

CHAPTER XIV FLUID DYNAMICS OF COMBUSTION

§128. Slow combustion

The speed of a chemical reaction (measured, say, by the number of molecules reacting in unit time) depends on the temperature of the mixture of gases in which it occurs, increasing with the temperature. In many cases this dependence is very marked.¹ The speed of the reaction may be so small at ordinary temperatures that the reaction hardly occurs, even though the gas mixture corresponding to a state of thermodynamic (chemical) equilibrium would be one in which the reaction had occurred. When the temperature rises sufficiently, the reaction proceeds rapidly. If it is endothermic, a continuous supply of heat from an external source is necessary for the reaction to be maintained; if the temperature is merely raised at the beginning of the reaction, only a small amount of matter reacts, and thereby reduces the gas temperature to a point where the reaction ceases. The situation is quite different for a strongly exothermic reaction, where a considerable quantity of heat is evolved. Here it is sufficient to raise the temperature at a single point; the reaction which begins at that point evolves heat and so raises the temperature of the surrounding gas, and the reaction, once having begun, will extend to the whole gas. This is called *slow combustion* or simply *combustion* of a gas mixture.²

The combustion of a gas mixture is necessarily accompanied by motion of the gas. The process of combustion is therefore not only a chemical phenomenon but also one of gas dynamics. In general, the nature of the combustion process has to be determined by a solution of simultaneous equations which include both those of chemical kinetics for the reaction and those of gas dynamics for the mixture concerned.

The situation is much simplified, however, in the very important case (the one usually encountered) where the characteristic dimension l of the problem is large (in a sense to be defined later). We shall see that, in such cases, the problems of gas dynamics and chemical kinetics can be, to a certain extent, considered separately.

The region of burnt gas (i.e., the region where the reaction is over and the gas is a mixture of combustion products) is separated from the gas where combustion has not yet begun by a transition layer, where the reaction is in progress (the *combustion zone* or *flame*); in the course of time, this layer moves forward, with a velocity which may be called the *velocity of propagation of combustion* in the gas. The magnitude of this velocity depends on the amount of heat transfer from the combustion zone to the cold gas mixture. The main mechanism of heat transfer is ordinary conduction (V. A. Mikhel'son 1890).

We denote by δ the order of magnitude of the width of the combustion zone. It is determined by the mean distance over which heat evolved in the reaction is propagated during the time τ for which the reaction lasts (at the point concerned). The time τ is characteristic of the reaction, and depends only on the thermodynamic state of the gas undergoing combustion (and not on the parameter l). If χ is the thermometric conductivity of the gas, we have (see (51.6))³

$$\delta \sim \sqrt{\chi \tau} \quad (128.1)$$

Let us now make more precise the above assumption: we shall suppose that the

¹ The reaction rate usually depends exponentially on the temperature, being nearly proportional to a factor of the form $e^{-U/T}$, where U is a constant for any given reaction and is called the *activation energy*. The greater U , the more strongly the reaction rate depends on the temperature.

² It should be borne in mind that, in a mixture capable of combustion, the spontaneous propagation of the combustion may be impossible in certain circumstances. This limitation is due to heat losses resulting from such factors as conduction through the walls of a pipe in which combustion occurs, radiation losses, etc. For this reason combustion is not possible in pipes with very small radius, for example.

³ To avoid misunderstanding, it should be mentioned that, when τ depends markedly on the temperature, a fairly large coefficient should appear in formula (128.1) if τ is the value for the temperature of the combustion products. The important fact for our purposes, however, is that δ does not depend on l .

characteristic dimension is large compared with the width of the combustion zone ($l \gg \delta$). When this condition holds, the problem of gas dynamics can be considered separately. In determining the gas flow, we can neglect the width of the combustion zone, regarding it as a surface which separates the combustion products from the unburnt gas. On this surface (the **flame front**) the state of the gas changes discontinuously, i.e., it is a surface of discontinuity.

The velocity v_1 of this discontinuity relative to the gas itself (in a direction normal to the front) is called the **normal velocity** of the flame. In a time τ , the combustion is propagated through a distance of the order of δ , and so the flame velocity is⁴

$$v_1 \sim \frac{\delta}{\tau} \sim \sqrt{\frac{\chi}{\tau}}. \quad (128.2)$$

The ordinary thermometric conductivity of the gas is of the order of the mean free path of the molecules multiplied by their thermal velocity or, what is the same thing, the mean free time τ_{fr} multiplied by the square of this velocity. Since the thermal velocity of the molecules is of

the same order as the velocity of sound, we have $\frac{v_1}{c} \sim \sqrt{\frac{\chi}{c^2}} \sim \sqrt{\frac{\tau_{fr}}{\tau}}$. Not every collision

between molecules results in a chemical reaction between them; on the contrary, only a very small fraction of colliding molecules react. This means that $\tau_{fr} \ll \tau$ and therefore $v_1 \ll c$.

Thus the flame velocity is, in this case, small compared with the velocity of sound.⁵

On the surface of discontinuity which replaces the combustion zone, the fluxes of mass, momentum and energy must be continuous, as at any discontinuity. The first of these conditions, as usual, determines the ratio of the components, normal to the surface, of the gas velocities relative to the discontinuity: $\rho_1 v_1 = \rho_2 v_2$, or

$$\frac{v_1}{v_2} = \frac{V_1}{V_2}, \quad (128.3)$$

where V_1 and V_2 are the specific volumes of the unburnt gas and the combustion products. According to the general results obtained in §84 for arbitrary discontinuities, the tangential velocity component must be continuous if the normal component is discontinuous. The streamlines are therefore refracted at the discontinuity.

On account of the smallness of the normal velocity of the flame relative to that of sound, the condition of continuity of the momentum flux reduces to the continuity of pressure, and that for the energy flux reduces to the continuity of the heat function:

$$p_1 = p_2, \quad w_1 = w_2. \quad (128.4)$$

In using these conditions, it must be remembered that the gases on the two sides of the discontinuity under consideration are chemically different, and so the thermodynamic quantities are not the same functions of one another.

For a polytropic gas we have $w_1 = w_{01} + c_{p1}T_1$, $w_2 = w_{02} + c_{p2}T_2$; the constant terms cannot be put equal to zero as for a single gas (by an appropriate choice of the zero of energy), since w_{01} and w_{02} are different. We put $w_{01} - w_{02} = q$; this is just the heat evolved (per unit mass) in the reaction, if the reaction occurs at a temperature of absolute zero. Then we obtain the following relations between the thermodynamic quantities for the unburnt gas (1) and the burnt gas (2):

⁴ As an example, it may be mentioned that the flame velocity in a mixture of methane (6 per cent) and air is only 5 cm/sec, whereas in detonating mixture ($2H_2 + O_2$) it is 1000 cm/sec; the widths of the combustion zones in these two cases are about 5×10^{-2} cm and 5×10^{-4} cm, respectively.

⁵ The diffusion of the components of the burning mixture also has a certain effect on the propagation of combustion; this does not alter the orders of magnitude of the flame velocity and width. We are, however, always considering the combustion of already mixed gases, not cases where the reactants are separated and combustion occurs only as they diffuse into one another.

$$\left. \begin{aligned} p_1 &= p_2 \\ T_2 &= \frac{q}{c_{p2}} + \frac{c_{p1}}{c_{p2}} T_1 \\ V_2 &= V_1 \frac{\gamma_1(\gamma_2 - 1)}{\gamma_2(\gamma_1 - 1)} \left(\frac{q}{c_{p1} T_1} + 1 \right) \end{aligned} \right\}. \quad (128.5)$$

Since the flame has a definite normal velocity, independent of the gas velocities themselves, the flame front has a definite form for steady combustion in a moving gas. An example is the combustion of gas leaving the end of a tube (a burner outlet). If v is the gas velocity averaged over the cross-section of the tube, it is evident that $v_1 S_1 = v S$, where S is the cross-sectional area of the tube and S_1 the total surface area of the flame front.

There arises the question of the **limits of stability** of this flow regime under small perturbations, that is, the conditions for it actually to occur. Since the gas velocity is much less than that of sound, in examining the stability of the flame front we can regard the gas as an incompressible ideal fluid, the normal velocity of the flame front being taken as a given constant. Such an investigation leads to the result that the flame front is unstable (L. D. Landau 1944; see Problem 1). In this form the investigation is valid only for large Reynolds numbers $\frac{v_1 l}{\nu_1}$ and $\frac{v_2 l}{\nu_2}$. When the viscosity of the gas is taken into account, however, it

cannot here result in very large critical Reynolds numbers.

Such instability would have to give rise to spontaneous turbulence of the flame. The experimental results, on the other hand, show that this does not occur, at least up to very large Reynolds numbers. This is because of the presence, in actual flows, of various factors of fluid dynamics and thermal diffusion which stabilize the flame. An account of these complex topics is outside the scope of the present book; here we shall give only some brief comments on certain possible causes of stabilization.

The influence of curvature of the front on the combustion rate may be important in this respect. If only thermal conduction is considered, v_1 increases on the parts of the front that are concave towards the original combustible mixture, because of the better heat transfer to unburnt mixture within the concavity, and decreases on the convex parts; this effect tends to straighten the front, i.e., has a stabilizing action. A change in the diffusion regime, however, can be shown by similar arguments to have a destabilizing action. Thus the overall sign of the effect depends on the relation between the thermometric conductivity and the diffusion coefficient (I. P. Drozdov and Ya. B. Zel'dovich 1943). For a phenomenological description of the influence of front curvature on the combustion rate v_1 , we can include in it a term proportional to the curvature (G. H. Markstein 1951); when the sign of this term is appropriately chosen, its inclusion in the boundary conditions eliminates the instability of short-wavelength perturbations.⁶ The development of perturbations that are unstable in the linear approximation can be stabilized at a certain steady-amplitude limit as a result of non-linear effects (R. E. Petersen and N. W. Emmons 1956; Ya. B. Zel'dovich 1966); this mechanism may give rise to a "cellular" structure of the flame.⁷

A flame propagated in a mixture of combustible gases results in a motion of the surrounding gas up to a considerable distance. The fact that a motion of the gas must

⁶ In the notation of Problem 1, the expression for v_1 including this effect is to be written as $v_1 = v_1^{(0)} (1 - \mu \frac{\partial^2 \zeta}{\partial y^2})$,

where $v_1^{(0)}$ is the combustion rate with a plane front, and μ is an empirical constant (having the dimensions of length) which is positive when there is stabilization.

⁷ A detailed account of these topics has been given by Ya. B. Zel'dovich, G. I. Barenblatt, V. B. Librovich, and G. M. Makhviladze, *Mathematical Theory of Combustion and Explosion* (Matematicheskaya teoriya goreniya i vzryva), Moscow 1980, chapters 4 and 6.

accompany combustion is evident from the fact that, because of the difference between the velocities v_1 and v_2 , the combustion products must move with velocity $v_1 - v_2$ relative to the unburnt gas. In some cases this motion results in the formation of shock waves. These shocks bear no direct relation to the process of combustion, and their occurrence is due to the necessity of satisfying the boundary conditions. Let us consider, for example, combustion propagated from the closed end of a pipe. In Fig. 131, ab is the combustion zone. The gas in regions 1 and 3 is the original unburnt mixture, while that in region 2 consists of combustion products. The velocity v_1 with which the combustion zone moves relative to the gas 1 in front of it is determined by the properties of the reaction and the conditions of heat transfer, and must be regarded as given. The velocity v_2 with which the flame moves relative to gas 2 is then determined at once by the condition (128.3). At the closed end of the pipe, the gas velocity must vanish, and so the gas in region 2 will be at rest. Gas 1, therefore, must move relative to the pipe with a constant velocity $v_2 - v_1$. In the forward part of the pipe, far from the flame, the gas is again at rest. This condition can be satisfied only by the presence of a shock wave (cd in Fig. 131), in which the gas velocity is discontinuous in such a way that gas 3 is at rest. From the given discontinuity of velocity we can find the discontinuities in the other quantities and the velocity of propagation of the shock itself. Thus we see that the flame front acts as a piston on the gas in front of it. The shock wave moves faster than the flame, so that the mass of gas set in motion increases in the course of time.⁸

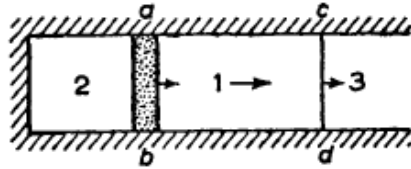


FIG. 131

For sufficiently large Reynolds numbers, the gas flow which accompanies combustion in a pipe becomes turbulent, and this in turn affects the flame which causes the motion. There are still many unsolved problems of turbulent combustion, and they will not be discussed here.

PROBLEMS

Problem 1. Investigate the stability of a plane flame front in slow combustion, with respect to infinitesimal disturbances.

Solution. We take the plane of discontinuity (flame front) as the yz -plane in coordinates where it is at rest, with the unperturbed gas velocity in the positive x -direction. On the flow with constant velocities v_1, v_2 (on the two sides of the discontinuity) we superpose a perturbation periodic in the y -direction and in time. From the equations of motion

$$\left. \begin{aligned} \operatorname{div} \mathbf{v}' &= 0 \\ \frac{\partial \mathbf{v}'}{\partial t} + (\mathbf{v} \cdot \operatorname{grad}) \mathbf{v}' &= -\frac{1}{\rho} \operatorname{grad} p' \end{aligned} \right\} \quad (1)$$

(\mathbf{v}, ρ being either \mathbf{v}_1, ρ_1 or \mathbf{v}_2, ρ_2), we obtain as in §29 the equation

$$\Delta p' = 0. \quad (2)$$

⁸ In actual cases, the combustion front in a pipe is usually convex towards the original gas mixture ahead of it. This gives rise to a specific mechanism of flame stabilization with respect to small-scale perturbations. The propagation of combustion along the normal to the front causes a "stretching" of the front, and perturbations arising anywhere on it are carried to the pipe walls, where they disappear; the steady shape of the front is maintained by the gas flow ahead of it. See Ya. B. Zel'dovich, A. G. Istratov, N. I. Kidin and V. B. Librovich, *Combustion Science and Technology* 24, 1, 1980.

On the surface of discontinuity (i.e., for $x \cong 0$) the following conditions must be satisfied: the equation of continuity of pressure

$$p'_1 = p'_2, \quad (3)$$

the condition of continuity of the velocity component tangential to the surface

$$v'_{1y} + v_1 \frac{\partial \zeta}{\partial y} = v'_{2y} + v_2 \frac{\partial \zeta}{\partial y} \quad (4)$$

(where $\zeta(y, t)$ is the small displacement of the surface of discontinuity along the x -axis due to the disturbance), and the condition that the gas velocity normal to the surface of discontinuity be unchanged,

$$v'_{1x} - \frac{\partial \zeta}{\partial t} = v'_{2x} - \frac{\partial \zeta}{\partial t} = 0. \quad (5)$$

In the region $x < 0$ (the unburnt gas 1), the solution of equations (1) and (2) can be written

$$\left. \begin{aligned} v'_{1x} &= A e^{iky+kx-i\omega t} \\ v'_{1y} &= iA e^{iky+kx-i\omega t} \\ p'_1 &= A \rho_1 \left(\frac{i\omega}{k} - v_1 \right) e^{iky+kx-i\omega t} \end{aligned} \right\}. \quad (6)$$

In the region $x > 0$ (the combustion products, gas 2), besides the solution having the form $\text{constant} \times e^{iky-kx-i\omega t}$, we must take into account another particular solution of equations (1) and (2), in which the dependence on y and t is given by the same factor $e^{iky-i\omega t}$. This solution is obtained by putting $p' = 0$; then the right-hand side of Euler's equation is zero, and the resulting homogeneous equation has a solution in which v'_x and v'_y are proportional to $e^{iky-i\omega t+i\alpha x/v}$. The reason why this solution need be taken into consideration only in gas 2, and not in gas 1, is that our ultimate purpose is to determine whether frequencies ω can exist having positive imaginary parts; for such ω , however, the factor $e^{i\alpha x/v}$ increases without limit with $|x|$ for $x < 0$, and so such a solution is not possible in region 1. Again choosing appropriate values of the constant coefficients, we seek a solution for $x > 0$ in the form

$$\left. \begin{aligned} v'_{2x} &= B e^{iky-kx-i\omega t} + C e^{iky-i\omega t+i\alpha x/v_2} \\ v'_{2y} &= -iB e^{iky-kx-i\omega t} - \frac{\omega}{kv_2} C e^{iky-i\omega t+i\alpha x/v_2} \\ p'_2 &= -B \rho_2 \left[v_2 + \frac{i\omega}{k} \right] e^{iky-kx-i\omega t} \end{aligned} \right\}. \quad (7)$$

Putting also

$$\zeta = D e^{iky-i\omega t}, \quad (8)$$

and substituting these expressions in the conditions (3)-(5), we obtain four homogeneous equations for the coefficients A, B, C, D .⁹ A simple calculation (using the fact that $j \equiv \rho_1 v_1 = \rho_2 v_2$) gives the following condition for these equations to be compatible:

$$\Omega^2 (v_1 + v_2) + 2\Omega k v_1 v_2 + k^2 v_1 v_2 (v_1 - v_2) = 0, \quad (9)$$

where $\Omega = -i\omega$. If $v_1 > v_2$, this equation has either two negative real roots or two complex conjugate roots with $\text{Re } \Omega < 0$; in this case, the flow is stable. If $v_1 < v_2$ (and accordingly $\rho_1 > \rho_2$), both roots of (9) are real, and one is positive:

$$\Omega = k v_1 \frac{\mu}{1 + \mu} \left[\sqrt{1 + \mu - \frac{1}{\mu}} - 1 \right]$$

⁹ The flow described by (6) is a potential flow; that described by (7) has $\text{curl } \mathbf{v}'_2 \neq 0$. Thus the movement of the combustion products behind the perturbed front is rotational.

where $\mu = \frac{\rho_1}{\rho_2}$, so that the flow is unstable. This case occurs for a combustion front, since the combustion product density ρ_2 is always less than the density ρ_1 of the unburnt gas, because of the considerable heating.

It should be noted that $\text{Im } \Omega = 0$. This means that the disturbances are not propagated along the front, but are amplified as stationary waves. Instability occurs for disturbances with any wavelength; the growth rate increases with k , but it should be remembered that the analysis in which the front is treated as a geometrical surface is valid only for disturbances whose wavelengths are much greater than δ ($k\delta \ll 1$). For given k , the growth rate increases with μ .

Problem 2. Combustion occurs on the surface of a liquid, the reaction taking place in vapour evaporating from the surface.¹⁰ Determine the stability condition in this case, taking into account the effect of the gravitational field and capillary forces (L. D. Landau 1944).

Solution. Let us consider the combustion zone in vapour near the liquid surface as a surface of discontinuity, but now let this surface have a surface tension α . The calculations are entirely similar to those of Problem 1, the only difference being that, instead of the boundary condition

(3), we now have $p'_1 - p'_2 = -\alpha \frac{\partial^2 \zeta}{\partial y^2} + (\rho_1 - \rho_2)g\zeta$; medium 1 is the liquid and medium 2

the burnt gas. The conditions (4) and (5) are unchanged. In place of equation (9) we obtain

$$\Omega^2(v_1 + v_2) + 2\Omega kv_1v_2 + \left[k^2(v_1 - v_2) + \frac{gk(\rho_1 - \rho_2) + \alpha k^3}{j} \right] v_1v_2 = 0.$$

The stability condition in this case is that the roots of this equation should have negative real parts, i.e., the free term must be positive for all k . This requirement gives the stability condition $j^4 < \frac{4\alpha g \rho_1^2 \rho_2^2}{\rho_1 - \rho_2}$. Since the density of the gaseous combustion products is small

compared with that of the liquid ($\rho_1 \gg \rho_2$), the condition becomes in practice

$$j^4 < 4\alpha g \rho_1 \rho_2^2.$$

Problem 3. Determine the temperature distribution in the gas in front of a plane flame.

Solution. In a system of coordinates moving with the front the temperature distribution is steady, and the gas moves with velocity $-v_1$. The equation of thermal conduction,

$$\mathbf{v} \cdot \text{grad} T = -v_1 \frac{dT}{dx} = \chi \frac{d^2 T}{dx^2},$$

has the solution $T = T_0 e^{-v_1 x / \chi}$, where T_0 is the temperature on the flame front, the temperature far from the front being taken as zero.

§129. Detonation

In the type of combustion (slow combustion) described above, the propagation through the gas is due to the heating which results from the direct transfer of heat from the burning gas to that which is still unburnt. Another entirely different mechanism of propagation of combustion, involving shock waves, is also possible. The shock wave heats the gas as it passes; the gas temperature behind the shock is higher than in front of it. If the shock wave is sufficiently strong, the rise in temperature which it causes may be sufficient for combustion to begin. The shock wave will then "ignite" the gas mixture as it moves, i.e., the combustion will be propagated with the velocity of the shock, or much faster than ordinary combustion. This

¹⁰ The reaction takes place in the vapour without involving any extraneous substances (such as atmospheric oxygen), i.e., it is a spontaneous decomposition reaction.

mechanism of propagation of combustion is called *detonation*.

When the shock wave passes some point in the gas, the reaction begins at that point, and continues until all the gas there is burnt, i.e., for a time τ which characterizes the kinetics of the reaction concerned.¹¹ It is therefore clear that the shock wave will be followed by a layer moving with it in which combustion is occurring, and the width of this layer is equal to the speed of propagation of the shock multiplied by the time t . It is of importance that the width does not depend on the dimensions of any bodies that are present. When the characteristic dimensions of the problem are sufficiently large, therefore, we can regard the shock wave and the combustion zone following it as a single surface of discontinuity which separates the burnt and unburnt gases. We call such a surface a *detonation wave*.

At a detonation wave the flux densities of mass, energy and momentum must be continuous, and the relations (85.1) - (85.10) derived previously, which follow from these continuity conditions alone, remain valid. In particular, the equation

$$w_1 - w_2 + \frac{1}{2}(V_1 + V_2)(p_2 - p_1) = 0 \quad (129.1)$$

holds; the suffix 1 always pertains to the unburnt gas and the suffix 2 to the combustion products. The curve of p_2 as a function of V_2 given by this equation is called the *detonation adiabetic*. Unlike the shock adiabetic considered earlier, this curve does not pass through the given initial point (p_1, V_1) . The fact that the shock adiabetic passes through this point is due to the fact that w_1 and w_2 are the same functions of p_1, V_1 and p_2, V_2 , respectively, whereas this does not now hold, on account of the chemical difference between the two gases. In Fig. 132 the continuous line shows the detonation adiabetic. The ordinary shock adiabetic for the unburnt gas mixture is drawn (dashed) through the point (p_1, V_1) . The detonation adiabetic always lies above the shock adiabetic, because a high temperature is reached in combustion, and the gas pressure is therefore greater than it would be in the unburnt gas for the same specific volume.

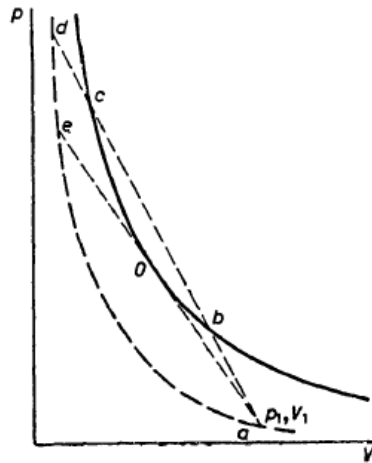


FIG. 132

The previous formula (85.6) holds for the mass flux density:

$$j^2 = \frac{p_2 - p_1}{V_1 - V_2}, \quad (129.2)$$

so that graphically j^2 is again the slope of the chord from the point (p_1, V_1) to any point (p_2, V_2) on the detonation adiabetic (for instance, the chord ac in Fig. 132). It is seen at once from the diagram that j^2 cannot be less than the slope of the tangent aO . The flux j is just

¹¹ This time is, however, itself dependent on the strength of the shock, decreasing rapidly as the shock becomes stronger, on account of the increase in the reaction rate with rising temperature.

the mass of gas which is ignited per unit time per unit area of the surface of the detonation wave; we see that, in a detonation, this quantity cannot be less than a certain limiting value j_{\min} (which depends on the initial state of the unburnt gas).

Formula (129.2) is a consequence only of the conditions of continuity of the fluxes of mass and momentum. It therefore holds (for a given initial state of the gas) not only for the final state of the combustion products, but also for all intermediate states, in which only part of the reaction energy has been evolved.¹² In other words, the pressure p and specific volume V of the gas in any state obey the linear relation

$$p = p_1 + j^2(V_1 - V), \quad (129.3)$$

which is shown graphically by the chord ad (V. A. Mikhel'son 1890).

Let us now use a procedure developed by Ya. B. Zel'dovich (1940) to investigate the variation of the state of the gas through the layer of finite width which a detonation wave actually is. The forward front of the detonation wave is a true shock wave in the unburnt gas 1. In it, the gas is compressed and heated to a state represented by the point d (Fig. 132) on the shock adiabat of gas 1. The chemical reaction begins in the compressed gas, and as the reaction proceeds the state of the gas is represented by a point which moves down the chord da ; heat is evolved, the gas expands, and its pressure decreases. This continues until combustion is complete and the whole heat of the reaction has been evolved. The corresponding point is c , which lies on the detonation adiabat representing the final state of the combustion products. The lower point b at which the chord ad intersects the detonation adiabat cannot be reached for a gas in which combustion is caused by compression and heating in a shock wave.¹³

Thus we conclude that the detonation is represented, not by the whole of the detonation adiabat, but only by the upper part, lying above the point O where this adiabat touches the straight line aO drawn from the initial point a .

It has been shown in §87 that, at the point where $\frac{d(j^2)}{dp_2} = 0$, i.e., where the shock adiabat touches the chord 12, the velocity v_2 is equal to the corresponding velocity of sound c_2 . This result has been obtained only from the conservation laws for the surface of discontinuity, and is therefore entirely applicable to the detonation wave also. On the ordinary shock adiabat for a single gas there are no points with $\frac{d(j^2)}{dp_2} = 0$, as has been shown in §87.

On the detonation adiabat, however, there is such a point, namely the point O . At the same time as $v_2 = c_2$, we have at such a point the inequality (87.10) $\frac{d(v_2/c_2)}{dp_2} < 0$, and therefore

$v_2 < c_2$ at higher values of p_2 , above O . Since detonation corresponds to the upper part only of the adiabat, above the point O , we conclude that

$$v_2 \leq c_2, \quad (129.4)$$

i.e., a detonation wave moves relative to the gas just behind it with a velocity equal to or less than that of sound; the equality $v_2 = c_2$ holds for a detonation corresponding to the point O (called the *Chapman-Jouguet point*).¹⁴

The velocity of the detonation wave relative to gas 1 is always supersonic (even for the point O):

$$v_1 > c_1. \quad (129.5)$$

¹² Here it is assumed that diffusion and viscosity may be neglected in the combustion zone, so that mass and momentum transfer take place only by fluid flow.

¹³ For completeness, it should also be mentioned that a discontinuous transition from state c to state b in another shock wave is also impossible, since the gas would have to cross such a shock from high pressure to low pressure.

¹⁴ It should be recalled that the velocities v_1, v_2 always signify the velocities normal to the surface of discontinuity.

This is most simply seen directly from Fig. 132. The velocity of sound c_1 is graphically by the slope of the tangent to the shock adiabat for gas 1 (dashed curve) at the point a . The velocity v_1 , on the other hand, is given by the slope of the chord ac . Since all the chords concerned are steeper than the tangent, we always have $v_1 > c_1$. Moving with supersonic velocity, the detonation wave, like a shock wave, does not affect the state of the gas in front of it. The velocity v_1 with which the detonation wave moves relative to the unburnt gas at rest is the velocity of propagation of the detonation.

Since $\frac{v_1}{V_1} = \frac{v_2}{V_2} \equiv j$, and $V_1 > V_2$, it follows that $v_1 > v_2$. The difference $v_1 - v_2$ is evidently the velocity of the combustion products relative to the unburnt gas. This difference is positive, i.e., the combustion products move in the direction of propagation of the detonation wave.

We may note also the following. In §87 it was also shown that $\frac{ds_2}{d(j^2)} > 0$. At the point

where j^2 has a minimum, s_2 therefore also has a minimum. This point is O , and we conclude that it corresponds to the least value of the entropy s_2 on the detonation adiabat. The entropy s_2 also has an extremum at O if we consider the change in state along the line ae (since the slopes of the curve and the tangent at O are the same). This extremum, however, is a maximum (V. A. Mikhel'son). For a displacement from e to O corresponds to the change of state as the combustion reaction occurs in the compressed gas, and this is accompanied by the evolution of heat and an increase in entropy; a passage from O to a , however, would correspond to the endothermic conversion of the combustion products into the original gases, with a decrease in entropy.

If the detonation is caused by a shock wave which is produced by some external source and is then incident on the gas, any point on the upper part of the detonation adiabat may correspond to the detonation. It is of particular interest, however, to consider a detonation which is due to the combustion process itself. We shall see in §130 that, in a number of important cases, such a detonation must correspond to the Chapman-Jouguet point, so that the velocity of the detonation wave relative to the combustion products just behind it is exactly equal to the velocity of sound, while the velocity $v_1 = jV_1$ relative to the unburnt gas has its least possible value.¹⁵

Let us now derive the relations between the various quantities in a detonation wave in a polytropic gas. Substituting in the general equation (129.1) the heat function in the form

$$w = w_0 + c_p T = w_0 + \frac{\gamma p V}{\gamma - 1},$$

we obtain

$$\frac{\gamma_2 + 1}{\gamma_2 - 1} p_2 V_2 - \frac{\gamma_1 + 1}{\gamma_1 - 1} p_1 V_1 - V_1 p_2 + V_2 p_1 = 2q, \quad (129.6)$$

where $q = w_{01} - w_{02}$ again denotes the heat of the reaction, reduced to the absolute zero of temperature. The curve $p_2(V_2)$ given by this equation is a rectangular hyperbola. For

$\frac{p_2}{p_1} \rightarrow \infty$, the ratio of densities tends to a finite limit $\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{\gamma_2 + 1}{\gamma_2 - 1}$; this is the greatest compression that can be achieved in a detonation wave.

The formulae are much simplified in the important case of strong detonation waves, which are obtained when the heat evolved in the reaction is large compared with the internal heat energy of the original gas, i.e., $q \gg c_{v1} T_1$. In this case we can neglect the terms containing

¹⁵ This result was put forward as a hypothesis by D. L. Chapman (1899) and E. Jouguet (1905); its theoretical justification is due to Ya. B. Zel'dovich (1940), and independently to J. von Neumann (1942) and W. Döring (1943).

p_1 in (129.6), obtaining

$$p_2 \left(\frac{\gamma_2 + 1}{\gamma_2 - 1} V_2 - V_1 \right) = 2q. \quad (129.7)$$

Let us consider in more detail a detonation corresponding to the Chapman-Jouguet point, which is of particular interest, from the above discussion. At this point $j^2 = \frac{c_2^2}{V_2^2} = \frac{\gamma_2 p_2}{V_2}$.

From this relation and (129.2) we can express p_2 and V_2 in the form

$$\left. \begin{aligned} p_2 &= \frac{p_1 + j^2 V_1}{\gamma_2 + 1} \\ V_2 &= \frac{\gamma_2 (p_1 + j^2 V_1)}{j^2 (\gamma_2 + 1)} \end{aligned} \right\}. \quad (129.8)$$

Substituting these expressions in equation (129.6) and replacing j by $\frac{v_1}{V_1}$, we have after a

simple reduction the following biquadratic equation for the velocity v_1 :

$$v_1^4 - 2v_1^2 \left[(\gamma_2^2 - 1)q + (\gamma_2^2 - \gamma_1)c_{v1}T_1 \right] + \gamma_2^2 (\gamma_1 - 1)^2 c_{v1}^2 T_1^2 = 0,$$

where the temperature has been introduced by $T = \frac{pV}{c_p - c_v} = \frac{pV}{c_v(\gamma - 1)}$. Hence¹⁶

$$v_1 = \sqrt{\frac{\gamma_2 - 1}{2} [(\gamma_2 + 1)q + (\gamma_1 + \gamma_2)c_{v1}T_1]} + \sqrt{\frac{\gamma_2 + 1}{2} [(\gamma_2 - 1)q + (\gamma_2 - \gamma_1)c_{v1}T_1]}. \quad (129.9)$$

This formula determines the velocity of propagation of the detonation in terms of the temperature T_1 of the original gas mixture.

We can rewrite formulae (129.8) in the form

$$\left. \begin{aligned} \frac{p_2}{p_1} &= \frac{v_1^2 + (\gamma_1 - 1)c_{v1}T_1}{(\gamma_2 + 1)(\gamma_1 - 1)c_{v1}T_1} \\ \frac{V_2}{V_1} &= \frac{\gamma_2 [v_1^2 + (\gamma_1 - 1)c_{v1}T_1]}{(\gamma_2 + 1)v_1^2} \end{aligned} \right\}. \quad (129.10)$$

Together with (129.9), they determine the ratios of pressure and density between the combustion products and the unburnt gas at temperature T_1 .

The velocity v_2 is calculated as $v_2 = \frac{V_2 v_1}{V_1}$, using formulae (129.9) and (129.10). The result is

$$v_2 = \sqrt{\frac{\gamma_2 - 1}{2} [(\gamma_2 + 1)q + (\gamma_1 + \gamma_2)c_{v1}T_1]} + \frac{\gamma_2 - 1}{\gamma_2 + 1} \sqrt{\frac{\gamma_2 + 1}{2} [(\gamma_2 - 1)q + (\gamma_2 - \gamma_1)c_{v1}T_1]}. \quad (129.11)$$

The difference $v_1 - v_2$, i.e., the velocity of the combustion products relative to the unburnt gas, is

$$v_1 - v_2 = \sqrt{\frac{2[(\gamma_2 - 1)q + (\gamma_2 - \gamma_1)c_{v1}T_1]}{\gamma_2 + 1}}. \quad (129.12)$$

¹⁶ If $x^4 - 2px^2 + q = 0$, then $x = \sqrt{p \pm \sqrt{p^2 - q}} = \sqrt{\frac{p + \sqrt{q}}{2}} \pm \sqrt{\frac{p - \sqrt{q}}{2}}$.

The two signs in this case correspond to the fact that two tangents can be drawn from the point a to the detonation adiabat: one upwards, as shown in Fig. 132, and the other downwards. The upward tangent, in which we are interested, has the steeper slope, and we accordingly take the plus sign.

The temperature of the combustion products is calculated from the formula

$$c_{v2}T_2 = \frac{v_2^2}{\gamma_2(\gamma_2 - 1)} \quad (129.13)$$

(since $v_2 = c_2$).

All these somewhat complex formulae are much simplified for strong detonation waves. In this case the velocities are given by the simple formulae

$$\left. \begin{aligned} v_1 &= \sqrt{2(\gamma_2^2 - 1)q} \\ v_1 - v_2 &= \frac{v_1}{\gamma_2 + 1} \end{aligned} \right\}. \quad (129.14)$$

The thermodynamic state of the combustion products is given by the formulae

$$\left. \begin{aligned} \frac{V_2}{V_1} &= \frac{\gamma_2}{\gamma_2 + 1} \\ T_2 &= \frac{2\gamma_2 q}{c_{v2}(\gamma_2 + 1)} \\ \frac{p_2}{p_1} &= \frac{2(\gamma_2 - 1)}{\gamma_1 - 1} \frac{q}{c_{v1}T_1} = \frac{\gamma_1 v_1^2}{(\gamma_2 + 1)c_1^2} \end{aligned} \right\}. \quad (129.15)$$

If we compare formulae (129.15) with the corresponding formulae (128.5) for slow combustion, we notice that, in the limiting case $q \gg c_{v1}T_1$, the ratio of the temperatures of the combustion products after detonation and slow combustion is $\frac{T_{2,\text{det}}}{T_{2,\text{com}}} = \frac{2\gamma_2^2}{\gamma_2 + 1}$. This ratio always exceeds unity (since $\gamma_2 > 1$).

PROBLEM

Determine the thermodynamic quantities for the gas immediately behind the shock wave which is the forward front of a strong detonation wave corresponding to the Chapman-Jouguet point.

Solution. Immediately behind the shock wave we have unburnt gas, and its state is represented by the point e where the tangent aO produced (Fig. 132) intersects the shock adiabetic of gas 1, shown dashed. Denoting the coordinates of this point by (p_1', V_1') , we have, firstly, by equation (89.1) for the shock adiabetic of gas 1,

$$\frac{V_1'}{V_1} = \frac{(\gamma_1 + 1)p_1 + (\gamma_1 - 1)p_1'}{(\gamma_1 - 1)p_1 + (\gamma_1 + 1)p_1'}$$

and, secondly, $\frac{p_1' - p_1}{V_1 - V_1'} = j^2 = \frac{v_1^2}{V_1^2}$. Taking v_1 from (129.14), we obtain

$$p_1' = p_1 \frac{4(\gamma_2^2 - 1)}{\gamma_1^2 - 1} \frac{q}{c_{v1}T_1},$$

$$V_1' = V_1 \frac{\gamma_1 - 1}{\gamma_1 + 1},$$

$$T_1' = \frac{q}{c_{v1}} \frac{4(\gamma_2^2 - 1)}{(\gamma_1 + 1)^2}.$$

The ratio of the pressure p_1' to the pressure p_2 behind the detonation wave is

$$\frac{p_1'}{p_2} = \frac{2(\gamma_2 + 1)}{\gamma_1 + 1}.$$

§130. The propagation of a detonation wave

Let us now consider some actual cases of the propagation of detonation waves in a gas initially at rest. We take first the case of detonation in a gas in a pipe closed at one end ($x = 0$). The boundary conditions in this case are that the gas velocity is zero both in front of the detonation wave (which does not affect the state of the gas in front of it) and at the closed end of the pipe. Since the gas acquires a non-zero velocity when the detonation wave passes, the velocity must diminish in the region between the detonation wave and the closed end of the pipe. In order to determine the resulting flow pattern, we notice that in this case there is no length parameter which might characterize the conditions of flow along the pipe (the x -direction). We have seen in §99 that, in such cases, the gas velocity can change either in a shock wave (separating two regions where the velocity is constant) or in a similarity rarefaction wave.

Let us first assume that the detonation wave does not correspond to the Chapman-Jouguet point on the adiabetic. Then its velocity of propagation relative to the gas behind it is $v_2 < c_2$. It is easy to see that, in this case, neither a shock wave nor a weak discontinuity (the forward front of a rarefaction wave) can follow the detonation wave. For the former would have to move, relative to the gas in front of it, with a velocity exceeding c_2 , and the latter with a velocity equal to c_2 , and either would overtake the detonation wave. Thus, on the above assumption, the velocity of the gas moving behind the detonation wave cannot decrease, i.e. the boundary condition at $x = 0$ cannot be satisfied.

This condition can be satisfied only for a detonation wave corresponding to the Chapman-Jouguet point. Then $v_2 = c_2$, and a rarefaction wave can follow the detonation wave. It is formed at $x = 0$ when the detonation begins, and its forward front coincides with the detonation wave.

Thus we reach the important result that a detonation wave propagated in a pipe, with the gas ignited at the closed end, must correspond to the Chapman-Jouguet point. It moves relative to the gas just behind it with a velocity equal to the local velocity of sound. The detonation wave adjoins a rarefaction wave, in which the gas velocity (relative to the pipe) falls monotonically to zero. The point where the velocity becomes zero is a weak discontinuity. Behind this discontinuity the gas is at rest (Fig. 133a).

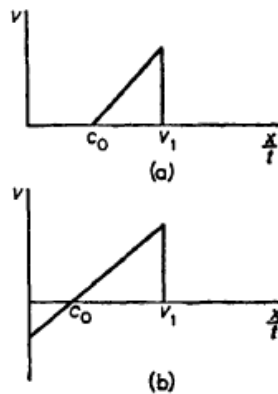


FIG. 133

Let us now consider a detonation wave propagated from the open end of a pipe. The pressure of the gas in front of the detonation wave must be equal to the original pressure, which clearly equals the external pressure. It is evident that, in this case also, the velocity must decrease somewhere behind the detonation wave. If the gas velocity were constant between the end of the pipe and the detonation wave, it would follow that gas was being sucked into the open end of the pipe from outside; this would be impossible, since the gas pressure in the pipe would be greater than the external pressure on account of the pressure increase in a detonation wave. For the same reasons as in the previous case, the detonation wave must correspond to the Chapman-Jouguet point. The resulting flow pattern is shown schematically in Fig. 133b. Immediately behind the detonation wave is a similarity rarefaction wave, in

which the velocity decreases monotonically towards the end of the pipe, changing sign at some point. This means that, in the end section of the pipe, the gas moves towards the open end and flows out of it; the velocity with which it leaves the pipe equals the local velocity of sound, and its pressure exceeds the external pressure. We have seen in §97 that such a flow is possible.¹⁷

Let us next consider the important case of a **spherically symmetrical** outgoing detonation wave whose centre is the point where the gas is first ignited (Ya. B. Zel'dovich 1942). Since the gas must be at rest both in front of the detonation wave and near the centre, the gas velocity must decrease from the detonation wave towards the centre. As with the flow in a pipe, there are no characteristic parameters having the dimensions of length. The result must therefore be a similarity flow, the coordinate x being replaced by the distance r from the centre. Thus all quantities are functions only of the ratio r/t .¹⁸

For centrally symmetrical flow ($v_r = v(r, t)$, $v_\phi = v_\theta = 0$), the equations of motion are as follows. The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(v\rho)}{\partial r} + \frac{2v\rho}{r} = 0;$$

Euler's equation is

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r},$$

and the equation of conservation of entropy is

$$\frac{\partial s}{\partial t} + v \frac{\partial s}{\partial r} = 0.$$

Introducing the variable $\xi = \frac{r}{t}$ (> 0) and assuming that all quantities are functions of ξ only, we obtain

$$\frac{(\xi - v)\rho'}{\rho} = v' + \frac{2v}{\xi}, \quad (130.1)$$

$$(\xi - v)v' = \frac{p'}{\rho}, \quad (130.2)$$

$$(\xi - v)s' = 0, \quad (130.3)$$

the prime denoting differentiation with respect to ξ . We cannot have $v = \xi$, since this contradicts the first equation. From the third equation, therefore, $s' = 0$, i.e., $s = \text{constant}$. We can therefore write $p' = c^2 \rho'$, and equation (130.2) becomes

$$(\xi - v)s' = \frac{c^2 \rho'}{\rho}. \quad (130.4)$$

Substituting $\frac{\rho'}{\rho}$ from (130.1), we obtain the relation

$$\left[\frac{(\xi - v)^2}{c^2} - 1 \right] v' = \frac{2v}{\xi}. \quad (130.5)$$

Equations (130.4) and (130.5) cannot be integrated analytically, but the properties of their solutions can be investigated.

The region where the gas flow is of the type considered is bounded, as we shall see below,

¹⁷ We everywhere ignore the heat losses that may accompany the propagation of a detonation wave. As in the case of slow combustion, these losses may prevent the propagation. For detonation in a pipe, the losses are due mainly to heat loss through the pipe walls and the retardation of the gas by friction.

¹⁸ The dimensionless similarity variable in this problem may be taken as $\frac{r}{t\sqrt{q}}$, where the constant parameter q is the

heat of reaction per unit mass.

by two spheres, of which the outer is the surface of the detonation wave itself, and the inner is the surface of a weak discontinuity, where the velocity is zero.

Let us first examine the properties of the solution near the point where v is zero. It is easy to see that, where $v = 0$, $\xi = 0$ also:

$$v = 0, \quad \xi = c. \quad (130.6)$$

For, when v tends to zero, $\log v \rightarrow -\infty$; hence, when ξ decreases to the value corresponding to the inner boundary of the region in question, the derivative $\frac{d \log v}{d \xi}$ must tend to $+\infty$.

From (130.5), however, we have for $v = 0$

$$\frac{d \log v}{d \xi} = \frac{2c^2}{\xi(\xi^2 - c^2)}.$$

This expression can tend to $+\infty$ only if $\xi \rightarrow c$.

At the origin, the radial velocity must vanish, by symmetry. Thus there is a region of gas at rest round the origin; this is the region inside the sphere $\xi = c_0$, where c_0 is the velocity of sound for the point where $v = 0$.

Let us ascertain the properties of the function $v(\xi)$ near the point (130.6). From (130.5) we have

$$v \frac{d \xi}{d v} = \frac{1}{2} \xi \left[\frac{(\xi - v)^2}{c^2} - 1 \right].$$

As far as quantities of the first order (such as v , $\xi - c_0$ and $c - c_0$), we have after a simple

calculation $v \frac{d(\xi - c_0)}{d v} = (\xi - c_0) - (v + c - c_0)$. According to (102.1) we have $v + c - c_0 = \alpha_0 v$, where α_0 is a positive constant, the value of (102.2) for $v = 0$, and we obtain the following linear first-order differential equation for $\xi - c_0$ as a function of v :

$$v \frac{d(\xi - c_0)}{d v} - (\xi - c_0) = \alpha_0 v.$$

The solution of this equation is

$$\xi - c_0 = \alpha_0 v \log \frac{\text{constant}}{v}. \quad (130.7)$$

This implicitly determines the function $v(\xi)$ near the point where $v = 0$.

We see that the inner boundary is a surface of weak discontinuity: the velocity tends continuously to zero. The curve of $v(\xi)$ has a horizontal tangent at this point ($\frac{d v}{d \xi} = 0$). The

weak discontinuity involved is very unusual: the first derivative is continuous, but all higher derivatives are infinite (as is easily seen from (130.7)). The ratio r/t for $v = 0$ is clearly just the velocity of motion of the boundary relative to the gas; according to (130.6), it is equal to the local velocity of sound, as it should be for a weak discontinuity.

We have also for small v , by (130.7),

$$\xi - v - c = (\xi - c_0) - (v + c - c_0) = \alpha_0 v \left[\log \frac{\text{constant}}{v} - 1 \right].$$

For small v , this quantity is positive: $\xi - v - c > 0$. We shall show that the difference $(\xi - v) - c$ cannot change sign anywhere in the region of the flow considered. Let us consider a point, if there is one, where

$$\xi - v = c, \quad v \neq 0. \quad (130.8)$$

We see from (130.5) that the derivative v' must be infinite at this point, i.e.,

$$\frac{d \xi}{d v} = 0. \quad (130.9)$$

The second derivative $\frac{d^2 \xi}{d v^2}$ is shown by a simple calculation (using the conditions (130.8))

and (130.9)) to be $\frac{d^2\xi}{dv^2} = -\frac{\alpha_0\xi}{c_0v}$, which is not zero. This means that ξ as a function of v has a maximum at the point in question. Thus the function $v(\xi)$ exists only for ξ less than the value corresponding to the conditions (130.8), and this value is the other boundary of the region considered. Since $\xi - v - c$ can vanish only at the boundary of the region, and $\xi - v - c > 0$ for small v , we conclude that

$$\xi - v > c \quad (130.10)$$

everywhere in the region.

It is now easy to see that the outer boundary of the region of the flow considered must in fact be at the point where the conditions (130.8) hold. To see this, we notice that the difference $r/t - v$, where r is the coordinate of the boundary, is just the velocity of the boundary relative to the gas behind it. A surface on which $r/t - v > c$, however, cannot be the surface of a detonation wave (where we must have $r/t - v \leq c$). We therefore conclude that the outer boundary of the region considered can only be the point where (130.8) holds. On this boundary v falls discontinuously to zero, and the velocity of the boundary relative to the gas just behind it is equal to the local velocity of sound. This means that the detonation wave must correspond to the Chapman-Jouguet point on the detonation adiabetic.¹⁹

We thus have the following flow pattern for spherical propagation of a detonation. The detonation wave, like that in a pipe, must correspond to the Chapman-Jouguet point. Immediately behind it is a **spherical similarity rarefaction wave**, in which the gas velocity decreases to zero. The decrease is monotonic, since, by (130.5), the derivative $\frac{dv}{d\xi}$ can vanish only if $v = 0$ also. The gas pressure and density also decrease monotonically, since by (130.4) and (130.10) the derivative p' always has the same sign as v' . The curve giving v as a function of r/t has a vertical tangent at the outer boundary (by (130.9)) and a horizontal tangent at the inner boundary (Fig. 134). The inner boundary is a weak discontinuity, near which the dependence of v on r/t is given by equation (130.7). The gas within the sphere bounded by the weak discontinuity is at rest. The total mass of gas at rest is, however, very small (cf. the remarks at the end of §106).

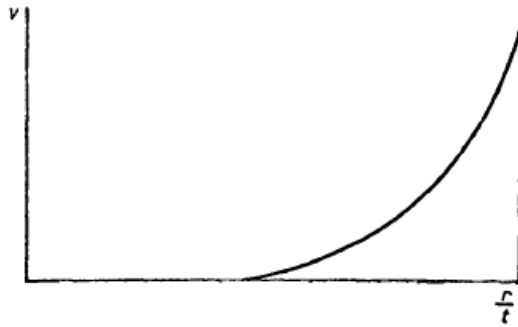


FIG. 134

Thus, in all the typical cases of spontaneous one-dimensional spherical propagation of detonation which we have considered, the boundary conditions in the region behind the detonation wave give a unique velocity for the latter, which corresponds to the Chapman-Jouguet point (the whole of the detonation adiabetic below this point being excluded by the arguments of §129). The achievement, in a pipe with constant cross-section, of a detonation corresponding to the part of the adiabetic above the Chapman-Jouguet point would require an artificial compression of the combustion products by a piston moving with a supersonic velocity (see Problem 3). Such detonation waves are said to be *over-compressed*.

¹⁹ We may notice for completeness that $v = \text{constant}$ is not a solution of the equations of centrally symmetrical flow. Hence the detonation wave cannot be followed by a region of constant velocity.

It should be emphasized, however, that these conclusions are not universally valid, and there are cases of propagation of a detonation where an over-compressed detonation wave occurs spontaneously. In particular, an over-compressed detonation wave is formed when an ordinary detonation wave goes from a wide pipe into a narrow one. This phenomenon occurs because, when a detonation wave reaches a narrowing of the pipe, it is partly reflected, and the pressure of the combustion products moving from the wide part to the narrow part is considerably increased (cf. Problem 4) (B. V. Aivazov and Ya. B. Zel'dovich 1947).²⁰

A general remark should be made regarding the theory given in §§129 and 130. The structure of the detonation wave has been assumed steady, and uniform over its area; it is one-dimensional, in the sense that the propagation of all quantities in the combustion zone is assumed to depend only on the transverse coordinate. However, the experimental results now available indicate that this picture is highly idealized and could serve only to give an averaged description of the process; the picture actually observed is usually quite different. The structure is in fact **far from steady**, and basically **three-dimensional**; the wave area has a complex **small-scale structure** which varies rapidly with time. This is due to the **instability** which arises in particular from the strong (exponential) **temperature dependence of the reaction rate**: even a slight change in temperature when the shock front undergoes distortion has a large effect on the progress of the reaction, and this instability becomes stronger with increasing ratio of the reaction activation energy to the gas temperature behind the shock. The non-uniform and non-steady form of the structure of the detonation wave is particularly clear in conditions close to the limit of propagation in a pipe: the combustible mixture is ignited almost entirely at isolated, eccentrically located and spirally moving, highly deformed parts of the shock front (*spinning detonation*). The study of the possible mechanisms of all these complex phenomena is outside the scope of the present book.²¹

PROBLEMS

Problem 1. Determine the gas flow when a detonation wave is propagated from the closed end of a pipe.

Solution. The velocity v_1 of the detonation wave relative to the gas at rest in front of it, and its velocity v_2 relative to the burnt gas just behind it, are given in terms of the temperature T_1 by formulae (129.11), (129.12); v_1 is also the velocity of the wave relative to the pipe, so that its coordinate is $x = v_1 t$. The velocity (relative to the pipe) of the combustion products at the detonation wave is $v_1 - v_2$. The velocity v_2 equals the local velocity of sound. Since the velocity of sound is related to the gas velocity v in a similarity rarefaction wave by $c = c_0 + \frac{\gamma-1}{2}v$, we have $v_2 = c_0 + \frac{\gamma_2-1}{2}(v_1 - v_2)$, whence $c_0 = \frac{\gamma_2+1}{2}v_2 - \frac{\gamma_2-1}{2}v_1$. For a strong detonation wave we have, by (129.14), simply $c_0 = \frac{1}{2}v_1$. The quantity c_0 is the velocity of the backward boundary of the rarefaction wave. The velocity varies linearly between the two boundaries (Fig. 133a).

Problem 2. The same as Problem 1, but for a pipe with an open end.

Solution. The velocities v_1 and v_2 are determined as in the previous case, and so c_0 is the same also. The rarefaction wave, however, now extends, not to the point where $v = 0$, but

²⁰ Over-compression arises also in the propagation of an ingoing cylindrical or spherical detonation wave; see Ya. B. Zel'dovich, *Soviet Physics JETP* 9, 550, 1959.

²¹ Some relevant monographs and review articles are: K. I. Shchelkin and Ya. K. Troshin, *Gasdynamics of Combustion*, Baltimore 1965; R. I. Soloukhin, *Shock Waves and Detonations in Gases*, Baltimore 1966; R. I. Soloukhin, *Soviet Physics Uspekhi* 6, 523, 1964; A. K. Oppenheim and R. I. Soloukhin, *Annual Review of Fluid Mechanics* 5, 31, 1973.

to the end of the pipe ($x = 0$, Fig. 133b). We see from the formula $x/t = v + c$ (99.5) that the gas leaves the open end of the pipe with a velocity $v = -c$ equal to the local velocity of sound.

Putting $-v = c = c_0 + \frac{\gamma_2 - 1}{2} v$, we therefore find the velocity of outflow to be

$[-v]_{x=0} = \frac{2c_0}{\gamma_2 + 1}$. For a strong detonation wave this velocity is $\frac{v_1}{\gamma_2 + 1}$, equal in magnitude to

that of the gas just behind it.

Problem 3. The same as Problem 1, but for a detonation wave propagated in a pipe whose end is closed by a piston which begins to move forward with a constant velocity U .

Solution. If $U < v_1$, the velocity distribution in the gas has the form shown in Fig. 135a. The gas velocity decreases from $v_1 - v_2$ at $x/t = v_1$ to U at $x/t = c_0 + \frac{\gamma + 1}{2} U$, with the same value of c_0 as before. Then follows a region in which the gas moves with constant velocity U .

If $U > v_1$, however, the detonation wave cannot correspond to the Chapman-Jouguet point (since the piston would overtake it). In this case we have an **over-compressed detonation wave**, corresponding to a point on the adiabat above the Chapman-Jouguet point. It is determined by the fact that the discontinuity of velocity in the detonation wave must equal the velocity of the piston: $v_1 - v_2 = U$. Throughout the region between the detonation wave and the piston, the gas moves with constant velocity U (Fig. 135b).

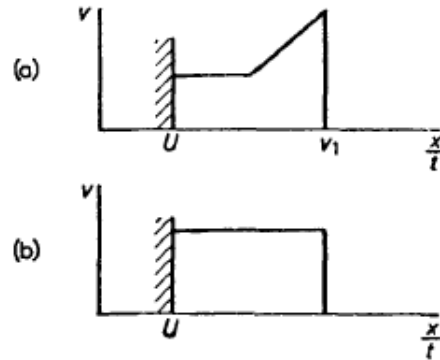


FIG. 135

Problem 4. Determine the pressure at a perfectly rigid wall when a strong plane detonation wave normally incident is reflected from it (K. P. Stanyukovich 1946).

Solution. When a detonation wave is incident on a wall, a reflected shock wave is formed and propagated in the opposite direction, through the combustion products. The calculations are entirely similar to those in §100, Problem 1. With the same notation, we obtain the three relations

$$p_2(V_1 - V_2) = (p_3 - p_2)(V_2 - V_3),$$

$$\frac{V_2}{V_1} = \frac{\gamma_2}{\gamma_2 + 1},$$

$$\frac{V_3}{V_2} = \frac{(\gamma_2 + 1)p_2 + (\gamma_2 - 1)p_3}{(\gamma_2 - 1)p_2 + (\gamma_2 + 1)p_3},$$

here we have neglected p_1 , in comparison with p_2 , but p_2 and p_3 are of the same order of magnitude. Eliminating the volumes, we obtain a quadratic equation for p_3 , and must take

the root which is greater than p_2 :

$$\frac{p_3}{p_2} = \frac{5\gamma_2 + 1 + \sqrt{17\gamma_2^2 + 2\gamma_2 + 1}}{4\gamma_2}.$$

It should be noted that this quantity is almost independent of γ_2 , varying from 2.6 to 2.3 as γ_2 varies from 1 to ∞ .

§131. The relation between the different modes of combustion

It has been shown in §129 that detonation corresponds to points on the upper part of the detonation adiabetic for the combustion process concerned. Since the equation of this adiabetic is a consequence only of the conservation laws for mass, momentum and energy (applied to the initial and final states of the burning gas), it is clear that the points representing the state of the reaction products must lie on the same curve for any other mode of combustion in which the combustion zone can be regarded as a surface of discontinuity of some kind. Let us now ascertain the physical significance of the remainder of the curve.

We draw through the point (p_1, V_1) (point 1 in Fig. 136) vertical and horizontal lines 1A and 1A', and the two tangents 1O and 1O' to the adiabetic. The points A, A', O, O' where these lines intersect or touch the curve divide the adiabetic into five parts. The part lying above O corresponds to detonation, as we have said. We shall now consider the other parts of the curve.

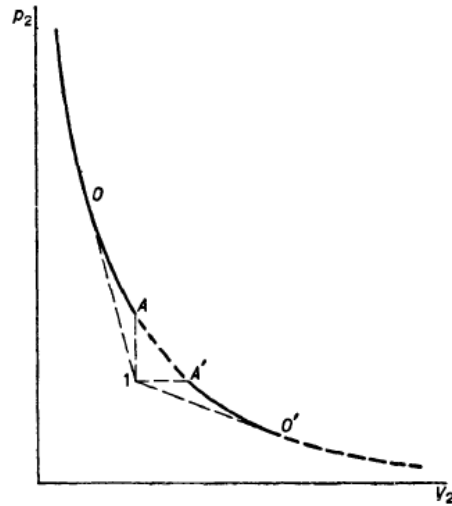


FIG. 136

First of all, it is easy to see that the section AA' has no physical significance. For we have on this section $p_2 > p_1$, $V_2 > V_1$, and so the mass flux j is imaginary; cf. (129.2).

At the points of contact O and O', the derivative $\frac{d(j^2)}{dp_2}$ is zero; it has been shown in §129 (referring to §87) that at such points we have $v_2/c_2 = 1$ and $d(v_2/c_2)/dp_2 < 0$. Hence it follows that above the points of contact $\frac{v_2}{c_2} < 1$, and below them $\frac{v_2}{c_2} > 1$. The relation

between v_1 and c_1 is always easily found by considering the slopes of the corresponding chords and tangents, as was done in §129 for the part above O. The result is that the following inequalities hold on the various sections of the adiabetic:

above O	$v_1 > c_1$,	$v_2 < c_2$;
on AO	$v_1 > c_1$	$v_2 > c_2$;
on A'O'	$v_1 < c_1$	$v_2 < c_2$;
below O'	$v_1 < c_1$	$v_2 > c_2$.

(131.1)

At O and O' , $v_2 = c_2$. As we approach A , the flux j , and therefore the velocities v_1, v_2 , tend to infinity. As we approach A' , however, j and the velocities v_1, v_2 tend to zero.

In §88 we have defined the condition for shock waves to be evolutionary, as a necessary condition for their existence. We have seen that the condition depends on the relation between the number of parameters determining the perturbation and the number of boundary conditions which the perturbation must satisfy at the surface of discontinuity.

All these considerations can also be applied to the surfaces of discontinuity here considered. In particular, the calculation made in §88 of the number of parameters of the perturbation for each case (131.1), shown in Fig. 57, remains valid. For a detonation (the adiabatic above O) the number of boundary conditions is the same as for an ordinary shock wave, and the condition for an evolutionary shock remains as before. In the absence of detonation (below O) the situation is different, because of the change in the number of boundary conditions. Here, the velocity of propagation is determined only by the properties of the chemical reaction and by the conditions of heat transfer from the combustion zone to the cold gas in front of it. This means that the mass flux j through the combustion zone is a given quantity (more precisely, a given function of the state of the unburnt gas 1), whereas in a shock or detonation wave j can have any value. Hence it follows that, on a discontinuity which is a zone of combustion without detonation, the number of boundary conditions is one more than at a shock wave: the condition that j have a given value is added. Thus there are altogether four conditions, and we now conclude in the same manner as in §87 that the discontinuity is absolutely unstable only in the case $v_1 < c_1$, $v_2 > c_2$, which corresponds to points below O' on the adiabatic. Consequently, this part of the curve does not correspond to any mode of combustion that can be realized in practice.

The section $A'O'$ of the adiabatic, on which both velocities v_1 and v_2 are subsonic, corresponds to the ordinary slow combustion. An increase in the rate of propagation of combustion, i.e., in j , corresponds to a movement from A' (where $j = 0$) towards O' . The formulae (128.5) correspond to the point A' (where $p_1 = p_2$), and are valid if j is sufficiently small, viz. if the velocity of propagation is small compared with that of sound. The point O' corresponds to the "most rapid" combustion of this type. We shall give the formulae pertaining to this limiting case.

The point O' , like O , is a point of contact between the curve and the tangent from the point 1. Hence the formulae relating to O' can be obtained immediately from formulae (129.8) - (129.11) for O by appropriately changing the signs (see the footnote to (129.9)). In formulae (129.9) and (129.11) for v_1 and v_2 we change the sign of the second radical, and the sign of the expression (129.12) for $v_1 - v_2$ is therefore changed also. Formulae (129.10) are unchanged if v_1 is taken to have its new value. All these formulae are much simplified if the heat of reaction is large ($q \gg c_{v1}T_1$). We then obtain

$$\left. \begin{aligned} v_1 &= \frac{\gamma_2 p_1 V_1}{\sqrt{2(\gamma_2^2 - 1)q}} \\ v_2 &= \sqrt{\frac{2(\gamma_2 - 1)q}{\gamma_2 + 1}} \\ \frac{p_2}{p_1} &= \frac{1}{\gamma_2 + 1} \\ c_{v2}T_2 &= \frac{2q}{\gamma_2(\gamma_2 + 1)} \end{aligned} \right\}. \quad (131.2)$$

The following remark must be made here. We have seen that, in slow combustion in a closed pipe, a shock wave must be formed in front of the combustion zone. For large velocities of propagation of combustion, this shock wave is strong, and it may considerably affect the state of the gas which enters the combustion zone. It is therefore, strictly speaking, useless to investigate the change in the manner of combustion with increasing velocity, the state p_1 , V_1 of the unburnt gas remaining unchanged. In order to reach the point O' we must create conditions of combustion in which no shock wave is formed. This can be done, for instance, in

combustion in a pipe open at both ends, with a continuous removal of combustion products at the rear end. The rate of removal must be such that the combustion zone remains at rest, and so no shock wave is formed.²²

The part AO of the adiabetic corresponds to combustion without detonation, propagated at supersonic speed. This can in principle occur in the presence of very high heat transfer, e.g., by radiation, which gives combustion rates j exceeding the value corresponding to O' .

In conclusion, we may call attention to the following general differences (besides those contained in the inequalities (131.1)) between the **modes of combustion** corresponding to the upper and lower parts of the adiabetic. Above A we have $p_2 > p_1$, $V_2 < V_1$, $v_2 < v_1$. That is, the reaction products have a pressure and density greater than that of the original gas, and move behind the combustion front with velocity $v_1 - v_2$. In the region below A , however, the inequalities are reversed: $p_2 < p_1$, $V_2 > V_1$, $v_2 > v_1$, and the combustion products are less dense than the original gas.

§132. Condensation discontinuities

There is a formal similarity between detonation waves and what are called **condensation discontinuities**; these occur, for instance, in the flow of a gas containing supersaturated water vapour.²³ The discontinuities are the result of a sudden condensation of vapour occurring very rapidly in a narrow region, which can be regarded as a surface of discontinuity separating the original gas from a gas containing condensed vapour (a *fog*). It should be emphasized that condensation discontinuities are a distinct physical phenomenon, and do not result from the compression of gas in an ordinary shock wave; the latter effect cannot lead to condensation, since the increase of pressure in the shock wave has less effect on the degree of supersaturation than the increase of temperature.

Like combustion, the condensation of a vapour is an exothermic process. The heat of reaction q is represented by the heat evolved per unit mass of gas by the condensation of the vapour.²⁴ The condensation adiabetic which gives p_2 as a function of V_2 for a given state p_1, V_1 of the original uncondensed gas has the same form as the combustion adiabetic shown in Fig. 136. The relations between the velocities of propagation of the discontinuity v_1, v_2 and the velocities of sound c_1, c_2 for the various parts of the condensation adiabetic are given by the inequalities (131.1). However, not all the four cases enumerated in (131.1) can actually occur.

First of all, the question arises whether condensation discontinuities are evolutionary. In this respect their properties are entirely similar to those of combustion zones. We have seen (§131) that the difference in stability between combustion zones and ordinary shock waves is due to the existence of a further condition (that the flux j have a given value) which must be satisfied at the surface. In the case of condensation discontinuities there is again a further condition: the thermodynamic state of the gas 1 in front of the discontinuity must be one for which rapid condensation of the vapour begins. This condition gives a relation between the pressure and temperature of gas 1. We therefore conclude immediately that the whole of the adiabetic below O' , for which $v_1 < c_1$, $v_2 > c_2$, is excluded, since it does not correspond to stable discontinuities.

It is easy to see that discontinuities corresponding to the part above O , for which $v_1 > c_1$, $v_2 < c_2$, also cannot occur in practice. Such a discontinuity would move with supersonic velocity relative to the gas in front of it, and so its presence would have no effect on the state of that gas. Consequently, the discontinuity would have to be formed along a surface

²² Ordinary slow combustion in a tube may spontaneously change into detonation. This is preceded by a spontaneous acceleration of the flame, the detonation wave being formed ahead of it. The possible mechanisms of these processes are discussed in the works cited in the last footnote and in the penultimate footnote to §128.

²³ The theoretical analysis was begun by K. Oswatitsch (1942) and S. Z. Belen'kii (1945).

²⁴ The heat q is not, strictly speaking, the usual latent heat of condensation, since the process occurring in the condensation zone includes not only the isothermal condensation of the vapour, but also a general change in the gas temperature. However, if the degree of supersaturation is not too small (a condition usually satisfied), the difference is unimportant.

determined by the conditions of flow, namely the surface on which the necessary conditions for the onset of rapid condensation would be fulfilled in continuous flow. The velocity of the discontinuity relative to the gas behind it, on the other hand, would be subsonic in this case. The equations of subsonic flow, however, in general have no solution for which all quantities take prescribed values on a given surface.²⁵

Thus only two types of condensation discontinuity are possible:

(1) **supersonic discontinuities** (the section AO of the adiabetic) for which

$$v_1 > c_1, \quad v_2 > c_2, \quad p_2 > p_1, \quad V_2 < V_1 \quad (132.1)$$

and the condensation involves a **compression**,

(2) **subsonic discontinuities** (the section $A'O'$ of the adiabetic), for which

$$v_1 < c_1, \quad v_2 < c_2, \quad p_2 < p_1, \quad V_2 > V_1 \quad (132.2)$$

and the condensation involves a **rarefaction**.

The value of the flux j increases monotonically along the section $A'O'$ from A' (where $j = 0$) to O' , and decreases monotonically along AO from A (where $j = \infty$) to O . The range of values of j (and therefore the range of values of the velocity $v_1 = jV_1$) between those corresponding to O and O' is "forbidden", and cannot occur in a condensation discontinuity. The total mass of condensed vapour is usually very small compared with the mass of the original gas. We can therefore regard both gases 1 and 2 as perfect gases; for the same reason, the specific heats of the two gases may be supposed equal. Then the value of v_1 at the point O is determined by formula (129.9), and its value at O' by the same formula with the sign of the second radical changed; putting $\gamma_1 = \gamma_2 \equiv \gamma$ and $c_1^2 = \gamma(\gamma - 1)c_v T_1$, we find the forbidden range of values of v_1 to be

$$\sqrt{c_1^2 + \frac{\gamma^2 - 1}{2}q} - \sqrt{\frac{\gamma^2 - 1}{2}q} < v_1 < \sqrt{c_1^2 + \frac{\gamma^2 - 1}{2}q} + \sqrt{\frac{\gamma^2 - 1}{2}q}. \quad (132.3)$$

PROBLEM

Determine the limiting values of the ratio of pressures $\frac{p_2}{p_1}$ in a condensation

discontinuity, assuming that $\frac{q}{c_1^2} \ll 1$.

Solution. On the section $A'O'$ of the condensation adiabetic (Fig. 136), the ratio $\frac{p_2}{p_1}$

increases monotonically from O' to A' , taking values in the range

$$1 - \gamma \sqrt{\frac{2(\gamma - 1)q}{(\gamma + 1)c_1^2}} \leq \frac{p_2}{p_1} \leq 1.$$

On the section AO , this ratio increases from A to O , taking values in the range

$$1 + \frac{\gamma(\gamma - 1)q}{c_1^2} \leq \frac{p_2}{p_1} \leq 1 + \gamma \sqrt{\frac{2(\gamma - 1)q}{(\gamma + 1)c_1^2}}.$$

²⁵ Similar arguments hold in the case where the total velocity v_2 (of which $v_2 < c_2$ is the component normal to the discontinuity) is supersonic.

To avoid misunderstanding, it should be mentioned that a condensation discontinuity with $v_1 > c_1$, $v_2 < c_2$ may actually (for certain conditions of vapour content and shape of the surface past which the flow occurs) be simulated by a true condensation discontinuity with $v_1 > c_1$, $v_2 > c_2$, closely followed by a shock wave which renders the flow subsonic.