relative order $1/M_1 \sin^2 \phi$, where ϕ is the angle between v_1 and the surface of discontinuity; at large distances, where the shock wave is weak, this angle tends to the Mach angle $\arcsin \frac{1}{M_1} \cong \frac{1}{M_1}$, and so the expansion parameter is no longer small: $1/{M_1}^2 \sin^2 \phi \sim 1$.[6]

PROBLEM

[5] The similarity law for hypersonic flow was formulated by H. S. Tsien (1946). Its relationship to the sonic analogy extended to the non-linear problem was noted by W. D. Hayes (1947). In the specialist literature, this is called the "***piston analogy***".

[6] The detailed proof is given by G. G. Chernyi, *Introduction to Hypersonic Flow*, New York 1961, chapter I, §4.

Determine the lift force on a flat wing with infinite span inclined at a small angle of attack α to the direction of flow, for $M_1 \geq 1/\alpha$ (R. D. Linnell 1949).

Solution. The flow pattern is as shown in Fig. 130: a shock wave and a rarefaction wave leave each of the two edges of the plate, and the stream is turned in them through an angle α in opposite directions.

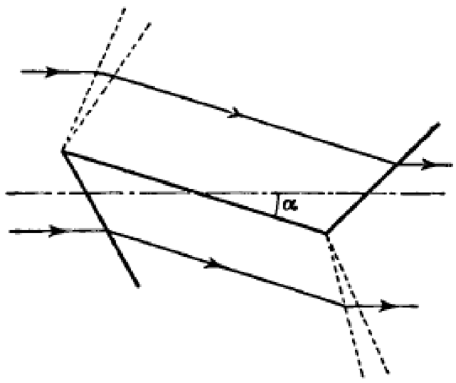

FIG. 130

According to the sonic analogy, the problem of steady flow past such a plate is equivalent to that of non-steady one-dimensional gas flow on each side of a piston moving with uniform velocity αv_1. In front of the piston a shock wave is formed, and behind it a rarefaction wave (see §99, Problems 1 and 2). Using the results there obtained, we find the required lift force as the difference of the pressures on the two sides of the plate. The lift coefficient is

$$C_y = \alpha^2 \left\{ \frac{2}{\gamma K^2} + \frac{\gamma+1}{2} + \sqrt{\frac{4}{K^2} + \left(\frac{\gamma+1}{2}\right)^2} \right\} - \frac{2\alpha^2}{\gamma K^2}\left[1 - \frac{\gamma-1}{2}K\right]^{2\gamma/(\gamma-1)},$$

where $K = \alpha M_1$. For $K \geq \dfrac{2}{\gamma-1}$, a vacuum is formed under the plate, and the second term must be omitted. In the range $1 << M_1 << \dfrac{1}{\alpha}$, this formula becomes $C_y = \dfrac{4\alpha}{M_1}$, as given by the linearized theory, in accordance with the fact that both procedures are applicable in that range.