CHAPTER I

FUNDAMENTAL EQUATIONS

§1. The strain tensor

The mechanics of solid bodies, regarded as continuous media, forms the content of the ***theory of elasticity***.[1]

Under the action of applied forces, solid bodies exhibit deformation to some extent, i.e., they change in shape and volume. The deformation of a body is described mathematically in the following way. The position of any point in the body is defined by its radius vector ***r*** (with components $x_1 = x$, $x_2 = y$, $x_3 = z$) in some co-ordinate system. When the body is deformed, every point in it is in general displaced. Let us consider some particular point; let its radius vector before the deformation be ***r***, and after the deformation have a different value ***r'*** (with components x'_i). The displacement of this point due to the deformation is then given by the vector ***r'*** - ***r***, which we shall denote by ***u***:

$$u_i = x'_i - x_i . \qquad (1\text{-}1)$$

The vector ***u*** is called the ***displacement vector***. The co-ordinates x'_i of the displaced point are, of course, functions of the co-ordinates x_i of the point before displacement. The displacement vector u_i is therefore also a function of the co-ordinates x_i. If the vector ***u*** is given as a function of x_i, the deformation of the body is entirely determined.

When a body is deformed, the distances between its points change. Let us consider two points very close together. If the radius vector joining them before the deformation is dx_i, the radius vector joining the same two points in the deformed body is $dx'_i = dx_i + du_i$. The distance between the points is $dl = \sqrt{dx_1^2 + dx_2^2 + dx_3^2}$ before the deformation, and $dl' = \sqrt{dx'^2_1 + dx'^2_2 + dx'^2_3}$ after it. Using the general summation rule,[2] we can write $dl^2 = dx_i^2$, $dl'^2 = dx'^2_i = (dx_i + du_i)^2$. Substituting $du_i = \dfrac{\partial u_i}{\partial x_k} dx_k$, we can write

$$dl'^2 = dl^2 + 2\frac{\partial u_i}{\partial x_k} dx_i dx_k + \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l} dx_k dx_l .$$

Since the summation is taken over both suffixes *i* and *k* in the second term on

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[1] The basic equations of elasticity theory were established in the 1820's by Cauchy and by Poisson.

[2] In accordance with the usual rule, we omit the sign of summation over vector and tensor suffixes. Summation over the values 1 , 2, 3 is understood with respect to all suffixes which appear twice in a given term.

the right, we can put $\frac{\partial u_i}{\partial x_k} dx_i dx_k = \frac{\partial u_k}{\partial x_i} dx_i dx_k$. In the third term, we interchange the suffixes *i* and *l*. Then dl'^2 takes the final form

$$dl'^2 = dl^2 + 2u_{ik} dx_i dx_k, \tag{1.2}$$

where the tensor u_{ik} is defined as

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right). \tag{1.3}$$

These expressions give the change in an element of length when the body is deformed.

The tensor u_{ik} is called the ***strain tensor***. We see from its definition that it is symmetrical, i.e.,

$$u_{ik} = u_{ki} \tag{1.4}$$

This result has been obtained by writing the term $2\frac{\partial u_i}{\partial x_k} dx_i dx_k$ in dl'^2 in the explicitly symmetrical form

$$\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) dx_i dx_k .$$

Like any symmetrical tensor, u_{ik} can be ***diagonalised*** at any given point. This means that, at any given point, we can choose co-ordinate axes (the ***principal axes*** of the tensor) in such a way that only the diagonal components u_{11}, u_{22}, u_{33} of the tensor u_{ik} are different from zero. These components, the ***principal values*** of the strain tensor, will be denoted by $u^{(1)}, u^{(2)}, u^{(3)}$. It should be remembered, of course, that, if the tensor u_{ik} is diagonalised at any point in the body, it will not in general be diagonal at any other point.

If the strain tensor is diagonalised at a given point, the element of length (1.2) near it becomes

$$\begin{aligned} dl'^2 &= (\delta_{ik} + 2u_{ik}) dx_i dx_k \\ &= (1 + 2u^{(1)}) dx_1^2 + (1 + 2u^{(2)}) dx_2^2 + (1 + 2u^{(3)}) dx_3^2 \end{aligned}.$$

We see that the expression is the sum of three independent terms. This means that the strain in any volume element may be regarded as composed of independent strains in three mutually perpendicular directions, namely those of the principal axes of the strain tensor. Each of these strains is a simple extension (or compression) in the corresponding direction : the length dx_1 along the first principal axis becomes $dx'_1 = \sqrt{1 + 2u^{(1)}}\, dx_1$, and similarly for the other two axes. The quantity $\sqrt{1 + 2u^{(i)}} - 1$ is consequently equal to the

relative extension $\frac{dx'_i - dx_i}{dx_i}$ along the ith principal axis.

In almost all cases occurring in practice, the strains are small. This means that the change in any distance in the body is small compared with the distance itself. In other words, the relative extensions are small compared with unity. In what follows we shall suppose that all strains are small.

If a body is subjected to a small deformation, all the components of the strain tensor are small, since they give, as we have seen, the relative changes in lengths in the body. The displacement vector u_i, however, may sometimes be large, even for small strains. For example, let us consider a long thin rod. Even for a large deflection, in which the ends of the rod move a considerable distance, the extensions and compressions in the rod itself will be small.

Except in such special cases,[3] the displacement vector for a small deformation is itself small. For it is evident that a three-dimensional body (i.e., one whose dimension in no direction is small) cannot be deformed in such a way that parts of it move a considerable distance without the occurrence of considerable extensions and compressions in the body.

Thin rods will be discussed in Chapter II. In other cases u_i is small for small deformations, and we can therefore neglect the last term in the general expression (1.3), as being of the second order of smallness. Thus, for small deformations, the strain tensor is given by

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right). \tag{1.5}$$

The relative extensions of the elements of length along the principal axes of the strain tensor (at a given point) are, to within higher-order quantities, $\sqrt{1+2u^{(i)}} - 1 \approx u^{(i)}$, i.e., they are the principal values of the tensor u_{ik}.

Let us consider an infinitesimal volume element dV, and find its volume dV' after the deformation. To do so, we take the principal axes of the strain tensor, at the point considered, as the co-ordinate axes. Then the elements of length dx_1, dx_2, dx_3 along these axes become, after the deformation, $dx'_1 = (1+u^{(1)})dx_1$, etc. The volume dV is the product $dx_1 dx_2 dx_3$, while dV' is $dx'_1\, dx'_2\, dx'_3$. Thus $dV' = dV(1+u^{(1)})(1+u^{(2)})(1+u^{(3)})$. Neglecting higher-order terms, we therefore have $dV' = dV(1+u^{(1)}+u^{(2)}+u^{(3)})$. The sum $u^{(1)}+u^{(2)}+u^{(3)}$ of the principal values of a tensor is well known to be invariant, and is equal to the sum of the diagonal components

[3] Which include, besides deformations of thin rods, those of thin plates to form cylindrical

$u_{ii} = u_{11} + u_{22} + u_{33}$ in any co-ordinate system. Thus

$$dV' = dV(1 + u_{ii}). \tag{1.6}$$

We see that the sum of the diagonal components of the strain tensor is the relative volume change $(dV' - dV)/dV$.

It is often convenient to use the components of the strain tensor in spherical or cylindrical co-ordinates. We give here, for reference, the corresponding formulae, which express the components in terms of the derivatives of the components of the displacement vector in the same co-ordinates. In **spherical** co-ordinates r, θ, ϕ, we have

$$\left.\begin{aligned}
u_{rr} &= \frac{\partial u_r}{\partial r} \\
u_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\
u_{\phi\phi} &= \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r} \\
2u_{\theta\phi} &= \frac{1}{r}\left(\frac{\partial u_\phi}{\partial \theta} - u_\phi \cot\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} \\
2u_{r\theta} &= \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \\
2u_{\phi r} &= \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}
\end{aligned}\right\}. \tag{1.7}$$

In **cylindrical** co-ordinates r, ϕ, z,

$$\left.\begin{aligned}
u_{rr} &= \frac{\partial u_r}{\partial r} \\
u_{\phi\phi} &= \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} \\
u_{zz} &= \frac{\partial u_z}{\partial z} \\
2u_{\phi z} &= \frac{1}{r}\frac{\partial u_z}{\partial \phi} + \frac{\partial u_\phi}{\partial z} \\
2u_{rz} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \\
2u_{r\phi} &= \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \phi}
\end{aligned}\right\}. \tag{1.8}$$

§2. The stress tensor

In a body that is not deformed, the arrangement of the molecules corresponds to a state of thermal equilibrium. All parts of the body are in mechanical equilibrium. This means that, if some portion of the body is considered, the resultant of the forces on that portion is zero.

When a deformation occurs, the arrangement of the molecules is changed, and the body ceases to be in its original state of equilibrium. Forces therefore

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surfaces.

arise which tend to return the body to equilibrium. These internal forces which occur when a body is deformed are called ***internal stresses***. If no deformation occurs, there are no internal stresses.

The internal stresses are due to molecular forces, i.e., the forces of interaction between the molecules. An important fact in the theory of elasticity is that the molecular forces have a very short range of action. Their effect extends only to the neighbourhood of the molecule exerting them, over a distance of the same order as that between the molecules, whereas in the theory of elasticity, which is a macroscopic theory, the only distances considered are those large compared with the distances between the molecules. The range of action of the molecular forces should therefore be taken as zero in the theory of elasticity. We can say that the forces which cause the internal stresses are, as regards the theory of elasticity, "near-action" forces, which act from any point only to neighbouring points. Hence it follows that the forces exerted on any part of the body by surrounding parts act only on the surface of that part.

The following reservation should be made here. The above assertion is not valid in cases where the deformation of the body results in macroscopic electric fields in it (pyroelectric and piezoelectric bodies). We shall not discuss such bodies in this book, however.

Let us consider the total force on some portion of the body. Firstly, this total force is equal to the sum of all the forces on all the volume elements in that portion of the body, i.e., it can be written as the volume integral $\int \boldsymbol{F} dV$, where $\boldsymbol{F}$ is the force per unit volume and $\boldsymbol{F}\mathrm{d}V$ the force on the volume element dV. Secondly, the forces with which various parts of the portion considered act on one another cannot give anything but zero in the total resultant force, since they cancel by Newton's third law. The required total force can therefore be regarded as the sum of the forces exerted on the given portion of the body by the portions surrounding it. From above, however, these forces act on the surface of that portion, and so the resultant force can be represented as the sum of forces acting on all the surface elements, i.e., as an integral over the surface.

Thus, for any portion of the body, each of the three components $\int F_i dV$ of the resultant of all the internal stresses can be transformed into an integral over the surface. As we know from vector analysis, the integral of a scalar over an arbitrary volume can be transformed into an integral over the surface

if the scalar is the divergence of a vector. In the present case we have the integral of a vector, and not of a scalar. Hence the vector F_i must be the divergence of a tensor of rank two, i.e., be of the form

$$F_i = \frac{\partial \sigma_{ik}}{\partial x_k}. \tag{2.1}$$

Then the force on any volume can be written as an integral over the closed surface bounding that volume:[4]

$$\int F_i dV = \int \frac{\partial \sigma_{ik}}{\partial x_k} dV = \oint \sigma_{ik} df_k , \tag{2.2}$$

where df_i are the components of the surface element vector ***df***, directed (as usual) along the outward normal.[5]

The tensor σ_{ik} is called the ***stress tensor***. As we see from (2.2), $\sigma_{ik} df_k$ is the *i*th component of the force on the surface element ***df***. By taking elements of area in the planes of *xy*, *yz*, *zx*, we find that the component σ_{ik} of the stress tensor is the *i*th. component of the force on unit area perpendicular to the x_k -axis. For instance, the force on unit area perpendicular to the *x*-axis, normal to the area (i.e., along the *x*-axis) is σ_{xx}, and the tangential forces (along the *y* and *z* axes) are σ_{yx} and σ_{zx}.

The following remark should be made concerning the sign of the force $\sigma_{ik} df_k$. The surface integral in (2.2) is the force exerted on the volume enclosed by the surface by the surrounding parts of the body. The force which this volume exerts on the surface surrounding it is the same with the opposite sign. Hence, for example, the force exerted by the internal stresses on the surface of the body itself is $-\oint \sigma_{ik} df_k$, where the integral is taken over the surface of the body and ***df*** is along the outward normal.

Let us determine the moment of the forces on a portion of the body. The moment of the force ***F*** can be written as an antisymmetrical tensor of rank two, whose components are $F_i x_k - F_k x_i$, where x_i are the co-ordinates of the point where the force is applied.[6] Hence the moment of the forces on the

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[4] The integral over a closed surface is transformed into one over the volume enclosed by the surface by replacing the surface element df_i by the operator $dV \frac{\partial}{\partial x_i}$.

[5] Strictly speaking, to determine the total force on a deformed portion of the body we should integrate, not over the old co-ordinates x_i, but over the co-ordinates x'_i of the points of the deformed body. The derivatives (2.1) should therefore be taken with respect to x'_i. However, in view of the smallness of the deformation, the derivatives with respect to x_i and x'_i differ only by higher-order quantities, and so the derivatives can be taken with respect to the co-ordinates x_i.

[6] The moment of the force ***F*** is defined as the vector product $\boldsymbol{F} \times \boldsymbol{r}$, and we know from vector analysis that the components of a vector product form an antisymmetrical tensor of rank two as

volume element dV is $(F_i x_k - F_k x_i)dV$, and the moment of the forces on the whole volume is $M_{ik} = \int (F_i x_k - F_k x_i)dV$. Like the total force on any volume, this moment can be expressed as an integral over the surface bounding the volume. Substituting the expression (2.1) for F_i, we find

$$M_{ik} = \int \left(\frac{\partial \sigma_{il}}{\partial x_l} x_k - \frac{\partial \sigma_{kl}}{\partial x_l} x_i \right) dV$$
$$= \int \frac{\partial(\sigma_{il} x_k - \sigma_{kl} x_i)}{\partial x_l} dV - \int \left(\sigma_{il} \frac{\partial x_k}{\partial x_l} - \sigma_{kl} \frac{\partial x_i}{\partial x_l} \right) dV .$$

In the second term we use the fact that the derivative of a co-ordinate with respect to itself is unity, and with respect to another co-ordinate is zero (since the three co-ordinates are independent variables). Thus $\frac{\partial x_k}{\partial x_l} = \delta_{kl}$, where δ_{kl} is the unit tensor; the multiplication gives $\sigma_{il}\delta_{kl} = \sigma_{ik}$, $\sigma_{kl}\delta_{il} = \sigma_{ki}$. In the first term, the integrand is the divergence of a tensor; the integral can be transformed into one over the surface. The result is

$$M_{ik} = \oint (\sigma_{il} x_k - \sigma_{kl} x_i) df_l + \int (\sigma_{ki} - \sigma_{ik}) dV .$$

If M_{ik} is to be an integral over the surface only, the second term must vanish identically, i.e., we must have

$$\sigma_{ik} = \sigma_{ki} . \qquad (2.3)$$

Thus we reach the important result that the **stress tensor is symmetrical**. The moment of the forces on a portion of the body can then be written simply as

$$M_{ik} = \int (F_i x_k - F_k x_i) dV = \oint (\sigma_{il} x_k - \sigma_{kl} x_i) df_l . \qquad (2.4)$$

It is easy to find the stress tensor for a body undergoing uniform compression from all sides (***hydrostatic*** compression). In this case a pressure of the same magnitude acts on every unit area on the surface of the body, and its direction is along the inward normal. If this pressure is denoted by p, a force $-pdf_i$ acts on the surface element df_i. This force, in terms of the stress tensor, must be $\sigma_{ik} df_k$. Writing $-pdf_i = -p\delta_{ik} df_k$, we see that the stress tensor in hydrostatic compression is

$$\sigma_{ik} = -p\delta_{ik} . \qquad (2.5)$$

Its non-zero components are simply equal to the pressure.

In the general case of an arbitrary deformation, the non-diagonal components of the stress tensor are also non-zero. This means that not only a

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written here.

normal force but also tangential (shearing) stresses act on each surface element. These latter stresses tend to move the surface elements relative to each other.

In equilibrium the internal stresses in every volume element must balance, i.e., we must have $F_i = 0$. Thus the equations of equilibrium for a deformed body are

$$\frac{\partial \sigma_{ik}}{\partial x_k} = 0. \qquad (2.6)$$

If the body is in a gravitational field, the sum $\boldsymbol{F} + \rho \boldsymbol{g}$ of the internal stresses and the force of gravity ($\rho \boldsymbol{g}$ per unit volume) must vanish ; ρ is the density[7] and $\boldsymbol{g}$ the gravitational acceleration vector, directed vertically downwards. In this case the equations of equilibrium are

$$\frac{\partial \sigma_{ik}}{\partial x_k} + \rho g_i = 0. \qquad (2.7)$$

The external forces applied to the surface of the body (which are the usual cause of deformation) appear in the boundary conditions on the equations of equilibrium. Let $\boldsymbol{P}$ be the external force on unit area of the surface of the body, so that a force $\boldsymbol{P}df$ acts on a surface element *df*. In equilibrium, this must be balanced by the force - $\sigma_{ik} df_k$ of the internal stresses acting on that element. Thus we must have $P_i df - \sigma_{ik} df_k = 0$. Writing $df_k = n_k df$, where $\boldsymbol{n}$ is a unit vector along the outward normal to the surface, we find

$$\sigma_{ik} n_k = P_i. \qquad (2.8)$$

This is the condition which must be satisfied at every point on the surface of a body in equilibrium.

We shall derive also a formula giving the mean value of the stress tensor in a deformed body. To do so, we multiply equation (2.6) by x_k and integrate over the whole volume:

$$\int \frac{\partial \sigma_{il}}{\partial x_l} x_k dV = \int \frac{\partial (\sigma_{il} x_k)}{\partial x_l} dV - \int \sigma_{il} \frac{\partial x_k}{\partial x_l} dV = 0.$$

The first integral on the right is transformed into a surface integral; in the second integral we put $\frac{\partial x_k}{\partial x_l} = \delta_{kl}$. The result is $\oint \sigma_{il} x_k df_l - \int \sigma_{ik} dV = 0$.

Substituting (2.8) in the first integral, we find $\oint P_i x_k df = \int \sigma_{ik} dV = V \overline{\sigma}_{ik}$,

where V is the volume of the body and $\overline{\sigma}_{ik}$ the mean value of the stress tensor. Since $\sigma_{ik} = \sigma_{ki}$, this formula can be written in the symmetrical form

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[7] Strictly speaking, the density of a body changes when it is deformed. An allowance for this change, however, involves higher-order quantities in the case of small deformations, and is therefore unimportant.

$$\overline{\sigma}_{ik} = \frac{1}{2V}\oint (P_i x_k + P_k x_i) df \,. \qquad (2.9)$$

Thus the mean value of the stress tensor can be found immediately from the external forces acting on the body, without solving the equations of equilibrium.

§3. The thermodynamics of deformation

Let us consider some deformed body, and suppose that the deformation is changed in such a way that the displacement vector u_i changes by a small amount δu_i ; and let us determine the work done by the internal stresses in this change. Multiplying the force $F_i = \frac{\partial \sigma_{ik}}{\partial x_k}$ by the displacement δu_i and integrating over the volume of the body, we have $\int \delta R dV = \int \frac{\partial \sigma_{ik}}{\partial x_k} \delta u_i dV$, where δR denotes the work done by the internal stresses per unit volume. We integrate by parts, obtaining

$$\int \delta R dV = \oint \sigma_{ik} \delta u_i df_k - \int \sigma_{ik} \frac{\partial \delta u_i}{\partial x_k} dV \,.$$

By considering an infinite medium which is not deformed at infinity, we make the surface of integration in the first integral tend to infinity; then $\sigma_{ik} = 0$ on the surface, and the integral is zero. The second integral can, by virtue of the symmetry of the tensor σ_{ik} , be written

$$\begin{aligned}\int \delta R dV &= -\frac{1}{2}\int \sigma_{ik}\left(\frac{\partial \delta u_i}{\partial x_k} + \frac{\partial \delta u_k}{\partial x_i}\right) dV \\ &= -\frac{1}{2}\int \sigma_{ik}\,\delta\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) dV \,. \\ &= -\int \sigma_{ik}\,\delta u_{ik}\, dV\end{aligned}$$

Thus we find

$$\delta R = -\sigma_{ik}\,\delta u_{ik} \,. \qquad (3.1)$$

This formula gives the work δR in terms of the change in the strain tensor.

If the deformation of the body is fairly small, it returns to its original undeformed state when the external forces causing the deformation cease to act. Such deformations are said to be ***elastic***. For large deformations, the removal of the external forces does not result in the total disappearance of the deformation ; a ***residual deformation*** remains, so that the state of the body is not that which existed before the forces were applied. Such deformations are said to be ***plastic***. In what follows we shall consider only elastic deformations.

We shall also suppose that the process of deformation occurs so **slowly** that thermodynamic equilibrium is established in the body at every instant, in accordance with the external conditions. This assumption is almost always justified in practice. The process will then be **thermodynamically reversible**.

In what follows we shall take all such thermodynamic quantities as the entropy S, the internal energy E, etc., relative to unit volume of the body,[8] and not relative to unit mass as in fluid mechanics, and denote them by the corresponding capital letters.

An infinitesimal change dE in the internal energy is equal to the difference between the heat acquired by the unit volume considered and the work dR done by the internal stresses. The amount of heat is, for a reversible process, TdS, where T is the temperature. Thus $dE = TdS\text{-}dR$; with dR given by (3.1), we obtain

$$dE = TdS + \sigma_{ik} du_{ik} . \qquad (3.2)$$

This is the **fundamental thermodynamic relation** for deformed bodies.

In hydrostatic compression, the stress tensor is $\sigma_{ik} = -p\delta_{ik}$ (2.5). Then $\sigma_{ik} du_{ik} = -p\delta_{ik} du_{ik} = -pdu_{ii}$. We have seen, however (cf. (1.6)), that the sum u_{ii} is the relative volume change due to the deformation. If we consider unit volume, therefore, u_{ii} is simply the change in that volume, and du_{ii} is the volume element dV. The thermodynamic relation then takes its usual form

$$dE = TdS - pdV .$$

Introducing the **free energy** of the body, $F = E - TS$, we find the form

$$dF = -SdT + \sigma_{ik} du_{ik} \qquad (3.3)$$

of the relation (3.2). Finally, the **thermodynamic potential** Φ is defined as

$$\Phi = E - TS - \sigma_{ik} u_{ik} = F - \sigma_{ik} u_{ik} . \qquad (3.4)$$

This is a generalisation of the usual expression $\Phi = E - TS + pV$.[9] Substituting (3.4) in (3.3), we find

$$d\Phi = -SdT - u_{ik} d\sigma_{ik} . \qquad (3.5)$$

The independent variables in (3.2) and (3.3) are, respectively, S, u_{ik} and T, u_{ik}. The components of the stress tensor can be obtained by

8 The following remark should be made here. Strictly speaking, the unit volumes before and after the deformation should be distinguished, since they in general contain different amounts of matter. We shall always relate the thermodynamic quantities to unit volume of the undeformed body, i.e., to the amount of matter therein, which may occupy a different volume after the deformation. Accordingly, the total energy of the body, for example, is obtained by integrating E over the volume of the undeformed body.

9 For hydrostatic compression, the expression (3.4) becomes $\Phi = F + pu_{ii} = F + p(V - V_0)$, where $V - V_0$ is the volume change resulting from the deformation. Hence we see that the definition of Φ used here differs by a term $-pV_0$ from the usual definition $\Phi = F + pV$.

differentiating E or F with respect to the components of the strain tensor, for constant entropy S or temperature T, respectively:

$$\sigma_{ik} = \left(\frac{\partial E}{\partial u_{ik}}\right)_S = \left(\frac{\partial F}{\partial u_{ik}}\right)_T . \tag{3.6}$$

Similarly, by differentiating Φ with respect to the components σ_{ik}, we can obtain the components u_{ik}:

$$u_{ik} = -\left(\frac{\partial \Phi}{\partial \sigma_{ik}}\right)_T . \tag{3.7}$$

§4. Hooke's law

In order to be able to apply the general formulae of thermodynamics to any particular case, we must know the free energy F of the body as a function of the strain tensor. This expression is easily obtained by using the fact that the deformation is small and expanding the free energy in powers of u_{ik}. We shall at present consider only **isotropic bodies**. The corresponding results for crystals will be obtained in §10.

In considering a deformed body at some temperature (constant throughout the body), we shall take the undeformed state to be the state of the body in the absence of external forces and at the same temperature; this last condition is necessary on account of the thermal expansion (see §6). Then, for $u_{ik} = 0$, the internal stresses are zero also, i.e., $\sigma_{ik} = 0$. Since $\sigma_{ik} = \frac{\partial F}{\partial u_{ik}}$, it follows that there is no linear term in the expansion of F in powers of u_{ik}.

Next, since the free energy is a scalar, each term in the expansion of F must be a scalar also. Two independent scalars of the second degree can be formed from the components of the symmetrical tensor u_{ik}: they can be taken as the squared sum of the diagonal components $(u_{ii}{}^2)$ and the sum of the squares of all the components $(u_{ik}{}^2)$. Expanding F in powers of u_{ik} we therefore have as far as terms of the second order

$$F = F_0 + \frac{1}{2}\lambda u_{ii}{}^2 + \mu u_{ik}{}^2 . \tag{4.1}$$

This is the general expression for the free energy of a deformed isotropic body. The quantities λ and μ are called ***Lamé coefficients***.

We have seen in §1 that the change in volume in the deformation is given by the sum u_{ii}. If this sum is zero, then the volume of the body is unchanged by the deformation, only its shape being altered. Such a deformation is called a ***pure shear***.

The opposite case is that of a deformation which causes a change in the

volume of the body but no change in its shape. Each volume element of the body retains its shape also. We have seen in §1 that the tensor of such a deformation is $u_{ik} = \text{constant} \times \delta_{ik}$. Such a deformation is called a ***hydrostatic compression***.

Any deformation can be represented as the sum of a pure shear and a hydrostatic compression. To do so, we need only use the identity

$$u_{ik} = (u_{ik} - \frac{1}{3}\delta_{ik}u_{ll}) + \frac{1}{3}\delta_{ik}u_{ll} . \qquad (4.2)$$

The first term on the right is evidently a pure shear, since the sum of its diagonal terms is zero ($\delta_{ii} = 3$). The second term is a hydrostatic compression.

As a general expression for the free energy of a deformed isotropic body, it is convenient to replace (4.1) by another formula, using this decomposition of an arbitrary deformation into a pure shear and a hydrostatic compression. We take as the two independent scalars of the second degree the sums of the squared components of the two terms in (4.2). Then F becomes[10]

$$F = \mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{ll})^2 + \frac{1}{2}Ku_{ll}{}^2 . \qquad (4.3)$$

The quantities K and μ . are called, respectively, the ***bulk modulus*** or ***modulus of hydrostatic compression*** (or simply the ***modulus of compression***) and the ***shear modulus*** or ***modulus of rigidity***. K is related to the Lamé coefficients by

$$K = \lambda + \frac{2}{3}\mu . \qquad (4.4)$$

In a state of thermodynamic equilibrium, the free energy is a minimum. If no external forces act on the body, then F as a function of u_{ik} must have a minimum for $u_{ik} = 0$. This means that the quadratic form (4.3) must be positive. If the tensor u_{ik} is such that $u_{ll} = 0$, only the first term remains in (4.3); if, on the other hand, the tensor is of the form $u_{ik} = \text{constant} \times \delta_{ik}$, then only the second term remains. Hence it follows that a necessary (and evidently sufficient) condition for the form (4.3) to be positive is that each of the coefficients K and μ is positive. Thus we conclude that the moduli of compression and rigidity are always positive:

$$K > 0 , \quad \mu > 0 . \qquad (4.5)$$

We now use the general thermodynamic relation (3.6) to determine the

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[10] The constant term F_0 is the free energy of the undeformed body, and is of no further interest. We shall therefore omit it, for brevity, taking F to be only the free energy of the deformation (the ***elastic free energy***, as it is called).

stress tensor. To calculate the derivatives $\frac{\partial F}{\partial u_{ik}}$, we write the total differential dF (for constant temperature):

$$dF = Ku_{ll}du_{ll} + 2\mu(u_{ik} - \frac{1}{3}u_{ll}\delta_{ik})d(u_{ik} - \frac{1}{3}u_{ll}\delta_{ik}) .$$

In the second term, multiplication of the first parenthesis by δ_{ik} gives zero, leaving $dF = Ku_{ll}du_{ll} + 2\mu(u_{ik} - \frac{1}{3}u_{ll}\delta_{ik})du_{ik}$, or writing $du_{ll} = \delta_{ik}du_{ik}$,

$$dF = \left[Ku_{ll}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}u_{ll}\delta_{ik}) \right] du_{ik} .$$

Hence the stress tensor is

$$\sigma_{ik} = Ku_{ll}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{ll}) . \qquad (4.6)$$

This expression determines the stress tensor in terms of the strain tensor for an isotropic body. It shows, in particular, that, if the deformation is a pure shear or a pure hydrostatic compression, the relation between σ_{ik} and u_{ik} is determined only by the modulus of rigidity or of hydrostatic compression, respectively.

It is not difficult to obtain the converse formula which expresses u_{ik} in terms of σ_{ik}. To do so, we find the sum σ_{ii} of the diagonal terms. Since this sum is zero for the second term of (4.6), we have $\sigma_{ii} = 3Ku_{ii}$, or

$$u_{ii} = \frac{\sigma_{ii}}{3K} . \qquad (4.7)$$

Substituting this expression in (4.6) and so determining u_{ik}, we find

$$u_{ik} = \frac{\delta_{ik}\sigma_{ll}}{9K} + \frac{\sigma_{ik} - (1/3)\delta_{ik}\sigma_{ll}}{2\mu} , \qquad (4.8)$$

which gives the strain tensor in terms of the stress tensor.

Equation (4.7) shows that the relative change in volume (u_{ii}) in any deformation of an isotropic body depends only on the sum σ_{ii} of the diagonal components of the stress tensor, and the relation between u_{ii} and σ_{ii} is determined only by the modulus of hydrostatic compression. In hydrostatic compression of a body, the stress tensor is $\sigma_{ik} = -p\delta_{ik}$. Hence we have in this case, from (4.7),

$$u_{ii} = -\frac{p}{K} . \qquad (4.9)$$

Since the deformations are small, u_{ii} and p are small quantities, and we can write the ratio $\frac{u_{ii}}{p}$ of the relative volume change to the pressure in the

differential form $\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$. Thus

$$\frac{1}{K} = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T .$$

The quantity $1/K$ is called the ***coefficient of hydrostatic compression*** (or simply the ***coefficient of compression***).

We see from (4.8) that the strain tensor u_{ik} is a linear function of the stress tensor σ_{ik}. That is, the deformation is proportional to the applied forces. This law, valid for small deformations, is called ***Hooke's law***.[11]

We may give also a useful form of the expression for the free energy of a deformed body, which is obtained immediately from the fact that F is quadratic in the strain tensor. According to Euler's theorem, $u_{ik}\frac{\partial F}{\partial u_{ik}} = 2F$, whence, since $\frac{\partial F}{\partial u_{ik}} = \sigma_{ik}$, we have

$$F = \frac{1}{2}\sigma_{ik}u_{ik} . \qquad (4.10)$$

If we substitute in this formula the u_{ik} as linear combinations of the components σ_{ik}, the elastic energy will be represented as a quadratic function of the σ_{ik}. Again applying Euler's theorem, we obtain $u_{ik}\frac{\partial F}{\partial u_{ik}} = 2F$, and a comparison with (4.10) shows that

$$u_{ik} = \frac{\partial F}{\partial \sigma_{ik}} . \qquad (4.11)$$

It should be emphasised, however, that, whereas the formula $\sigma_{ik} = \frac{\partial F}{\partial u_{ik}}$ is a general relation of thermodynamics, the inverse formula (4.11) is applicable only if Hooke's law is valid.

§5. Homogeneous deformations

Let us consider some simple cases of what are called ***homogeneous deformations***, i.e., those in which the strain tensor is constant throughout the volume of the body. For example, the hydrostatic compression already considered is a homogeneous deformation.

We first consider a ***simple extension*** (or compression) of a rod. Let the rod be along the *z*-axis, and let forces be applied to its ends which stretch it in

[11] Hooke's law is actually applicable to almost all elastic deformation. The reason is that deformations usually cease to be elastic when they are still so small that Hooke's law is a good approximation. Substances such as rubber form an exception.

both directions. These forces act uniformly over the end surfaces of the rod; let the force on unit area be p.

Since the deformation is homogeneous, i.e., u_{ik} is constant through the body, the stress tensor σ_{ik} is also constant, and so it can be determined at once from the boundary conditions (2.8). There is no external force on the sides of the rod, and therefore $\sigma_{ik} n_k = 0$. Since the unit vector $\boldsymbol{n}$ on the side of the rod is perpendicular to the z-axis, i.e., $n_z = 0$, it follows that all the components σ_{ik} except σ_{zz} are zero. On the end surface we have $\sigma_{zi} n_i = p$, or $\sigma_{zz} = p$.

From the general expression (4.8) which relates the components of the strain and stress tensors, we see that all the components u_{ik} with $i \neq k$ are zero. For the remaining components we find

$$u_{xx} = u_{yy} = -\frac{1}{3}\left(\frac{1}{2\mu} - \frac{1}{3K}\right)p, \quad u_{zz} = \frac{1}{3}\left(\frac{1}{3K} + \frac{1}{\mu}\right)p. \tag{5.1}$$

The component u_{zz} gives the relative lengthening of the rod. The coefficient of p is called the ***coefficient of extension***, and its reciprocal is the ***modulus of extension*** or ***Young's modulus***, E:

$$u_{zz} = \frac{p}{E}, \tag{5.2}$$

where

$$E = \frac{9K\mu}{3K + \mu}. \tag{5.3}$$

The components u_{xx} and u_{yy} give the relative compression of the rod in the transverse direction. The ratio of the transverse compression to the longitudinal extension is called Poisson's ratio, σ:[12]

$$u_{xx} = -\sigma u_{zz}, \tag{5.4}$$

where

$$\sigma = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}. \tag{5.5}$$

Since K and μ are always positive, Poisson's ratio can vary between -1 (for $K = 0$) and 1/2 (for $\mu = 0$). Thus[13]

$$-1 \leq \sigma \leq \frac{1}{2}. \tag{5.6}$$

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[12] The use of σ to denote Poisson's ratio and σ_{ik} to denote the components of the stress tensor cannot lead to ambiguity, since the latter always have suffixes.

[13] In practice, Poisson's ratio varies only between 0 and 1/2. There are no substances known for which $\sigma < 0$, i.e., which would expand transversely when stretched longitudinally. It may be mentioned that the inequality $\sigma > 0$ corresponds to $\lambda > 0$, where λ is the Lamé coefficient appearing in (4.1); in other words, both terms in (4.1), as well as in (4.3), are always positive in practice, although this is not thermodynamically necessary. Values of σ close to 1/2 (e.g., for rubber) correspond to a modulus of rigidity which is small compared with the modulus of

Finally, the relative increase in the volume of the rod is

$$u_{ii} = \frac{p}{3K}. \tag{5.7}$$

The free energy of a stretched rod can be obtained immediately from formula (4.10). Since only the component σ_{zz} is not zero, we have $F = \frac{1}{2}\sigma_{zz}u_{zz}$, whence

$$F = \frac{p^2}{2E}. \tag{5.8}$$

In what follows we shall, as is customary, use E and σ instead of K and μ. Inverting formulae (5.3) and (5.5), we have[14]

$$\mu = \frac{E}{2(1+\sigma)}, \quad K = \frac{E}{3(1-2\sigma)}. \tag{5.9}$$

We shall write out here the general formulae of §4, with the coefficients expressed in terms of E and σ. The free energy is

$$F = \frac{E}{2(1+\sigma)}\left(u_{ik}{}^2 + \frac{\sigma}{1-2\sigma}u_{ll}{}^2\right). \tag{5.10}$$

The stress tensor is given in terms of the strain tensor by

$$\sigma_{ik} = \frac{E}{1+\sigma}\left(u_{ik} + \frac{\sigma}{1-2\sigma}u_{ll}\delta_{ik}\right). \tag{5.11}$$

Conversely,

$$u_{ik} = \frac{(1+\sigma)\sigma_{ik} - \sigma\sigma_{ll}\delta_{ik}}{E}. \tag{5.12}$$

Since formulae (5.11) and (5.12) are in frequent use, we shall give them also in component form:

$$\left.\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\sigma)(1-2\sigma)}\left[(1-\sigma)u_{xx} + \sigma(u_{yy}+u_{zz})\right]\\
\sigma_{yy} &= \frac{E}{(1+\sigma)(1-2\sigma)}\left[(1-\sigma)u_{yy} + \sigma(u_{xx}+u_{zz})\right]\\
\sigma_{zz} &= \frac{E}{(1+\sigma)(1-2\sigma)}\left[(1-\sigma)u_{zz} + \sigma(u_{xx}+u_{yy})\right]\\
\sigma_{xy} &= \frac{E}{1+\sigma}u_{xy}\\
\sigma_{xz} &= \frac{E}{1+\sigma}u_{xz}\\
\sigma_{yz} &= \frac{E}{1+\sigma}u_{yz}
\end{aligned}\right\}. \tag{5.13}$$

and conversely

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compression.

[14] The second Lamé coefficient is $\lambda = \frac{E\sigma}{(1-2\sigma)(1+\sigma)}$.

$$\left.\begin{aligned} u_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \sigma(\sigma_{yy} + \sigma_{zz})\right] \\ u_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \sigma(\sigma_{xx} + \sigma_{zz})\right] \\ u_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \sigma(\sigma_{xx} + \sigma_{yy})\right] \\ u_{xy} &= \frac{1+\sigma}{E}\sigma_{xy} \\ u_{xz} &= \frac{1+\sigma}{E}\sigma_{xz} \\ u_{yz} &= \frac{1+\sigma}{E}\sigma_{yz} \end{aligned}\right\} . \tag{5.14}$$

Let us now consider the compression of a rod whose sides are fixed in such a way that they cannot move. The external forces which cause the compression of the rod are applied to its ends and act along its length, which we again take to be along the z-axis. Such a deformation is called a ***unilateral compression***. Since the rod is deformed only in the z-direction, only the component u_{zz} of u_{ik} is not zero. Then we have from (5.11)

$$\sigma_{xx} = \sigma_{yy} = \frac{E}{(1+\sigma)(1-2\sigma)}u_{zz},$$

$$\sigma_{zz} = \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}u_{zz}.$$

Again denoting the compressing force by p ($\sigma_{zz} = p$, which is negative for a compression), we have

$$u_{zz} = \frac{p(1L\sigma)(1-2\sigma)}{E(1-\sigma)}. \tag{5.15}$$

The coefficient of p is called the ***coefficient of unilateral compression***. For the transverse stresses we have

$$\sigma_{xx} = \sigma_{yy} = \frac{p\sigma}{1-\sigma}. \tag{5.16}$$

Finally, the free energy of the rod is

$$F = \frac{p^2(1+\sigma)(1-2\sigma)}{2E(1-\sigma)}. \tag{5.17}$$

§6. Deformations with change of temperature

Let us now consider deformations which are accompanied by a change in the temperature of the body; this can occur either as a result of the deformation process itself, or from external causes.

We shall regard as the undeformed state the state of the body in the absence of external forces at some given temperature T_0. If the body is at a temperature T different from T_0, then, even if there are no external forces, it

will in general be deformed, on account of thermal expansion. In the expansion of the free energy $F(T)$, there will therefore be terms linear, as well as quadratic, in the strain tensor. From the components of the tensor u_{ik}, of rank two, we can form only one linear scalar quantity, the sum u_{ii} of its diagonal components. We shall also assume that the temperature change $T-T_0$ which accompanies the deformation is small. We can then suppose that the coefficient of u_{ii} in the expansion of F (which must vanish for $T=T_0$) is simply proportional to the difference $T-T_0$. Thus we find the free energy to be (instead of (4.3))

$$F(T)=F_0(T)-K\alpha(T-T_0)u_{ll}+\mu(u_{ik}-\frac{1}{3}\delta_{ik}u_{ll})^2+\frac{1}{2}Ku_{ll}{}^2, \tag{6.1}$$

where the coefficient of $T-T_0$ has been written as $-K\alpha$. The quantities μ, K and α can here be supposed constant; an allowance for their temperature dependence would lead to terms of higher order.

Differentiating F with respect to u_{ik}, we obtain the stress tensor:

$$\sigma_{ik}=-K\alpha(T-T_0)\delta_{ik}+Ku_{ll}\delta_{ik}+2\mu(u_{ik}-\frac{1}{3}\delta_{ik}u_{ll}). \tag{6.2}$$

The first term gives the additional stresses caused by the change in temperature. In free thermal expansion of the body (external forces being absent), there can be no internal stresses. Equating σ_{ik} to zero, we find that u_{ik} is of the form $\text{constant}\times\delta_{ik}$, and

$$u_{ll}=\alpha(T-T_0). \tag{6.3}$$

But u_{ll} is the relative change in volume caused by the deformation. Thus α is just the ***thermal expansion coefficient*** of the body.

Among the various (thermodynamic) types of deformation, isothermal and adiabatic deformations are of importance. In isothermal deformations, the temperature of the body does not change. Accordingly, we must put $T=T_0$ in (6.1), returning to the usual formulae; the coefficients K and μ, may therefore be called ***isothermal moduli***.

A deformation is adiabatic if there is no exchange of heat between the various parts of the body (or, of course, between the body and the surrounding medium). The entropy S remains constant. It is the derivative $-\frac{\partial F}{\partial T}$ of the free energy with respect to temperature. Differentiating the expression (6.1), we have as far as terms of the first order in u_{ik}

$$S(T)=S_0(T)+K\alpha u_{ll}. \tag{6.4}$$

Putting S constant, we can determine the change of temperature $T - T_0$ due to the deformation, which is therefore proportional to u_{ll}. Substituting this expression for $T - T_0$ in (6.2), we obtain for σ_{ik} an expression of the usual kind,

$$\sigma_{ik} = K_{ad} u_{ll} \delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik} u_{ll}), \qquad (6.5)$$

with the same modulus of rigidity μ but a different modulus of compression K_{ad}. The relation between the adiabatic modulus K_{ad} and the ordinary isothermal modulus K can also be found directly from the thermodynamic formula

$$\left(\frac{\partial V}{\partial p}\right)_S = \left(\frac{\partial V}{\partial p}\right)_T + \frac{T(\partial V/\partial T)p^2}{C_p},$$

where C_p is the specific heat per unit volume at constant pressure. If V is taken to be the volume occupied by matter which before the deformation occupied unit volume, the derivatives $\frac{\partial V}{\partial T}$ and $\frac{\partial V}{\partial p}$ give the relative volume changes in heating and compression, respectively. That is,

$$\left.\begin{aligned} \left(\frac{\partial V}{\partial T}\right)_p &= \alpha \\ \left(\frac{\partial V}{\partial p}\right)_S &= -\frac{1}{K_{ad}} \\ \left(\frac{\partial V}{\partial p}\right)_T &= -\frac{1}{K} \end{aligned}\right\}.$$

Thus we find the relation between the adiabatic and isothermal moduli to be

$$\left.\begin{aligned} \frac{1}{K_{ad}} &= \frac{1}{K} - \frac{T\alpha^2}{C_p} \\ \mu_{ad} &= \mu \end{aligned}\right\}. \qquad (6.6)$$

For the adiabatic Young's modulus and Poisson's ratio we easily obtain

$$\left.\begin{aligned} E_{ad} &= \frac{E}{1 - ET\alpha^2/9C_p} \\ \sigma_{ad} &= \frac{\sigma + ET\alpha^2/9C_p}{1 - ET\alpha^2/9C_p} \end{aligned}\right\}. \qquad (6.7)$$

In practice, $ET\alpha^2/C_p$ is usually small, and it is therefore sufficiently accurate to put

$$\left.\begin{aligned} E_{ad} &= E + \frac{E^2 T\alpha^2}{9C_p} \\ \sigma_{ad} &= \sigma + \frac{(1+\sigma)ET\alpha^2}{9C_p} \end{aligned}\right\}. \qquad (6.8)$$

In isothermal deformation, the stress tensor is given in terms of the derivatives of the free energy:

$$\sigma_{ik} = \left(\frac{\partial F}{\partial u_{ik}}\right)_T .$$

For constant entropy, on the other hand, we have (see (3.6))

$$\sigma_{ik} = \left(\frac{\partial E}{\partial u_{ik}}\right)_S ,$$

where E is the internal energy. Accordingly, the expression analogous to (4.3) determines, for adiabatic deformations, not the free energy but the internal energy per unit volume

$$E = \frac{1}{2} K_{ad} u_{ll}{}^2 + \mu(u_{ik} - \frac{1}{3} u_{ll}\delta_{ik})^2 . \qquad (6.9)$$

§7. The equations of equilibrium for isotropic bodies

Let us now derive the equations of equilibrium for isotropic solid bodies. To do so, we substitute in the general equations (2.7)

$$\frac{\partial \sigma_{ik}}{\partial x_k} + \rho g_i = 0$$

the expression (5.11) for the stress tensor. We have

$$\frac{\partial \sigma_{ik}}{\partial x_k} = \frac{E\sigma}{(1+\sigma)(1-2\sigma)} \frac{\partial u_{ll}}{\partial x_i} + \frac{E}{1+\sigma} \frac{\partial u_{ik}}{\partial x_k} .$$

Substituting

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right),$$

we obtain the equations of equilibrium in the form

$$\frac{E}{2(1+\sigma)} \frac{\partial^2 u_i}{\partial x_k{}^2} + \frac{E}{2(1+\sigma)(1-2\sigma)} \frac{\partial^2 u_l}{\partial x_i \partial x_l} + \rho g_i = 0 . \qquad (7.1)$$

These equations can be conveniently rewritten in vector notation. The quantities $\frac{\partial^2 u_i}{\partial x_k{}^2}$ are components of the vector $\varDelta u$, and $\frac{\partial u_l}{\partial x_l} \equiv \mathrm{div}\boldsymbol{u}$. Thus the equations of equilibrium become

$$\varDelta \boldsymbol{u} + \frac{1}{1-2\sigma} \boldsymbol{grad}\mathrm{div}\boldsymbol{u} = -\rho \boldsymbol{g} \frac{2(1+\sigma)}{E} . \qquad (7.2)$$

It is sometimes useful to transform this equation by using the vector identity

$\boldsymbol{grad}\,\mathrm{div}\boldsymbol{u} = \Delta\boldsymbol{u} + \boldsymbol{curl}\,\boldsymbol{curl}\boldsymbol{u}$. Then (7.2) becomes

$$\boldsymbol{grad}\,\mathrm{div}\boldsymbol{u} - \frac{1-2\sigma}{2(1-\sigma)}\boldsymbol{curl}\,\boldsymbol{curl}\boldsymbol{u} = -\rho\boldsymbol{g}\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}. \tag{7.3}$$

We have written the equations of equilibrium for a uniform gravitational field, since this is the body force most usually encountered in the theory of elasticity. If there are other body forces, the vector $\rho\boldsymbol{g}$ on the right-hand side of the equation must be replaced accordingly.

A very important case is that where the deformation of the body is caused, not by body forces, but by forces applied to its surface. The equation of equilibrium then becomes

$$(1-2\sigma)\Delta\boldsymbol{u} + \boldsymbol{grad}\,\mathrm{div}\boldsymbol{u} = 0 \tag{7.4}$$

or

$$2(1-\sigma)\boldsymbol{grad}\,\mathrm{div}\boldsymbol{u} - (1-2\sigma)\boldsymbol{curl}\,\boldsymbol{curl}\boldsymbol{u} = 0. \tag{7.5}$$

The external forces appear in the solution only through the boundary conditions.

Taking the divergence of equation (7.4) and using the identity

$$\mathrm{div}\boldsymbol{grad} \equiv \Delta,$$

we find

$$\Delta\mathrm{div}\boldsymbol{u} = 0, \tag{7.6}$$

i.e., $\mathrm{div}\boldsymbol{u}$ (which determines the volume change due to the deformation) is a **harmonic function**. Taking the Laplacian of equation (7.4), we then obtain

$$\Delta\Delta\boldsymbol{u} = 0, \tag{7.7}$$

i.e., in equilibrium the displacement vector satisfies the **biharmonic equation**. These results remain valid in a uniform gravitational field (since the right-hand side of equation (7.2) gives zero on differentiation), but not in the general case of external forces which vary through the body.

The fact that the displacement vector satisfies the biharmonic equation does not, of course, mean that the general integral of the equations of equilibrium (in the absence of body forces) is an arbitrary biharmonic vector; it must be remembered that the function $\boldsymbol{u}(x,y,z)$ also satisfies the lower-order differential equation (7.4). It is possible, however, to express the general integral of the equations of equilibrium in terms of the derivatives of an arbitrary biharmonic vector (see Problem 10).

If the body is non-uniformly heated, an additional term appears in the equation of equilibrium. The stress tensor must include the term

$$-K\alpha(T-T_0)\delta_{ik}$$

(see (6.2)), and $\frac{\partial \sigma_{ik}}{\partial x_k}$ accordingly contains a term

$$-K\alpha \frac{\partial T}{\partial x_i} = -\frac{E\alpha}{3(1-2\sigma)} \frac{\partial T}{\partial x_i} .$$

The equation of equilibrium thus takes the form

$$\frac{3(1-\sigma)}{1+\sigma} \boldsymbol{grad}\,\mathrm{div}\,\boldsymbol{u} - \frac{3(1-2\sigma)}{2(1+\sigma)} \boldsymbol{curl}\,\boldsymbol{curl}\,\boldsymbol{u} = \alpha\,\boldsymbol{grad}\,T \ . \tag{7.8}$$

Let us consider the particular case of a ***plane deformation***, in which one component of the displacement vector (u_z) is zero throughout the body, while the components u_x, u_y depend only on *x* and *y*. The components u_{zz}, u_{xz}, u_{yz} of the strain tensor then vanish identically, and therefore so do the components σ_{xz}, σ_{yz} of the stress tensor (but not the longitudinal stress σ_{zz}, the existence of which is implied by the constancy of the length of the body in the *z*-direction).[15]

Since all quantities are independent of the co-ordinate *z*, the equations of equilibrium (in the absence of external body forces) $\frac{\partial \sigma_{ik}}{\partial x_k} = 0$ reduce in this case to two equations:

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \end{aligned}\right\} . \tag{7.9}$$

The most general functions $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ satisfying these equations are of the form

$$\left.\begin{aligned} \sigma_{xx} &= \frac{\partial^2 \chi}{\partial y^2} \\ \sigma_{xy} &= -\frac{\partial^2 \chi}{\partial x \partial y} \\ \sigma_{yy} &= \frac{\partial^2 \chi}{\partial x^2} \end{aligned}\right\} , \tag{7.10}$$

where χ is an arbitrary function of *x* and *y*. It is easy to obtain an equation which must be satisfied by this function. Such an equation must exist, since the three quantities $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ can be expressed in terms of the two quantities u_x, u_y, and are therefore not independent. Using formulae (5.13), we find, for a plane deformation,

[15] The use of the theory of functions of a complex variable provides very powerful methods of solving plane problems in the theory of elasticity. See N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, 2nd English ed., P. Noordhoff, Groningen

$$\sigma_{xx}+\sigma_{yy}=\frac{E(u_{xx}+u_{yy})}{(1+\sigma)(1-2\sigma)}\,.$$

But

$$\sigma_{xx}+\sigma_{yy}=\Delta\chi\,,\;\; u_{xx}+u_{yy}=\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\equiv \mathrm{div}\boldsymbol{u}\,,$$

and, since by (7.6) div$\boldsymbol{u}$ is harmonic, we conclude that the function χ satisfies the equation

$$\Delta\Delta\chi=0\,, \qquad (7.11)$$

i.e., it is **biharmonic**. This function is called the ***stress function***. When the plane problem has been solved and the function χ is known, the longitudinal stress σ_{zz} is determined at once from the formula

$$\sigma_{zz}=\frac{\sigma E(u_{xx}+u_{yy})}{(1+\sigma)(1-2\sigma)}=\sigma(\sigma_{xx}+\sigma_{yy})\,,$$

or

$$\sigma_{zz}=\sigma\Delta\chi\,. \qquad (7.12)$$

PROBLEMS

Problem 1. Determine the deformation of a long rod (of length *l*) standing vertically in a gravitational field.

Solution. We take the *z*-axis along the axis of the rod, and the *xy*-plane in the plane of its lower end. The equations of equilibrium are $\frac{\partial\sigma_{xi}}{\partial x_i}=\frac{\partial\sigma_{yi}}{\partial x_i}=0$, $\frac{\partial\sigma_{zi}}{\partial x_i}=\rho g$. On the sides of the rod all the components σ_{ik} except σ_{zz} must vanish, and on the upper end ($z=l$) $\sigma_{xz}=\sigma_{yz}=\sigma_{zz}=0$. The solution of the equations of equilibrium satisfying these conditions is $\sigma_{zz}=-\rho g(l-z)$, with all other σ_{ik} zero. From σ_{ik} we find u_{ik} to be $u_{xx}=u_{yy}=\frac{\sigma\rho g(l-z)}{E}$, $u_{zz}=-\frac{\rho g(l-z)}{E}$, $u_{xy}=u_{xz}=u_{yz}=0$, and hence by integration we have the components of the displacement vector,

$$\begin{cases} u_x=\dfrac{\sigma\rho g(l-z)x}{E} \\ u_y=\dfrac{\sigma\rho g(l-z)y}{E} \\ u_z=-\dfrac{\rho g}{2E}\left\{l^2-(l-z)^2-\sigma(x^2+y^2)\right\} \end{cases}.$$

The expression for u_z satisfies the boundary condition $u_z=0$ only at one point on the lower end of the rod. Hence the solution obtained is not valid

1963.

near the lower end.

Problem 2. Determine the deformation of a hollow sphere (of external and internal radii R_2 and R_1) with a pressure p_1 inside and p_2 outside.

Solution. We use spherical co-ordinates, with the origin at the centre of the sphere. The displacement vector ***u*** is everywhere radial, and is a function of r alone. Hence **curl *u***=0, and equation (7.5) becomes **grad** div ***u*** = 0. Hence

$$\mathrm{div}\boldsymbol{u} = \frac{1}{r^2}\frac{d(r^2 u)}{dr} = \text{constant} \equiv 3a \ ,$$

or $u = ar + \frac{b}{r^2}$. The components of the strain tensor are (see formulae (1.7))

$u_{rr} = a - \frac{2b}{r^3}$, $u_{\theta\theta} = u_{\phi\phi} = a + \frac{b}{r^3}$. The radial stress is

$$\sigma_{rr} = \frac{E}{(1+\sigma)(1-2\sigma)}\left\{(1-\sigma)u_{rr} + 2\sigma u_{\theta\theta}\right\} = \frac{E}{1-2\sigma}a - \frac{2E}{1+\sigma}\frac{b}{r^3} \ .$$

The constants a and b are determined from the boundary conditions: $\sigma_{rr} = -p_1$, at $r = R_1$, and $\sigma_{rr} = -p_2$, at $r = R_2$. Hence we find

$$\begin{cases} a = \dfrac{p_1 R_1^{\ 3} - p_2 R_2^{\ 3}}{R_2^{\ 3} - R_1^{\ 3}} \cdot \dfrac{1-2\sigma}{E} \\ b = \dfrac{R_1^{\ 3} R_2^{\ 3}(p_1 - p_2)}{R_2^{\ 3} - R_1^{\ 3}} \cdot \dfrac{1+\sigma}{2E} \end{cases} .$$

For example, the stress distribution in a **spherical shell** with a pressure $p_1 = p$ inside and $p_2 = 0$ outside is given by

$$\begin{cases} \sigma_{rr} = \dfrac{pR_1^{\ 3}}{R_2^{\ 3} - R_1^{\ 3}}\left(1 - \dfrac{R_2^{\ 3}}{r^3}\right) \\ \sigma_{\theta\theta} = \sigma_{\phi\phi} = \dfrac{pR_1^{\ 3}}{R_2^{\ 3} - R_1^{\ 3}}\left(1 + \dfrac{R_2^{\ 3}}{r^3}\right) \end{cases} .$$

For a **thin spherical shell** of thickness $h = R_2 - R_1 << R$ we have approximately

$$\begin{cases} u = \dfrac{pR^2(1-\sigma)}{2Eh} \\ \sigma_{\theta\theta} = \sigma_{\phi\phi} = \dfrac{pR}{2h} \\ \bar{\sigma}_{rr} = \dfrac{p}{2} \end{cases} ,$$

where $\bar{\sigma}_{rr}$ is the mean value of the radial stress over the thickness of the shell.

The stress distribution in an infinite elastic medium with a spherical cavity (of radius R) subjected to hydrostatic compression is obtained by putting $R_1 = R$, $R_2 = \infty$, $p_1 = 0$, $p_2 = 0$:

$$\begin{cases} \sigma_{rr} = -p\left(1 - \dfrac{R^3}{r^3}\right) \\ \sigma_{\theta\theta} = \sigma_{\phi\phi} = -p\left(1 + \dfrac{R^3}{2r^3}\right) \end{cases}.$$

At the surface of the cavity the tangential stresses $\sigma_{\theta\theta} = \sigma_{\phi\phi} = -\frac{3p}{2}$, i.e., they exceed the pressure at infinity.

Problem 3. Determine the deformation of a solid sphere (of radius *R*) in its own gravitational field.

Solution. The force of gravity on unit mass in a spherical body is $-g\boldsymbol{r}/R$. Substituting this expression in place of $\boldsymbol{g}$ in equation (7.3), we obtain the following equation for the radial displacement:

$$\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\frac{d}{dr}\left(\frac{1}{r^2}\frac{d(r^2u)}{dr}\right) = \rho g\frac{r}{R}.$$

The solution finite for $r = 0$ which satisfies the condition $\sigma_{rr} = 0$ for $r = R$ is

$$u = -\frac{g\rho R(1-2\sigma)(1+\sigma)}{1-E(1-\sigma)}r\left(\frac{3-\sigma}{1+\sigma} - \frac{r^2}{R^2}\right).$$

It should be noticed that the substance is compressed ($u_{rr} < 0$) inside a spherical surface of radius $R\sqrt{\dfrac{3-\sigma}{3(1+\sigma)}}$ and stretched outside it ($u_{rr} > 0$).

The pressure at the centre of the sphere is $\dfrac{(3-\sigma)g\rho R}{10(1-\sigma)}$.

Problem 4. Determine the deformation of a cylindrical pipe (of external and internal radii R_2 and R_1), with a pressure p inside and no pressure outside.[16]

Solution. We use cylindrical co-ordinates, with the *z*-axis along the axis of the pipe. When the pressure is uniform along the pipe, the deformation is a purely radial displacement $u_r = u(r)$. Similarly to Problem 2, we have

$$\mathrm{div}\boldsymbol{u} = \frac{1}{r}\frac{d(ru)}{dr} = \text{constant} \equiv 2a.$$

Hence $u = ar + \dfrac{b}{r}$. The non-zero components of the strain tensor are (see formulae (1.8)) $u_{rr} = \dfrac{du}{dr} = a - \dfrac{b}{r^2}$, $u_{\phi\phi} = \dfrac{u}{r} = a + \dfrac{b}{r^2}$. From the conditions $\sigma_{rr} = 0$ at $r = R_2$, and $\sigma_{rr} = -p$ at $r = R_1$, we find

[16] In Problems 4, 5 and 7 it is assumed that the length of the cylinder is maintained constant, so that there is no longitudinal deformation.

$$\begin{cases} a = \dfrac{pR_1{}^2}{R_2{}^2 - R_1{}^2} \cdot \dfrac{(1+\sigma)(1-2\sigma)}{E} \\ b = \dfrac{pR_1{}^2 R_2{}^2}{R_2{}^2 - R_1{}^2} \cdot \dfrac{1+\sigma}{E} \end{cases}.$$

The stress distribution is given by the formulae

$$\begin{cases} \sigma_{rr} = \dfrac{pR_1{}^2}{R_2{}^2 - R_1{}^2}\left(1 - \dfrac{R_2{}^2}{r^2}\right) \\ \sigma_{\phi\phi} = \dfrac{pR_1{}^2}{R_2{}^2 - R_1{}^2}\left(1 + \dfrac{R_2{}^2}{r^2}\right). \\ \sigma_{zz} = \dfrac{2p\sigma R_1{}^2}{R_2{}^2 - R_1{}^2} \end{cases}$$

Problem 5. Determine the deformation of a cylinder rotating uniformly about its axis.

Solution. Replacing the gravitational force in (7.3) by the centrifugal force $\rho\Omega^2 r$ (where Ω is the angular velocity), we have in cylindrical co-ordinates the following equation for the displacement $u_r = u(r)$:

$$\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} \frac{d}{dr}\left(\frac{1}{r}\frac{d(ru)}{dr}\right) = -\rho\Omega^2 r .$$

The solution which is finite for*0* $r = 0$ and satisfies the condition $\sigma_{rr} = 0$ for $r = R$ is

$$u = \frac{\rho\Omega^2 (1+\sigma)(1-2\sigma)}{8E(1-\sigma)} r\left[(3-2\sigma)R^2 - r^2\right].$$

Problem 6. Determine the deformation of a non-uniformly heated sphere with a spherically symmetrical temperature distribution.

Solution. In spherical co-ordinates, equation (7.8) for a purely radial deformation is

$$\frac{d}{dr}\left(\frac{1}{r^2}\frac{d(r^2 u)}{dr}\right) = \alpha \frac{1+\sigma}{3(1-\sigma)} \frac{dT}{dr} .$$

The solution which is finite for $r = 0$ and satisfies the condition $\sigma_{rr} = 0$ for $r = R$ is

$$u = \alpha \frac{1+\sigma}{3(1-\sigma)} \left\{ \frac{1}{r^2} \int_0^r T(r) r^2 dr + \frac{2(1-2\sigma)}{1+\sigma} \frac{r}{R^3} \int_0^R T(r) r^2 dr \right\}.$$

The temperature $T(r)$ is measured from the value for which the sphere, if uniformly heated, is regarded as undeformed. In the above formula the temperature in question is taken as that of the outer surface of the sphere, so that $T(R) = 0$.

Problem 7. The same as Problem 6, but for a non-uniformly heated cylinder with an axially symmetrical temperature distribution.

Solution. We similarly have in cylindrical co-ordinates

$$u = \alpha \frac{1+\sigma}{3(1-\sigma)} \left\{ \frac{1}{r} \int_0^r T(r) r dr + (1-2\sigma) \frac{r}{R^2} \int_0^R T(r) r dr \right\}.$$

Problem 8. Determine the deformation of an infinite elastic medium with a given temperature distribution $T(x, y, z)$ which is such that the temperature tends to a constant value T_0 at infinity, there being no deformation there.

Solution. Equation (7.8) has an obvious solution for which **curl** $\boldsymbol{u}$ = and

$$\mathrm{div}\boldsymbol{u} = \alpha(1+\sigma)\frac{T(x,y,z)-T_0}{3(1-\sigma)}.$$

The vector $\boldsymbol{u}$, whose divergence is a given function defined in all space and vanishing at infinity, and whose curl is zero identically, can be written, as we know from vector analysis, in the form

$$\boldsymbol{u}(x,y,z) = -\frac{1}{4\pi} \boldsymbol{grad} \int \frac{\mathrm{div}' \boldsymbol{u}(x',y',z')}{r} dV',$$

where

$$r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}.$$

We therefore obtain the general solution of the problem in the form

$$\boldsymbol{u} = -\frac{\alpha(1+\sigma)}{12\pi(1-\sigma)} \boldsymbol{grad} \int \frac{T' - T_0}{r} dV', \qquad (1)$$

where $T' \equiv T(x', y', z')$.

If a finite quantity of heat q is evolved in a very small volume at the origin, the temperature distribution can be written $T - T_0 = \frac{q}{C}\delta(x)\delta(y)\delta(z)$, where C is the specific heat of the medium. The integral in (1) is then q/Cr, and the deformation is given by

$$\boldsymbol{u} = \frac{\alpha(1+\sigma)q}{12\pi(1-\sigma)C} \cdot \frac{\boldsymbol{r}}{r^3}.$$

Problem 9. Derive the equations of equilibrium for an isotropic body (in the absence of body forces) in terms of the components of the stress tensor.

Solution. The required system of equations contains the three equations

$$\frac{\partial \sigma_{ik}}{\partial x_k} = 0 \qquad (1)$$

and also the equations resulting from the fact that the six different components of u_{ik} are not independent quantities. To derive these equations, we first write down the system of differential relations satisfied by the components of the tensor u_{ik}. It is easy to see that the quantities

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

satisfy identically the relations

$$\frac{\partial^2 u_{ik}}{\partial x_l \partial x_m} + \frac{\partial^2 u_{lm}}{\partial x_i \partial x_k} = \frac{\partial^2 u_{il}}{\partial x_k \partial x_m} + \frac{\partial^2 u_{km}}{\partial x_i \partial x_l}.$$

Here there are only six essentially different relations, namely those corresponding to the following values of *i*, *k*, *l*, *m*: 1122, 1133, 2233, 1123, 2213, 3312. All these are retained if the above tensor equation is contracted with respect to *l* and *m*:

$$\Delta u_{ik} + \frac{\partial^2 u_{ll}}{\partial x_i \partial x_k} = \frac{\partial^2 u_{il}}{\partial x_k \partial x_l} + \frac{\partial^2 u_{kl}}{\partial x_i \partial x_l}. \qquad (2)$$

Substituting here u_{ik} in terms of σ_{ik} : according to (5.12) and using (1), we obtain the required equations:

$$(1+\sigma)\Delta\sigma_{ik} + \frac{\partial^2 u_{ll}}{\partial x_i \partial x_k} = 0. \qquad (3)$$

These equations remain valid in the presence of external forces constant throughout the body.

Contracting equation (3) with respect to the suffixes *i* and *k*, we find that $\Delta\sigma_{ll} = 0$, i.e., σ_{ll} is a **harmonic function**. Taking the Laplacian of equation (3), we then find that $\Delta\Delta\sigma_{ik} = 0$, i.e., the components σ_{ik} are **biharmonic functions**. These results follow also from (7.6) and (7.7), since σ_{ik} and u_{ik} are linearly related.

Problem 10. Express the general integral of the equations of equilibrium (in the absence of body forces) in terms of an arbitrary biharmonic vector (B. G. Galerkin 1930).

Solution. It is natural to seek a solution of equation (7.4) in the form

$$\boldsymbol{u} = \Delta\boldsymbol{f} + A\,\boldsymbol{grad}\,\mathrm{div}\boldsymbol{f}.$$

Hence $\mathrm{div}\boldsymbol{u} = (1+A)\mathrm{div}\Delta\boldsymbol{f}$. Substituting in (7.4), we obtain

$$(1-2\sigma)\Delta\Delta\boldsymbol{f} + [2(1-\sigma)A+1]\boldsymbol{grad}\,\mathrm{div}\Delta\boldsymbol{f} = 0.$$

From this we see that, if $\boldsymbol{f}$ is an arbitrary biharmonic vector ($\Delta\Delta\boldsymbol{f} = 0$), then

$$\boldsymbol{u} = \Delta\boldsymbol{f} - \frac{1}{2(\sigma-1)}\boldsymbol{grad}\,\mathrm{div}\boldsymbol{f}.$$

Problem 11. Express the stresses $\sigma_{rr}, \sigma_{\phi\phi}, \sigma_{r\phi}$ for a plane deformation (in polar co-ordinates *r*, ϕ) as derivatives of the stress function.

Solution. Since the required expressions cannot depend on the choice of the initial line of ϕ, they do not contain ϕ explicitly. Hence we can proceed as follows : we transform the Cartesian derivatives (7.10) into derivatives with respect to *r*, ϕ, and use the results that $\sigma_{rr} = (\sigma_{xx})_{\phi=0}$, $\sigma_{\phi\phi} = (\sigma_{yy})_{\phi=0}$,

$\sigma_{r\phi} = (\sigma_{xy})_{\phi=0}$, the angle ϕ being measured from the x-axis. Thus

$$\begin{cases} \sigma_{rr} = \dfrac{1}{r}\dfrac{\partial \chi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \chi}{\partial \phi^2} \\ \sigma_{\phi\phi} = \dfrac{\partial^2 \chi}{\partial r^2} \\ \sigma_{r\phi} = -\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial \chi}{\partial \phi}\right) \end{cases} .$$

Problem 12. Determine the stress distribution in an infinite elastic medium containing a spherical cavity and subjected to a homogeneous deformation at infinity.

Solution. A general homogeneous deformation can be represented as a combination of a homogeneous hydrostatic extension (or compression) and a homogeneous shear. The former has been considered in Problem 2, so that we need only consider a homogeneous shear.

Let $\sigma_{ik}{}^{(0)}$ be the homogeneous stress field which would be found in all space if the cavity were absent: in a pure shear $\sigma_{ii}{}^{(0)} = 0$. The corresponding displacement vector is denoted by $\boldsymbol{u}^{(0)}$, and we seek the required solution in the form $\boldsymbol{u} = \boldsymbol{u}^{(0)} + \boldsymbol{u}^{(1)}$, where the function $\boldsymbol{u}^{(1)}$ arising from the presence of the cavity is zero at infinity.

Any solution of the biharmonic equation can be written as a linear combination of centrally symmetrical solutions and their spatial derivatives of various orders. The functions r^2, r, 1, $1/r$ are independent centrally symmetrical solutions. Hence the most general form of a biharmonic vector $\boldsymbol{u}^{(1)}$, depending only on the components of the constant tensor $\sigma_{ik}{}^{(0)}$ as parameters and vanishing at infinity, is

$$u_i{}^{(1)} = A\sigma_{ik}{}^{(0)}\frac{\partial}{\partial x_k}\left(\frac{1}{r}\right) + B\sigma_{ik}{}^{(0)}\frac{\partial^3}{\partial x_i \partial x_k \partial x_l}\left(\frac{1}{r}\right) + C\sigma_{ik}{}^{(0)}\frac{\partial^3}{\partial x_i \partial x_k \partial x_l} r . \tag{1}$$

Substituting this expression in equation (7.4), we obtain

$$(1-2\sigma)\frac{\partial^2 u_i}{\partial x_l{}^2} + \frac{\partial}{\partial x_i}\frac{\partial u_l}{\partial x_l} = \left[2(1-2\sigma)C + (A+2C)\right]\sigma_{kl}{}^{(0)}\frac{\partial^3}{\partial x_i \partial x_k \partial x_l}\frac{1}{r} = 0 ,$$

whence $A = -4C(1-\sigma)$. Two further relations between the constants A, B, C are obtained from the condition at the surface of the cavity: $(\sigma_{ik}{}^{(0)} + \sigma_{ik}{}^{(1)})n_k = 0$ for $r = R$ (R being the radius of the cavity, the origin at its centre, and $\boldsymbol{n}$ a unit vector parallel to $\boldsymbol{r}$). A somewhat lengthy calculation, using (1), gives the following values:

$$\begin{cases} B = \dfrac{CR^2}{5} \\ C = \dfrac{5R^2(1+\sigma)}{2E(7-5\sigma)} \end{cases}.$$

The final expression for the stress distribution is

$$\begin{aligned}\sigma_{ik} &= \sigma_{ik}^{(0)}\left\{1 + \frac{5(1-2\sigma)}{7-5\sigma}\left(\frac{R}{r}\right)^3 + \frac{3}{7-5\sigma}\left(\frac{R}{r}\right)^5\right\} \\ &+ \frac{15}{7-5\sigma}\left(\frac{R}{r}\right)^3\left\{\sigma - \left(\frac{R}{r}\right)^2\right\}(\sigma_{il}^{(0)}n_k n_l + \sigma_{kl}^{(0)}n_l n_i) \\ &+ \frac{15}{2(7-5\sigma)}\left(\frac{R}{r}\right)^3\left\{-5 + 7\left(\frac{R}{r}\right)^2\right\}\sigma_{lm}^{(0)}n_l n_m n_i n_k \\ &+ \frac{15}{2(7-5\sigma)}\left(\frac{R}{r}\right)^3\left\{1 - 2\sigma - \left(\frac{R}{r}\right)^2\right\}\delta_{ik}\sigma_{lm}^{(0)}n_l n_m\end{aligned}.$$

In order to obtain the stress distribution for arbitrary $\sigma_{ik}^{(0)}$ (not a pure shear), $\sigma_{ik}^{(0)}$ in this expression must be replaced by $\sigma_{ik}^{(0)} - \frac{1}{3}\delta_{ik}\sigma_{ll}^{(0)}$, and the expression

$$\frac{1}{3}\sigma_{ll}^{(0)}\left[\delta_{ik} + \frac{R^3}{2r^3}(\delta_{ik} - 3n_i n_k)\right]$$

corresponding to a deformation homogeneous at infinity (cf. Problem 2) must be added. We may give here the general formula for the stresses at the surface of the cavity:

$$\begin{aligned}\sigma_{ik} = \frac{15}{7-5\sigma}[(1-\sigma)(\sigma_{ik}^{(0)} - \sigma_{il}^{(0)}n_l n_k - \sigma_{kl}^{(0)}n_l n_i) \\ + \sigma_{lm}^{(0)}n_l n_m n_i n_k - \sigma\sigma_{lm}^{(0)}n_l n_m \delta_{ik} + \frac{5\sigma - 1}{10}\sigma_{ll}^{(0)}(\delta_{ik} - n_i n_k)]\end{aligned}$$

Near the cavity, the stresses considerably exceed the stresses at infinity, but this extends over only a short distance (the ***concentration of stresses***). For example, if the medium is subjected to a homogeneous extension (only $\sigma_{zz}^{(0)}$ different from zero), the greatest stress occurs on the equator of the cavity, where

$$\sigma_{zz} = \frac{27 - 15\sigma}{2(7-5\sigma)}\sigma_{zz}^{(0)}.$$

§8. Equilibrium of an elastic medium bounded by a plane

Let us consider an elastic medium occupying a half-space, i.e., bounded on one side by an infinite plane, and determine the deformation of the

medium caused by forces applied to its free surface.[17] The distribution of these forces need satisfy only one condition: they must vanish at infinity in such a way that there is no deformation at infinity. In such a case the equations of equilibrium can be integrated in a general form.

The equation of equilibrium (7.4) holds throughout the space occupied by the medium:

$$\boldsymbol{grad}\,\mathrm{div}\boldsymbol{u} + (1-2\sigma)\Delta\boldsymbol{u} = 0 . \qquad (8.1)$$

We seek a solution of this equation in the form

$$\boldsymbol{u} = \boldsymbol{f} + \boldsymbol{grad}\phi , \qquad (8.2)$$

where ϕ is some scalar and the vector $\boldsymbol{f}$ satisfies Laplace's equation:

$$\Delta\boldsymbol{f} = 0 . \qquad (8.3)$$

Substituting (8.2) in (8.1), we then obtain the following equation for ϕ:

$$2(1-\sigma)\Delta\phi = -\mathrm{div}\boldsymbol{f} . \qquad (8.4)$$

We take the free surface of the elastic medium as the *xy*-plane; the medium is in $z > 0$. We write the functions f_x and f_y as the *z*-derivatives of some functions g_x and g_y:

$$f_x = \frac{\partial g_x}{\partial z}, \quad f_y = \frac{\partial g_y}{\partial z} . \qquad (8.5)$$

Since f_x and f_y are harmonic functions, we can always choose the functions g_x and g_y so as to satisfy Laplace's equation

$$\Delta g_x = 0 , \quad \Delta g_y = 0 . \qquad (8.6)$$

Equation (8.4) then becomes

$$2(1-\sigma)\Delta\phi = -\frac{\partial}{\partial z}\left(\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + f_z\right).$$

Since g_x, g_y and f_z are harmonic functions, we easily see that a function ϕ which satisfies this equation can be written as

$$\phi = -\frac{z}{4(1-\sigma)}\left(f_z + \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y}\right) + \psi , \qquad (8.7)$$

where ψ is again a harmonic function:

$$\Delta\psi = 0 . \qquad (8.8)$$

Thus the problem of determining the displacement $\boldsymbol{u}$ reduces to that of finding the functions g_x, g_y, f_z, ψ, all of which satisfy Laplace's equation.

We shall now write out the boundary conditions which must be satisfied at the free surface of the medium (the plane $z = 0$). Since the unit outward normal vector $\boldsymbol{n}$ is in the negative *z*-direction, it follows from the general

[17] The most direct and regular method of solving this problem is to use Fourier's method on equation (8.1). In that case, however, some fairly complicated integrals have to be calculated. The method given below is based on a number of artificial devices, but the calculations are simpler.

formula (2.8) that $\sigma_{iz} = -P_i$. Using for σ_{ik} the general expression (5.11) and expressing the components of the vector $\boldsymbol{u}$ in terms of the auxiliary quantities g_x, g_y, f_z, and ψ, we obtain, after a simple calculation, the boundary conditions

$$\left[\frac{\partial^2 g_x}{\partial z^2}\right]_{z=0} + \left[\frac{\partial}{\partial x}\left\{\frac{1-2\sigma}{2(1-\sigma)} f_z - \frac{1}{2(1-\sigma)}\left(\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y}\right) + 2\frac{\partial \psi}{\partial z}\right\}\right]_{z=0} = -\frac{2(1+\sigma)P_x}{E}$$

$$\left[\frac{\partial^2 g_y}{\partial z^2}\right]_{z=0} + \left[\frac{\partial}{\partial y}\left\{\frac{1-2\sigma}{2(1-\sigma)} f_z - \frac{1}{2(1-\sigma)}\left(\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y}\right) + 2\frac{\partial \psi}{\partial z}\right\}\right]_{z=0} = -\frac{2(1+\sigma)P_y}{E}$$

(8.9)

$$\left[\frac{\partial}{\partial z}\left\{f_z - \left(\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y}\right) + 2\frac{\partial \psi}{\partial z}\right\}\right]_{z=0} = -\frac{2(1+\sigma)P_z}{E}$$

(8.10)

The components P_x, P_y, P_z of the external forces applied to the surface are given functions of the co-ordinates *x* and *y*, and vanish at infinity.

The formulae by which the auxiliary quantities g_x, g_y, f_z, and ψ were defined do not determine them uniquely. We can therefore impose an arbitrary additional condition on these quantities, and it is convenient to make the quantity in the braces in equations (8.9) vanish:[18]

$$(1-2\sigma) f_z - \left(\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y}\right) + 4(1-\sigma)\frac{\partial \psi}{\partial z} = 0. \qquad (8.11)$$

Then the conditions (8.9) become simply

$$\left.\begin{aligned} \left[\frac{\partial^2 g_x}{\partial z^2}\right]_{z=0} &= -\frac{2(1+\sigma)}{E} P_x \\ \left[\frac{\partial^2 g_y}{\partial z^2}\right]_{z=0} &= -\frac{2(1+\sigma)}{E} P_y \end{aligned}\right\}. \qquad (8.12)$$

Equations (8.10)-(8.12) suffice to determine completely the harmonic functions g_x, g_y, f_z, and ψ.

For simplicity, we shall consider the case where the free surface of an elastic half-space is subjected to a concentrated force $\boldsymbol{F}$, i.e., one which is applied to an area so small that it can be regarded as a point. The effect of this force is the same as that of surface forces given by $\boldsymbol{P} = \boldsymbol{F}\delta(x)\delta(y)$, the origin being at the point of application of the force. If we know the solution for a concentrated force, we can immediately find the solution for any force distribution $\boldsymbol{P}(x, y)$. For, if

[18] We shall not prove here that this condition can in fact be imposed; this follows from the

$$u_i = G_{ik}(x, y, z)F_k \tag{8.13}$$

is the displacement due to the action of a concentrated force $\boldsymbol{F}$ applied at the origin, then the displacement caused by forces $\boldsymbol{P}\{x, y)$ is given by the integral[19]

$$u_i = \iint G_{ik}(x - x', y - y', z)P_k(x', y')dx'dy'. \tag{8.14}$$

We know from potential theory that a harmonic function f which is zero at infinity and has a given normal derivative $\frac{\partial f}{\partial z}$ on the plane $z = 0$ is given by the formula

$$f(x, y, z) = -\frac{1}{2\pi}\iint \left[\frac{\partial f(x', y', z)}{\partial z}\right]_{z=0} \frac{dx'dy'}{r},$$

where

$$r = \sqrt{(x - x')^2 + (y - y')^2 + z^2}.$$

Since the quantities $\frac{\partial g_x}{\partial z}$, $\frac{\partial g_y}{\partial z}$ and that in the braces in equation (8.10) satisfy Laplace's equation, while equations (8.10) and (8.12) determine the values of their normal derivatives on the plane $z = 0$, we have

$$\begin{aligned} f_z - \left(\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y}\right) + 2\frac{\partial \psi}{\partial z} &= \frac{1+\sigma}{\pi E}\iint \frac{P_z(x', y')}{r}dx'dy' \\ &= \frac{1+\sigma}{\pi E}\cdot\frac{F_z}{r} \end{aligned} \tag{8.15}$$

$$\frac{\partial g_x}{\partial z} = \frac{1+\sigma}{\pi E}\cdot\frac{F_x}{r}, \quad \frac{\partial g_y}{\partial z} = \frac{1+\sigma}{\pi E}\cdot\frac{F_y}{r}, \tag{8.16}$$

where now $r = \sqrt{x^2 + y^2 + z^2}$.

The expressions for the components of the required vector $\boldsymbol{u}$ involve the derivatives of g_x, g_y with respect to x, y, z, but not g_x, g_y themselves. To calculate $\frac{\partial g_x}{\partial z}$, $\frac{\partial g_y}{\partial z}$, we differentiate equations (8.16) with respect to x and y, respectively:

$$\frac{\partial^2 g_x}{\partial x \partial z} = -\frac{1+\sigma}{\pi E}\cdot\frac{F_x x}{r^3}, \quad \frac{\partial^2 g_y}{\partial y \partial z} = -\frac{1+\sigma}{\pi E}\cdot\frac{F_y y}{r^3}.$$

Now, integrating over z from ∞ to z, we obtain

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absence of contradiction in the result.

[19] In mathematical terms, G_{ik} is the ***Green's tensor*** for the equations of equilibrium of a semi-infinite medium.

$$\left.\begin{aligned}\frac{\partial g_x}{\partial x}=\frac{1+\sigma}{\pi E}\cdot\frac{F_x x}{r(r+z)}\\ \frac{\partial g_y}{\partial y}=\frac{1+\sigma}{\pi E}\cdot\frac{F_y y}{r(r+z)}\end{aligned}\right\}. \tag{8.17}$$

We shall not pause to complete the remaining calculations, which are elementary but laborious. We determine f_z and $\frac{\partial \psi}{\partial z}$ from equations (8.11), (8.15) and (8.17). Knowing $\frac{\partial \psi}{\partial z}$, it is easy to calculate $\frac{\partial \psi}{\partial x}$ and $\frac{\partial \psi}{\partial y}$ by integrating with respect to z and then differentiating with respect to x and y. We thus obtain all the quantities needed to calculate the displacement vector from (8.2), (8.5) and (8.7). The following are the final formulae:

$$u_x=\frac{1+\sigma}{2\pi E}\left\{\left[\frac{xz}{r^3}-\frac{(1-2\sigma)x}{r(r+z)}\right]F_z+\frac{2(1-\sigma)r+z}{r(r+z)}F_z+\frac{[2r(\sigma r+z)+z^2]x}{r^3(r+z)^2}(xF_x+yF_y)\right\}$$

$$u_y=\frac{1+\sigma}{2\pi E}\left\{\left[\frac{yz}{r^3}-\frac{(1-2\sigma)y}{r(r+z)}\right]F_z+\frac{2(1-\sigma)r+z}{r(r+z)}F_y+\frac{[2r(\sigma r+z)+z^2]x}{r^3(r+z)^2}(xF_x+yF_y)\right\}$$

$$u_z=\frac{1+\sigma}{2\pi E}\left\{\left[\frac{2(1-\sigma)}{r}+\frac{z^2}{r^3}\right]F_z+\left[\frac{1-2\sigma}{r(r+z)}+\frac{z}{r^3}\right](xF_x+yF_y)\right\}$$

(8.18)

In particular, the displacement of points on the surface of the medium is given by putting $z = 0$:

$$u_x=\frac{1+\sigma}{2\pi E}\cdot\frac{1}{r}\left\{-\frac{(1-2\sigma)x}{r}F_z+2(1-\sigma)F_z+\frac{2\sigma x}{r^2}(xF_x+yF_y)\right\},$$

$$u_y=\frac{1+\sigma}{2\pi E}\cdot\frac{1}{r}\left\{-\frac{(1-2\sigma)y}{r}F_z+2(1-\sigma)F_y+\frac{2\sigma y}{r^2}(xF_x+yF_y)\right\},$$

$$u_z=\frac{1+\sigma}{2\pi E}\cdot\frac{1}{r}\left\{2(1-\sigma)F_z+(1-2\sigma)\frac{1}{r}(xF_x+yF_y)\right\}.$$

(8.19)

PROBLEM

Determine the deformation of an infinite elastic medium when a force ***F*** is applied to a small region in it.[20]

Solution. If we consider the deformation at distances ***r*** which are large compared with the dimension of the region where the force is applied, we can suppose that the force is applied at a point. The equation of equilibrium is (cf.

[20] The corresponding problem for an arbitrary infinite anisotropic medium has been solved by I. M. Lifshitz and L. N. Rozentsveig (*Zhurnal experimental'noi i teoreticheskoi fiziki* 17, 783, 1947).

(7.2))

$$\Delta \boldsymbol{u} + \frac{1}{1-2\sigma}\boldsymbol{grad}\,\mathrm{div}\boldsymbol{u} = -\frac{2(1+\sigma)}{E}\boldsymbol{F}\delta(\boldsymbol{r}), \qquad (1)$$

where $\delta(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z)$, the origin being at the point where the force is applied. We seek the solution in the form $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1$, where $\boldsymbol{u}_0$ satisfies the Poisson-type equation

$$\Delta \boldsymbol{u}_0 = -\frac{2(1+\sigma)}{E}\boldsymbol{F}\delta(\boldsymbol{r}). \qquad (2)$$

We then have for $\boldsymbol{u}_1$ the equation

$$\boldsymbol{grad}\,\mathrm{div}\boldsymbol{u}_1 + (1-2\sigma)\Delta\boldsymbol{u}_1 = -\boldsymbol{grad}\,\mathrm{div}\boldsymbol{u}_0. \quad (3)$$

The solution of equation (2) which vanishes at infinity is $\boldsymbol{u}_0 = \dfrac{(1+\sigma)\boldsymbol{F}}{2\pi E r}$.

Taking the curl of equation (3), we have $\Delta \mathrm{curl}\boldsymbol{u}_1 = 0$. At infinity we must have $\mathrm{curl}\boldsymbol{u}_1 = 0$. But a function harmonic in all space and zero at infinity must be zero identically. Thus $\mathrm{curl}\boldsymbol{u}_1 = 0$, and we can therefore write $\boldsymbol{u}_1 = \boldsymbol{grad}\phi$. From (3) we obtain $\boldsymbol{grad}\{2(1-\sigma)\Delta\phi + \mathrm{div}\boldsymbol{u}_0\} = 0$. Hence it follows that the quantity in braces is a constant, and it must be zero at infinity; we therefore have in all space

$$\Delta\phi = -\frac{1}{2(10\sigma)}\mathrm{div}\boldsymbol{u}_0 = -\frac{1+\sigma}{4\pi E(1-\sigma)}\boldsymbol{F}\cdot\boldsymbol{grad}\left(\frac{1}{r}\right).$$

If ψ is a solution of the equation $\Delta\psi = \dfrac{1}{r}$ then

$$\phi = -\frac{1+\sigma}{4\pi E(1-\sigma)}\boldsymbol{F}\cdot\boldsymbol{grad}\psi.$$

Taking the solution $\psi = \dfrac{r}{2}$, which has no singularities, we obtain

$$\boldsymbol{u}_1 = \boldsymbol{grad}\phi = \frac{1+\sigma}{8\pi E(1-\sigma)}\frac{(\boldsymbol{F}\cdot\boldsymbol{n})\boldsymbol{n} - \boldsymbol{F}}{r},$$

where $\boldsymbol{n}$ is a unit vector parallel to the radius vector $\boldsymbol{r}$. The final result is

$$\boldsymbol{u} = \frac{1+\sigma}{8\pi E(1-\sigma)}\cdot\frac{(3-4\sigma)\boldsymbol{F} + \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{F})}{r}.$$

On putting this formula into the form (8.13) we obtain the **Green's tensor** for the equations of equilibrium of an infinite isotropic medium:[21]

[21] The fact that the components of the tensor G_{ik} are first-order homogeneous functions of the coordinates *x*, *y*, *z* is evident from arguments of homogeneity applied to the form of equation (1), where the left-hand side is a linear combination of the second derivatives of the components of the vector $\boldsymbol{u}$, and the right-hand side is a third-order homogeneous function ($\delta(a\boldsymbol{r}) = a^{-3}\delta(\boldsymbol{r})$). This property remains valid in the general case of an arbitrary anisotropic

$$G_{ik} = \frac{1+\sigma}{8\pi E(1-\sigma)}\left[(3-4\sigma)\delta_{ik} + n_i n_k\right]\frac{1}{r}$$
$$= \frac{1}{4\pi\mu}\left[\frac{\delta_{ik}}{r} - \frac{1}{4(1-\sigma)}\frac{\partial^2 r}{\partial x_i \partial x_k}\right].$$

§9. Solid bodies in contact

Let two solid bodies be in contact at a point which is not a singular point on either surface. Fig. 1a shows a cross-section of the two surfaces near the point of contact *O*. The surfaces have a common tangent plane at *O*, which we take as the *xy*-plane. We regard the positive *z*-direction as being into either body (i.e., in opposite directions for the two bodies) and denote the corresponding co-ordinates by *z* and *z*'.

Near a point of ordinary contact with the *xy*-plane, the equation of the surface can be written

$$z = \kappa_{\alpha\beta} x_\alpha x_\beta, \tag{9.1}$$

where summation is understood over the values 1, 2, of the repeated suffixes α, β ($x_1 = x$, $x_2 = y$), and $\kappa_{\alpha\beta}$ is a symmetrical tensor of rank two, which characterises the curvature of the surface: the principal values of the tensor $\kappa_{\alpha\beta}$ are $\frac{1}{2R_1}$ and $\frac{1}{2R_2}$, where R_1 and R_2 are the principal radii of curvature of the surface at the point of contact. A similar relation for the surface of the other body near the point of contact can be written

$$z' = \kappa'_{\alpha\beta}\, x_\alpha x_\beta. \tag{9.2}$$

Let us now assume that the two bodies are pressed together by applied forces, and approach a short distance *h*.[22] Then a deformation occurs near the

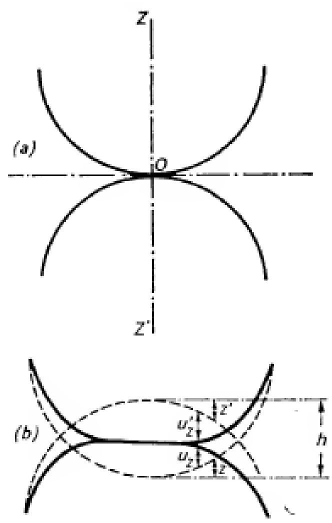

FIG. 1

medium.

original point of contact, and the two bodies will be in contact over a small but finite portion of their surfaces. Let u_z and u'_z be the components (along the z and z' axes, respectively) of the corresponding displacement vectors for points on the surfaces of the two bodies. The broken lines in Fig. 1b show the surfaces as they would be in the absence of any deformation, while the continuous lines show the surfaces of the deformed bodies; the letters z and z' denote the distances given by equations (9.1) and (9.2). It is seen at once from the figure that the equation

$$(z + u_z) + (z' + u'_z) = h,$$

or

$$(\kappa_{\alpha\beta} + \kappa'_{\alpha\beta}) x_\alpha x_\beta + u_z + u'_z = h, \qquad (9.3)$$

holds everywhere in the region of contact. At points outside the region of contact, we have

$$z + z' + u_z + u'_z < h.$$

We choose the x and y axes to be the principal axes of the tensor $\kappa_{\alpha\beta} + \kappa'_{\alpha\beta}$. Denoting the principal values of this tensor by A and B,[23] we can rewrite equation (9.3) as

$$Ax^2 + By^2 + u_z + u'_z = h. \qquad (9.4)$$

We denote by $P_z(x, y)$ the pressure between the two deformed bodies at points in the region of contact; outside this region, of course $P_z = 0$. To determine the relation between P_z and the displacements u_z, u'_z, we can with sufficient accuracy regard the surfaces as plane and use the formulae obtained in §8. According to the third of formulae (8.19) and (8.14), the displacement u_z under the action of normal forces $P_z(x, y)$ is given by

$$\left.\begin{aligned} u_z &= \frac{1-\sigma^2}{\pi E} \iint \frac{P_z(x', y')}{r} dx' dy' \\ u'_z &= \frac{1-\sigma'^2}{\pi E'} \iint \frac{P_z(x', y')}{r} dx' dy' \end{aligned}\right\}, \qquad (9.5)$$

where σ, σ' and E, E' are the Poisson's ratios and the Young's moduli of the two bodies. Since $P_z = 0$ outside the region of contact, the integration

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[22] This contact problem in the theory of elasticity was first solved by H. Hertz.

[23] The quantities A and B are related to the radii of curvature R_1, R_2 and R'_1, R'_2 by

$$2(A+B) = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R'_1} + \frac{1}{R'_2},$$

$$4(A-B)^2 = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)^2 + \left(\frac{1}{R'_1} - \frac{1}{R'_2}\right)^2 + 2\cos 2\phi \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{1}{R'_1} - \frac{1}{R'_2}\right),$$

where ϕ is the angle between the normal sections whose radii of curvature are R_1 and R'_1. The radii of curvature are regarded as positive if the centre of curvature lies within the body concerned, and negative in the contrary case.

extends only over this region. It may be noted that, from these formulae, the ratio $\frac{u_z}{u'_z}$ is constant:

$$\frac{u_z}{u'_z} = \frac{(1-\sigma^2)E'}{(1-\sigma'^2)E}. \tag{9.6}$$

The relations (9.4) and (9.6) together give the displacements u_z, u'_z at every point of the region of contact (although (9.5) and (9.6), of course, relate to points outside that region also).

Substituting the expressions (9.5) in (9.4), we obtain

$$\frac{1}{\pi}\left(\frac{1-\sigma^2}{E} + \frac{1-\sigma'^2}{E'}\right)\iint \frac{P_z(x', y')}{r}\,dx'\,dy' = h - Ax^2 - By^2. \tag{9.7}$$

This integral equation determines the distribution of the pressure P_z over the region of contact. Its solution can be found by analogy with the following results of potential theory. The idea of using this analogy arises as follows: firstly, the integral on the left-hand side of equation (9.7) is of a type commonly found in potential theory, where such integrals give the potential of a charge distribution; secondly, the potential inside a uniformly charged ellipsoid is a quadratic function of the co-ordinates.

If the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is uniformly charged (with volume charge density ρ), the potential in the ellipsoid is given by

$$\phi(x, y, z) = \pi\rho abc \int_0^\infty \left\{1 - \frac{x^2}{a^2+\xi} - \frac{y^2}{b^2+\xi} - \frac{z^2}{c^2+\xi}\right\} \frac{d\xi}{\sqrt{(a^2+\xi)(b^2+\xi)(c^2+\xi)}}$$

In the limiting case of an ellipsoid which is very much flattened in the z-direction ($c \to 0$), we have

$$\phi(x, y) = \pi\rho abc \int_0^\infty \left\{1 - \frac{x^2}{a^2+\xi} - \frac{y^2}{b^2+\xi}\right\} \frac{d\xi}{\sqrt{(a^2+\xi)(b^2+\xi)\xi}};$$

in passing to the limit $c \to 0$ we must, of course, put $z = 0$ for points inside the ellipsoid. The potential $\phi(x, y, z)$ can also be written as

$$\phi(x, y, z) = \iiint \frac{\rho\,dx'\,dy'\,dz'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}},$$

where the integration is over the volume of the ellipsoid. In passing to the limit $c \to 0$, we must put $z = z' = 0$ in the radicand; integrating over z' between the limits

$$\pm c\sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}\ ,$$

we obtain

$$\phi(x,y)=2\rho c\iint\frac{dx'dy'}{r}\sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}\ ,$$

where

$$r=\sqrt{(x-x')^2+(y-y')^2}\ ,$$

and the integration is over the area inside the ellipse

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1\ .$$

Equating the two expressions for $\phi(x,y)$, we obtain the identity

$$\iint\frac{dx'dy'}{r}\sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}=\frac{\pi ab}{2}\int_0^\infty\left(1-\frac{x^2}{a^2+\xi}-\frac{y^2}{b^2+\xi}\right)\frac{d\xi}{\sqrt{(a^2+\xi)(b^2+\xi)\xi}} \tag{9.8}$$

Comparing this relation with equation (9.7), we see that the right-hand sides are quadratic functions of x and y of the same form, and the left-hand sides are integrals of the same form. We can therefore deduce immediately that the region of contact (i.e., the region of integration in (9.7)) is bounded by an ellipse of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{9.9}$$

and that the function $P_z(x,y)$ must be of the form

$$P_z(x,y)=\text{constant}\times\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\ .$$

Taking the constant such that the integral $\iint P_z dxdy$ over the region of contact is equal to the given total force F which moves the bodies together, we obtain

$$P_z(x,y)=\frac{3F}{2\pi ab}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\ . \tag{9.10}$$

This formula gives the distribution of pressure over the area of the region of contact. It may be pointed out that the pressure at the centre of this region is 3/2 times the mean pressure $\dfrac{F}{\pi ab}$.

Substituting (9.10) in equation (9.7) and replacing the resulting integral in accordance with (9.8), we obtain

$$\frac{\dfrac{FD}{\pi}\displaystyle\int_0^{\infty}\left(1-\frac{x^2}{a^2+\xi}-\frac{y^2}{b^2+\xi}\right)d\xi}{\sqrt{(a^2+\xi)(b^2+\xi)\xi}}=h-Ax^2-By^2\,,$$

where

$$D=\frac{3}{4}\left(\frac{1-\sigma^2}{E}+\frac{1-\sigma'^2}{E'}\right).$$

This equation must hold identically for all values of x and y inside the ellipse (9.9) ; the coefficients of x and y and the free terms must therefore be respectively equal on each side. Hence we find

$$h=\frac{FD}{\pi}\int_0^{\infty}\frac{d\xi}{\sqrt{(a^2+\xi)(b^2+\xi)\xi}}\,, \qquad (9.11)$$

$$\left.\begin{aligned} A&=\frac{FD}{\pi}\int_0^{\infty}\frac{d\xi}{(a^2+\xi)\sqrt{(a^2+\xi)(b^2+\xi)\xi}}\\ B&=\frac{FD}{\pi}\int_0^{\infty}\frac{d\xi}{(b^2+\xi)\sqrt{(a^2+\xi)(b^2+\xi)\xi}}\end{aligned}\right\}. \qquad (9.12)$$

Equations (9.12) determine the semi-axes a and b of the region of contact from the given force F {A and B being known for given bodies). The relation (9.11) then gives the distance of approach h as a function of the force F. The right-hand sides of these equations involve elliptic integrals.

Thus the problem of bodies in contact can be regarded as completely solved. The form of the surfaces (i.e., the displacements u_z, u'_z) outside the region of contact is determined by the same formulae (9.5) and (9.10); the values of the integrals can be found immediately from the analogy with the potential outside a charged ellipsoid. Finally, the formulae of §8 enable us to find also the deformation at various points in the bodies (but only, of course, at distances small compared with the dimensions of the bodies).

Let us apply these formulae to the case of contact between two **spheres** of radii R and R'. Here $A = B = 1/2R+1/2R'$. It is clear from symmetry that $a = b$, i.e., the region of contact is a circle. From (9.12) we find the radius a of this circle to be

$$a=F^{1/3}\left\{\frac{DRR'}{R+R'}\right\}^{1/3}. \qquad (9.13)$$

h is in this case the difference between the sum $R+R'$ and the distance between the centres of the spheres. From (9.10) we obtain the following

relation between F and h:

$$h = F^{2/3}\left[D^2\left(\frac{1}{R}+\frac{1}{R'}\right)\right]^{1/3}. \qquad (9.14)$$

It should be noticed that h is proportional to $F^{2/3}$; conversely, the force F varies as $h^{3/2}$. We can write down also the potential energy U of the spheres in contact. Since $-F = -\frac{\partial U}{\partial h}$, we have

$$U = h^{5/2}\frac{2}{5D}\sqrt{\frac{RR'}{R+R'}}. \qquad (9.15)$$

Finally, it may be mentioned that a relation of the form $h = \text{constant} \times F^{2/3}$, or $F = \text{constant} \times h^{3/2}$, holds not only for spheres but also for other finite bodies in contact. This is easily seen from similarity arguments. If we make the substitution

$$a^2 \to \alpha a^2,\ b^2 \to \alpha b^2,\ F \to \alpha^{3/2}F,$$

where α is an arbitrary constant, equations (9.12) remain unchanged. In equation (9.11), the right-hand side is multiplied by α, and so h must be replaced by αh if this equation is to remain unchanged. Hence it follows that F must be proportional to $h^{3/2}$.

PROBLEMS

Problem 1. Determine the time for which two colliding elastic spheres remain in contact.

Solution. In a system of co-ordinates in which the centre of mass of the two spheres is at rest, the energy before the collision is equal to the kinetic energy of the relative motion $\frac{1}{2}\mu v^2$, where v is the relative velocity of the colliding spheres and $\mu = \frac{m_1 m_2}{m_1 + m_2}$ their reduced mass. During the collision, the total energy is the sum of the kinetic energy, which may be written $\frac{1}{2}\mu \dot{h}^2$, and the potential energy (9.15). By the law of conservation of energy we have

$$\mu\left(\frac{dh}{dt}\right)^2 + kh^{5/2} = \mu v^2,\ k = \frac{4}{5D}\sqrt{\frac{RR'}{R+R'}}.$$

The maximum approach h_0 of the spheres corresponds to the time when their relative velocity $\dot{h} = 0$, and is $h_0 = \left(\frac{\mu}{k}\right)^{2/5} v^{4/5}$.

The time τ during which the collision takes place (i.e., h varies from 0 to h_0 and back) is

$$\tau = 2\int_0^{h_0} \frac{dh}{\sqrt{v^2 - \dfrac{kh^{5/2}}{\mu}}} = 2\left(\frac{\mu^2}{k^2 v}\right)^{1/5} \int_0^1 \frac{dx}{\sqrt{1-x^{2/5}}},$$

or

$$\tau = \frac{4\sqrt{\pi}\Gamma(2/5)}{5\Gamma(9/10)}\left(\frac{\mu^2}{k^2 v}\right)^{1/5} = 2.94\left(\frac{\mu^2}{k^2 v}\right)^{1/5}.$$

By using the statical formulae obtained in the text to solve this problem, we have neglected elastic oscillations of the spheres resulting from the collision. If this is legitimate, the velocity v must be small compared with the velocity of sound. In practice, however, the validity of the theory is limited by the still more stringent requirement that the resulting deformations should not exceed the elastic limit of the substance.

Problem 2. Determine the dimensions of the region of contact and the pressure distribution when two cylinders are pressed together along a generator.

Solution. In this case the region of contact is a narrow strip along the length of the cylinders. Its width $2a$ and the pressure distribution across it can be found from the formulae in the text by going to the limit $b/a \to \infty$. The pressure distribution will be of the form $P_z(x) = \text{constant} \times \sqrt{1 - \frac{x^2}{a^2}}$, where x is the co-ordinate across the strip; normalizing the pressure to give a force F per unit length, we obtain

$$P_z(x) = \frac{2F}{\pi a}\sqrt{1 - \frac{x^2}{a^2}}.$$

Substituting this expression in (9.7) and effecting the integration by means of (9.8), we have

$$A = \frac{4DF}{3\pi}\int_0^\infty \frac{d\xi}{(a^2+\xi)^{3/2}\xi} = \frac{8DF}{3\pi a^2}.$$

One of the radii of curvature of a cylindrical surface is infinite, and the other is the radius of the cylinder; in this case, therefore, $A = \frac{1}{2R} + \frac{1}{2R'}$, $B = 0$. We have finally for the width of the region of contact

$$a = \sqrt{\frac{16DF}{3\pi} \cdot \frac{RR'}{R+R'}}.$$

§10. The elastic properties of crystals