CHAPTER II

THE EQUILIBRIUM OF RODS AND PLATES

§11. The energy of a bent plate

In this chapter we shall study some particular cases of the equilibrium of deformed bodies, and we begin with that of thin deformed plates. When we speak of a ***thin plate***, we mean that its thickness is small compared with its dimensions in the other two directions. The deformations themselves are supposed small, as before. In the present case the deformation is small if the displacements of points in the plate are small compared with its thickness.

The general equations of equilibrium are considerably simplified when applied to thin plates. It is more convenient, however, not to derive these simplified equations directly from the general ones, but to calculate afresh the free energy of a bent plate and then vary that energy.

When a plate is bent, it is stretched at some points and compressed at others: on the convex side there is evidently an extension, which decreases as we penetrate into the plate, finally becoming zero, after which a gradually increasing compression is found. The plate therefore contains a ***neutral surface***, on which there is no extension or compression, and on opposite sides of which the deformation has opposite signs. The neutral surface clearly lies midway through the plate.

We take a co-ordinate system with the origin on the neutral surface and the z-axis normal to the surface. The xy-plane is that of the undeformed plate. We denote by ζ the vertical displacement of a point on the neutral surface, i.e., its z co-ordinate (Fig. 2). The components of its displacement in the xy-plane are evidently of the second order of smallness relative to ζ, and can therefore be put equal to zero. Thus the displacement vector for points on the neutral surface is

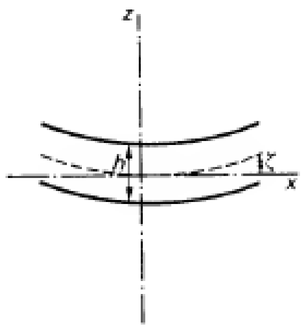

FIG. 2

$$u_x^{(0)} = u_y^{(0)} = 0\,, \quad u_z^{(0)} = \zeta(x, y)\,. \qquad (11.1)$$

For further calculations it is necessary to note the following property of the stresses in a deformed plate. Since the plate is thin, comparatively small forces on its surface are needed to bend it. These forces are always considerably less than the internal stresses caused in the deformed plate by the extension and compression of its parts. We can therefore neglect the

forces P_i in the boundary condition (2.8), leaving $\sigma_{ik} n_k = 0$. Since the plate is only slightly bent, we can suppose that the normal vector $\boldsymbol{n}$ is along the z-axis. Thus we must have on both surfaces of the plate $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$. Since the plate is thin, however, these quantities must be small within the plate if they are zero on each surface. We therefore conclude that the components σ_{xz}, σ_{yz}, σ_{zz} are small compared with the remaining components of the stress tensor everywhere in the plate. We can therefore equate them to zero and use this condition to determine the components of the strain tensor.

By the general formulae (5.13), we have

$$\left.\begin{aligned} \sigma_{zx} &= \frac{E}{1+\sigma} u_{zx} \\ \sigma_{zy} &= \frac{E}{1+\sigma} u_{zy} \\ \sigma_{zz} &= \frac{E}{(1+\sigma)(1-2\sigma)} \left\{(1-\sigma) u_{zz} + \sigma(u_{xx} + u_{yy})\right\} \end{aligned}\right\}. \quad (11.2)$$

Equating these expressions to zero, we obtain $\frac{\partial u_x}{\partial z} = -\frac{\partial u_z}{\partial x}$, $\frac{\partial u_y}{\partial z} = -\frac{\partial u_z}{\partial y}$, $u_{zz} = -\frac{\sigma}{1-\sigma}(u_{xx} + u_{yy})$. In the first two of these equations u_z can, with sufficient accuracy, be replaced by $\zeta(x, y)$: $\frac{\partial u_x}{\partial z} = -\frac{\partial \zeta}{\partial x}$, $\frac{\partial u_y}{\partial z} = -\frac{\partial \zeta}{\partial y}$, whence

$$\left.\begin{aligned} u_x &= -z \frac{\partial \zeta}{\partial x} \\ u_y &= -z \frac{\partial \zeta}{\partial y} \end{aligned}\right\}. \quad (11.3)$$

The constants of integration are put equal to zero in order to make

$$u_x = u_y = 0 \quad \text{for} \quad z = 0.$$

Knowing u_x and u_y, we can determine all the components of the strain tensor

$$\left.\begin{aligned} u_{xx} &= -z \frac{\partial^2 \zeta}{\partial x^2} \\ u_{yy} &= -z \frac{\partial^2 \zeta}{\partial y^2} \\ u_{xy} &= -z \frac{\partial^2 \zeta}{\partial x \partial y} \\ u_{xz} &= u_{yz} = 0 \\ u_{zz} &= \frac{\sigma}{1-\sigma} z \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right) \end{aligned}\right\}. \quad (11.4)$$

We can now calculate the free energy F per unit volume of the plate, using the general formula (5.10). A simple calculation gives the expression

$$F = z^2 \frac{E}{1+\sigma}\left\{\frac{1}{2(1-\sigma)}\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)^2+\left[\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2-\frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right]\right\}. \tag{11.5}$$

The total free energy of the plate is obtained by integrating over the volume. The integration over z is from $-h/2$ to $+h/2$ where h is the thickness of the plate, and that over x, y is over the surface of the plate. The result is that the total free energy $F_{pl} = \int F dV$ of a deformed plate is

$$F_{pl} = \frac{Eh^3}{24(1-\sigma^2)}\iint\left[\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)^2+2(1-\sigma)\left\{\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2-\frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right\}\right]dxdy \tag{11.6}$$

the element of area can with sufficient accuracy be written as $dx\,dy$ simply, since the deformation is small.

Having obtained the expression for the free energy, we can regard the plate as being of infinitesimal thickness, i.e., as being a geometrical surface, since we are interested only in the form which it takes under the action of the applied forces, and not in the distribution of deformations inside it. The quantity ζ is then the displacement of points on the plate, regarded as a surface, when it is bent.

§12. The equation of equilibrium for a plate

The equation of equilibrium for a plate can be derived from the condition that its free energy is a minimum. To do so, we must calculate the variation of the expression (11.6).

We divide the integral in (11.6) into two, and vary the two parts separately. The first integral can be written in the form $\int(\Delta\zeta)^2 df$, where $df = dxdy$ is a surface element and $\Delta \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is here (and in §§13, 14) the two-dimensional Laplacian. Varying this integral, we have

$$\begin{aligned}\delta\frac{1}{2}\int(\Delta\zeta)^2 df &= \int \Delta\zeta\Delta(\delta\zeta)df \\ &= \int \Delta\zeta \operatorname{div}\boldsymbol{grad}\,\delta\zeta df \\ &= \int \operatorname{div}(\Delta\zeta\,\boldsymbol{grad}\,\delta\zeta)df - \int \boldsymbol{grad}\,\delta\zeta\cdot\boldsymbol{grad}\,\Delta\zeta df\end{aligned}.$$

All the vector operators, of course, relate to the two-dimensional co-ordinate system (x, y). The first integral on the right can be transformed into an integral along a closed contour enclosing the plate:[1]

$$\int \operatorname{div}(\Delta\zeta \boldsymbol{grad}\delta\zeta)df = \oint \Delta\zeta(\boldsymbol{n}\cdot\boldsymbol{grad}\delta\zeta)dl = \oint \Delta\zeta\frac{\partial\delta\zeta}{\partial n}dl,$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the outward normal to the contour.

In the second integral we use the same transformation to obtain

$$\begin{aligned}\int \boldsymbol{grad}\delta\zeta\cdot\boldsymbol{grad}\Delta\zeta df &= \int \operatorname{div}(\delta\zeta\boldsymbol{grad}\Delta\zeta)df - \int \delta\zeta\Delta^2\zeta df\\ &= \oint \delta\zeta(n\cdot\boldsymbol{grad}\Delta\zeta)dl - \int \delta\zeta\Delta^2 df\\ &= \oint \delta\zeta\frac{\partial\Delta\zeta}{\partial n}dl - \int \delta\zeta\Delta^2 df\end{aligned}.$$

Substituting these results, we find that

$$\delta\frac{1}{2}\int(\Delta\zeta)^2 df = \int \delta\zeta\Delta^2\zeta df - \oint \delta\zeta\frac{\partial\Delta\zeta}{\partial n}dl + \oint \Delta\zeta\frac{\partial\delta\zeta}{\partial n}dl. \tag{12.1}$$

The transformation of the variation of the second integral in (11.6) is somewhat more lengthy. This transformation is conveniently effected in components, and not in vector form. We have

$$\delta\int\left\{\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2 - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right\}df = \int\left\{2\frac{\partial^2\zeta}{\partial x\partial y}\frac{\partial^2\delta\zeta}{\partial x\partial y} - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\delta\zeta}{\partial y^2} - \frac{\partial^2\delta\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right\}df$$

The integrand can be written

$$\frac{\partial}{\partial x}\left(\frac{\partial\delta\zeta}{\partial y}\frac{\partial^2\zeta}{\partial x\partial y} - \frac{\partial\delta\zeta}{\partial x}\frac{\partial^2\zeta}{\partial y^2}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\delta\zeta}{\partial x}\frac{\partial^2\zeta}{\partial x\partial y} - \frac{\partial\delta\zeta}{\partial y}\frac{\partial^2\zeta}{\partial x^2}\right),$$

i.e., as the (two-dimensional) divergence of a certain vector. The variation

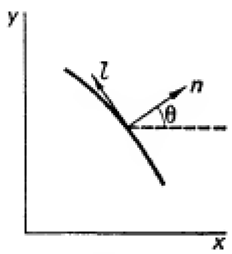

FIG. 3

[1] The transformation formula for two-dimensional integrals is exactly analogous to the one for three dimensions. The volume element dV is replaced by the surface element df (a scalar), and the surface element $\boldsymbol{df}$ is replaced by a contour element dl multiplied by the vector $\boldsymbol{n}$ along the outward normal to the contour. The integral over df is converted into one over dl by replacing the operator $df\frac{\partial}{\partial x_i}$ by $n_i dl$. For instance, if ϕ is a scalar, we have $\int \boldsymbol{grad}\phi df = \oint \phi\boldsymbol{n}dl$.

can therefore be written as a contour integral:

$$\delta\iint\left\{\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2-\frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right\}df=\oint dl\sin\theta\left\{\frac{\partial\delta\zeta}{\partial x}\frac{\partial^2\zeta}{\partial x\partial y}-\frac{\partial\delta\zeta}{\partial y}\frac{\partial^2\zeta}{\partial x^2}\right\}+\oint dl\cos\theta\left\{\frac{\partial\delta\zeta}{\partial y}\frac{\partial^2\zeta}{\partial x\partial y}-\frac{\partial\delta\zeta}{\partial x}\frac{\partial^2\zeta}{\partial y^2}\right\}$$

(12.2)

where θ is the angle between the x-axis and the normal to the contour (Fig. 3).

The derivatives of $\delta\zeta$ with respect to x and y are expressed in terms of its derivatives along the normal $\boldsymbol{n}$ and the tangent $\boldsymbol{l}$ to the contour:

$$\begin{cases}\dfrac{\partial}{\partial x}=\cos\theta\dfrac{\partial}{\partial n}-\sin\theta\dfrac{\partial}{\partial l}\\[2ex]\dfrac{\partial}{\partial y}=\sin\theta\dfrac{\partial}{\partial n}+\cos\theta\dfrac{\partial}{\partial l}\end{cases}.$$

Then formula (12.2) becomes

$$\begin{aligned}&\delta\iint\left\{\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2-\frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right\}df\\&=\oint dl\frac{\partial\delta\zeta}{\partial n}\left\{2\sin\theta\cos\theta\frac{\partial^2\zeta}{\partial x\partial y}-\sin^2\theta\frac{\partial^2\zeta}{\partial x^2}-\cos^2\theta\frac{\partial^2\zeta}{\partial y^2}\right\}\\&+\oint dl\frac{\partial\delta\zeta}{\partial l}\left\{\sin\theta\cos\theta\left(\frac{\partial^2\zeta}{\partial y^2}-\frac{\partial^2\zeta}{\partial x^2}\right)+(\cos^2\theta-\sin^2\theta)\frac{\partial^2\zeta}{\partial x\partial y}\right\}\end{aligned}.$$

The second integral may be integrated by parts. Since it is taken along a closed contour, the limits of integration are the same point, and we have simply

$$-\oint dl\delta\zeta\frac{\partial}{\partial l}\left\{\sin\theta\cos\theta\left(\frac{\partial^2\zeta}{\partial y^2}-\frac{\partial^2\zeta}{\partial x^2}\right)+(\cos^2\theta-\sin^2\theta)\frac{\partial^2\zeta}{\partial x\partial y}\right\}.$$

Collecting all the above expressions and multiplying by the coefficients shown in formula (11.6), we obtain the following final expression for the variation of the free energy:

$$\delta F_{pl}=\frac{Eh^3}{12(1-\sigma^2)}(A+B+C),$$

$$A=\int\Delta^2\zeta\delta\zeta df,$$

$$B=-\oint\delta\zeta dl\left[\frac{\partial\Delta\zeta}{\partial n}+(1-\sigma)\frac{\partial}{\partial l}\left\{\sin\theta\cos\theta\left(\frac{\partial^2\zeta}{\partial y^2}-\frac{\partial^2\zeta}{\partial x^2}\right)+(\cos^2\theta-\sin^2\theta)\frac{\partial^2\zeta}{\partial x\partial y}\right\}\right]$$

$$C = \oint \frac{\partial \delta\zeta}{\partial n} dl \left\{ \Delta\zeta + (1-\sigma)\left(2\sin\theta\cos\theta \frac{\partial^2\zeta}{\partial x \partial y} - \sin^2\theta \frac{\partial^2\zeta}{\partial x^2} - \cos^2\theta \frac{\partial^2\zeta}{\partial y^2} \right) \right\}.$$

(12.3)

In order to derive from this the equation of equilibrium for the plate, we must equate to zero the sum of the variation δF and the variation δU of the potential energy of the plate due to the external forces acting on it. This latter variation is minus the work done by the external forces in deforming the plate. Let P be the external force acting on the plate, per unit area[2] and normal to the surface. Then the work done by the external forces when the points on the plate are displaced a distance $\delta\zeta$ is $\int P\delta\zeta df$. Thus the condition for the total free energy of the plate to be a minimum is

$$\delta F_{pl} - \int P\delta\zeta df = 0 . \qquad (12.4)$$

On the left-hand side of this equation we have both surface and contour integrals. The surface integral is

$$\int \left\{ \frac{Eh^3}{12(1-\sigma^2)} \Delta^2\zeta - P \right\} \delta\zeta df .$$

The variation $\delta\zeta$ in this integral is arbitrary. The integral can therefore vanish only if the coefficient of $\delta\zeta$ is zero, i.e.,

$$\frac{Eh^3}{12(1-\sigma^2)} \Delta^2\zeta - P = 0 . \qquad (12.5)$$

This is the equation of equilibrium for a plate bent by external forces acting on it.[3]

The boundary conditions for this equation are obtained by equating to zero the contour integrals in (12.3). Here various particular cases have to be considered. Let us suppose that part of the edge of the plate is free, i.e., no external forces act on it. Then the variations $\delta\zeta$ and $\delta\frac{\partial\zeta}{\partial n}$ on this part of the edge are arbitrary, and their coefficients in the contour integrals must be zero. This gives the equations

$$-\frac{\partial\Delta\zeta}{\partial n} + (1-\sigma)\frac{\partial}{\partial l}\left\{ \cos\theta\sin\theta\left(\frac{\partial^2\zeta}{\partial x^2} - \frac{\partial^2\zeta}{\partial y^2} \right) + (\sin^2\theta - \cos^2\theta)\frac{\partial^2\zeta}{\partial x \partial y} \right\} = 0 ,$$

[2] The force P may be the result of body forces (e.g., the force of gravity), and is then equal to the integral of the body force over the thickness of the plate.

[3] The coefficient $D = \frac{Eh^3}{12(1-\sigma^2)}$ in this equation is called the ***flexural rigidity*** or ***cylindrical***

(12.6)

$$\Delta\zeta + (1-\sigma)\left\{2\sin\theta\cos\theta\frac{\partial^2\zeta}{\partial x\partial y} - \sin^2\theta\frac{\partial^2\zeta}{\partial x^2} - \cos^2\theta\frac{\partial^2\zeta}{\partial y^2}\right\} = 0,$$

(12.7)

which must hold at all free points on the edge of the plate.

The boundary conditions (12.6) and (12.7) are very complex. Considerable simplifications occur when the edge of the plate is clamped or supported. If it is clamped (Fig. 4a), no vertical displacement is possible, and moreover no bending is possible at the edge. The angle through which a given part of the edge turns from its initial position is (for small displacements ζ) the derivative $\frac{\partial\zeta}{\partial n}$. Thus the variations $\delta\zeta$ and $\delta\frac{\partial\zeta}{\partial n}$ must be zero at clamped edges, so that the contour integrals in (12.3) are zero identically. The boundary conditions have in this case the simple form

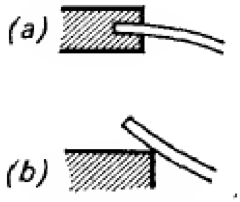

FIG. 4

$$\zeta = 0, \quad \frac{\partial\zeta}{\partial n} = 0. \tag{12.8}$$

The first of these expresses the fact that the edge of the plate undergoes no vertical displacement in the deformation, and the second that it remains horizontal.

It is easy to determine the reaction forces on a plate at a point where it is **clamped**. These are equal and opposite to the forces exerted by the plate on its support. As we know from mechanics, the force in any direction is equal to the space derivative, in that direction, of the energy. In particular, the force exerted by the plate on its support is given by minus the derivative of the energy with respect to the displacement ζ of the edge of the plate, and the reaction force by this derivative itself. The derivative in question, however, is just the coefficient of $\delta\zeta$ in the second integral in (12.3). Thus the reaction force per unit length is equal to the expression on the left of equation (12.6) (which, of course, is not now zero), multiplied by $\frac{Eh^3}{12(1-\sigma^2)}$.

Similarly, the moment of the reaction forces is given by the expression on the left of equation (12.7), multiplied by the same factor. This follows at once from the result of mechanics that the moment of the force is equal to the

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rigidity of the plate.

derivative of the energy with respect to the angle through which the body turns. This angle is $\frac{\partial \zeta}{\partial n}$, so that the corresponding moment is given by the coefficient of $\frac{\partial \delta\zeta}{\partial n}$ in the third integral in (12.3). Both these expressions (that for the force and that for the moment) can be very much simplified by virtue of the conditions (12.8). Since ζ and $\frac{\partial \zeta}{\partial n}$ are zero everywhere on the edge of the plate, their tangential derivatives of all orders are zero also. Using this and converting the derivatives with respect to x and y in (12.6) and (12.7) into those in the directions of $\boldsymbol{n}$ and $\boldsymbol{l}$, we obtain the following simple expressions for the reaction force F and the reaction moment M:

$$F = -\frac{Eh^3}{12(1-\sigma^2)}\left[\frac{\partial^3 \zeta}{\partial n^3} + \frac{d\theta}{dl}\frac{\partial^2 \zeta}{\partial n^2}\right], \qquad (12.9)$$

$$M = \frac{Eh^3}{12(1-\sigma^2)}\frac{\partial^2 \zeta}{\partial n^2}. \qquad (12.10)$$

Another important case is that where the plate is **supported** (Fig. 4b), i.e., the edge rests on a fixed support, but is not clamped to it. In this case there is again no vertical displacement at the edge of the plate (i.e., on the line where it rests on the support), but its direction can vary. Accordingly, we have in (12.3) $\delta\zeta = 0$ in the contour integral, but $\frac{\partial \delta\zeta}{\partial n} \neq 0$. Hence only the condition (12.7) remains valid, and not (12.6). The expression on the left of (12.6) gives as before the reaction force at the points where the plate is supported ; the moment of this force is zero in equilibrium. The boundary condition (12.7) can be simplified by converting to the derivatives in the direction of $\boldsymbol{n}$ and $\boldsymbol{l}$ and using the fact that, since $\zeta = 0$ everywhere on the edge, the derivatives $\frac{\partial \zeta}{\partial l}$ and $\frac{\partial^2 \zeta}{\partial l^2}$ are also zero. We then have the boundary conditions in the form

$$\zeta = 0, \quad \frac{\partial^2 \zeta}{\partial n^2} + \sigma\frac{d\theta}{dl}\frac{\partial \zeta}{\partial n} = 0. \qquad (12.11)$$

PROBLEMS

Problem 1. Determine the deflection of a circular plate (of radius R) with **clamped** edges, placed horizontally in a gravitational field.

Solution. We take polar co-ordinates, with the origin at the centre of the plate. The force on unit area of the surface of the plate is $P = \rho hg$. Equation (12.5)

becomes $\Delta^2\zeta = 64\beta$, where $\beta = \dfrac{3\rho g(1-\sigma^2)}{16h^2 E}$; positive values of ζ correspond to displacements downward. Since ζ is a function of r only, we can put $\Delta = \dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{d}{dr}\right)$. The general integral is $\zeta = \beta r^4 + ar^2 + b + cr^2 \log\dfrac{r}{R} + d\log\dfrac{r}{R}$. In the case in question we must put $d = 0$, since $\log\dfrac{r}{R}$ becomes infinite at $r = 0$, and $c = 0$, since this term gives a singularity in $\Delta\zeta$ at $r = 0$ corresponding to a force applied at the centre of the plate; see Problem 3). The constants a and b are determined from the boundary conditions $\zeta = 0$, $\dfrac{d\zeta}{dr} = 0$ for $r = R$. The result is

$$\zeta = \beta(R^2 - r^2)^2 .$$

Problem 2. The same as Problem 1, but for a plate with **supported** edges.

Solution. The boundary conditions (12.11) for a circular plate are

$$\zeta = 0, \quad \frac{d^2\zeta}{dr^2} + \frac{\sigma}{r}\frac{d\zeta}{dr} = 0 .$$

The solution is similar to that of Problem 1, and the result is

$$\zeta = \beta(R^2 - r^2)\left(\frac{5+\sigma}{1+\sigma}R^2 - r^2\right).$$

Problem 3. Determine the deflection of a circular plate with **clamped** edges when a force f is applied to its centre.

Solution. We have $\Delta^2\zeta = 0$ everywhere except at the origin. Integration gives

$$\zeta = ar^2 + b + cr^2 \log\frac{r}{R},$$

the $\log r$ term again being omitted. The total force on the plate is equal to the force f at its centre. The integral of $\Delta^2\zeta$ over the surface of the plate must therefore be

$$2\pi\int_0^R r\Delta^2\zeta dr = \frac{12(1-\sigma^2)}{Eh^3} f .$$

Hence $c = \dfrac{3(1-\sigma^2)f}{2\pi Eh^3}$. The constants a and b are determined from the boundary conditions. The result is

$$\zeta = \frac{3f(1-\sigma^2)}{2\pi Eh^3}\left[\frac{1}{2}(R^2 - r^2) - r^2 \log\frac{R}{r}\right].$$

Problem 4. The same as Problem 3, but for a plate with **supported** edges.

Solution.

$$\zeta = \frac{3f(1-\sigma^2)}{2\pi E h^3}\left[\frac{3+\sigma}{1\sigma}(R^2 - r^2) - 2r^2 \log\frac{R}{r}\right].$$

Problem 5. Determine the deflection of a circular plate suspended by its centre and in a gravitational field.

Solution. The equation for ζ and its general solution are the same as in Problem 1. Since the displacement at the centre is $\zeta = 0$, we have $c = 0$. The constants a and b are determined from the boundary conditions (12.6) and (12.7), which are, for circular symmetry,

$$\frac{d\Delta\zeta}{dr} = \frac{d}{dr}\left(\frac{d^2\zeta}{dr^2} + \frac{1}{r}\frac{d\zeta}{dr}\right) = 0, \quad \frac{d^2\zeta}{dr^2} + \frac{\sigma}{r}\frac{d\zeta}{dr} = 0.$$

The result is

$$\zeta = \beta r^2\left[r^2 + 8R^2 \log\frac{R}{r} + 2R^2\frac{3+\sigma}{1+\sigma}\right].$$

Problem 6. A thin layer (of thickness h) is torn off a body by external forces acting against surface tension forces at the surface of separation. With given external forces, equilibrium is established for a definite area of the surface separated and a definite shape of the layer removed (Fig. 5). Derive a formula relating the surface tension to the shape of the layer removed.[4]

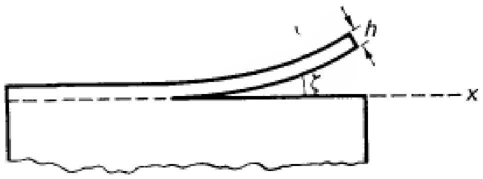

FIG. 5

Solution. The layer removed can be regarded as a plate with one edge (the line of separation) clamped. The bending moment on the layer is given by formula (12.10). The work done by this moment when the length of the separated surface increases by δx is

$$M\frac{\partial\delta\zeta}{\partial x} = M\delta x\frac{\partial^2\zeta}{\partial x^2}$$

(the work of the bending force F itself is a second-order quantity). The equilibrium condition is that this work should be equal to the change in the surface energy, i.e., to $2\alpha\delta x$, where α is the **surface-tension coefficient**, the factor 2 allowing for the creation of two free surfaces by the separation. Thus

[4] This problem was discussed by I. V. Obreimov (1930) in connection with a method which he developed for measuring the surface tension of mica. The measurements which he made by this

$$\alpha = \frac{Eh^3}{24(1-\sigma^2)}\left(\frac{\partial^2 \zeta}{\partial x^2}\right)^2 .$$

§13. Longitudinal deformations of plates

Longitudinal deformations occurring in the plane of the plate, and not resulting in any bending, form a special case of deformations of thin plates. Let us derive the equations of equilibrium for such deformations.

If the plate is sufficiently thin, the deformation may be regarded as uniform over its thickness. The strain tensor is then a function of *x* and *y* only (the *xy*-plane being that of the plate) and is independent of *z*. Longitudinal deformations of a plate are usually caused either by forces applied to its edges or by body forces in its plane. The boundary conditions on both surfaces of the plate are then $\sigma_{ik} n_k = 0$, or, since the normal vector is parallel to the *z*-axis, $\sigma_{iz} = 0$, i.e., $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$. It should be noticed, however, that in the approximate theory given below these conditions continue to hold even when the external tension forces are applied to the surfaces of the plate, since these forces are still small compared with the resulting longitudinal internal stresses (σ_{xx}, σ_{yy}, σ_{xy}) in the plate. Since they are zero at both surfaces, the quantities σ_{xz}, σ_{yz}, σ_{zz} must be small throughout the thickness of the plate, and we can therefore take them as approximately zero everywhere in the plate.

Equating to zero the expressions (11.2), we obtain the relations

$$u_{zz} = -\frac{\sigma}{1-\sigma}(u_{xx} + u_{yy}), \quad u_{xz} = u_{yz} = 0. \tag{13.1}$$

Substituting in the general formulae (5.13), we obtain for the non-zero components of the stress tensor

$$\left.\begin{aligned} \sigma_{xx} &= \frac{E}{1-\sigma^2}(u_{xx} + \sigma u_{yy}) \\ \sigma_{yy} &= \frac{E}{1-\sigma^2}(u_{yy} + \sigma u_{xx}) \\ \sigma_{xy} &= \frac{E}{1+\sigma} u_{xy} \end{aligned}\right\} \tag{13.2}$$

It should be noticed that the formal transformation

$$E \to \frac{E}{1-\sigma^2}, \quad \sigma \to \frac{\sigma}{1-\sigma} \tag{13.3}$$

converts these expressions into those which give the relation between the stresses σ_{xx}, σ_{xy}, σ_{yy}, and the strains u_{xx}, u_{yy}, u_{zz} for a plane

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method were the first direct measurements of the surface tension of solids.

deformation (formulae (5.13) with $u_{zz} = 0$).

Having thus eliminated the displacement u_z, we can regard the plate as a two-dimensional medium (an "elastic plane"), of zero thickness, and take the displacement vector $\boldsymbol{u}$ to be a two-dimensional vector with components u_x and u_y. If P_x and P_y are the components of the external body force per unit area of the plate, the general equations of equilibrium are

$$\left.\begin{aligned} h\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y}\right) + P_x = 0 \\ h\left(\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}\right) + P_y = 0 \end{aligned}\right\}.$$

Substituting the expressions (13.2), we obtain the equations of equilibrium in the form

$$\left.\begin{aligned} Eh\left\{\frac{1}{1-\sigma^2}\frac{\partial^2 u_x}{\partial x^2} + \frac{1}{2(1+\sigma)}\frac{\partial^2 u_x}{\partial y^2} + \frac{1}{2(1-\sigma)}\frac{\partial^2 u_y}{\partial x \partial y}\right\} + P_x = 0 \\ Eh\left\{\frac{1}{1-\sigma^2}\frac{\partial^2 u_y}{\partial y^2} + \frac{1}{2(1+\sigma)}\frac{\partial^2 u_y}{\partial x^2} + \frac{1}{2(1-\sigma)}\frac{\partial^2 u_x}{\partial x \partial y}\right\} + P_y = 0 \end{aligned}\right\} \tag{13.4}$$

These equations can be written in the two-dimensional vector form

$$\boldsymbol{grad}\,\mathrm{div}\boldsymbol{u} - \frac{1-\sigma}{2}\boldsymbol{curl}\,\boldsymbol{curl}\boldsymbol{u} = -\frac{1-\sigma^2}{Eh}\boldsymbol{P}, \tag{13.5}$$

where all the vector operators are two-dimensional.

In particular, the equation of equilibrium in the absence of body forces is

$$\boldsymbol{grad}\,\mathrm{div}\boldsymbol{u} - \frac{1-\sigma}{2}\boldsymbol{curl}\,\boldsymbol{curl}\boldsymbol{u} = 0. \tag{13.6}$$

It differs from the equation of equilibrium for a plane deformation of a body infinite in the *z*-direction (§7) only by the sign of the coefficient (in accordance with (13.3)).[5] As for a plane deformation, we can introduce the ***stress function*** defined by

$$\left.\begin{aligned} \sigma_{xx} &= \frac{\partial^2 \chi}{\partial y^2} \\ \sigma_{xy} &= -\frac{\partial^2 \chi}{\partial x \partial y} \\ \sigma_{yy} &= \frac{\partial^2 \chi}{\partial x^2} \end{aligned}\right\}, \tag{13.7}$$

[5] A deformation homogeneous in the *z*-direction for which $\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$ everywhere is sometimes called a state of ***plane stress***, as distinct from a plane deformation, for which $u_{zx} = u_{zy} = u_{zz} = 0$ everywhere.

whereby we automatically satisfy the equations of equilibrium in the form

$$\left.\begin{aligned}\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}=0\\ \frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=0\end{aligned}\right\}.$$

The stress function, as before, satisfies the biharmonic equation, since for $\Delta\chi$ we have

$$\Delta\chi=\sigma_{xx}+\sigma_{yy}=\frac{E}{1-\sigma}(u_{xx}+u_{yy})=\frac{E}{1-\sigma}\operatorname{div}\boldsymbol{u};$$

this differs only by a factor from the result for a plane deformation.

It may be pointed out that the stress distribution in a plate deformed by given forces applied to its edges is independent of the elastic constants of the material. For these constants appear neither in the biharmonic equation satisfied by the stress function, nor in the formulae (13.7) which determine the components σ_{ik} from that function (nor, therefore, in the boundary conditions at the edges of the plate).

PROBLEMS

Problem 1. Determine the deformation of a plane disc rotating uniformly about an axis through its centre perpendicular to its plane.

Solution. The required solution differs only in the constant coefficients from the solution obtained in §7, Problem 5, for the plane deformation of a rotating cylinder. The radial displacement $u_r=u(r)$ is given by the formula

$$u=\frac{\rho\Omega^2(1-\sigma^2)}{8E}r\left(\frac{3+\sigma}{1\sigma}R^2-r^2\right).$$

This is the expression which gives that of §7, Problem 5, if the substitution (13.3) is made.

Problem 2. Determine the deformation of a semi-infinite plate (with a straight edge) under the action of a concentrated force in its plane, applied to a point on the edge.

Solution. We take polar co-ordinates, with the angle ϕ measured from the direction of the applied force; it takes values from $-(\pi/2+\alpha)$ to $\pi/2-\alpha$, where α is the angle between the direction of the force and the normal to the edge of the plate (Fig. 6). At every point of the edge except that where the force is applied (the origin) we must

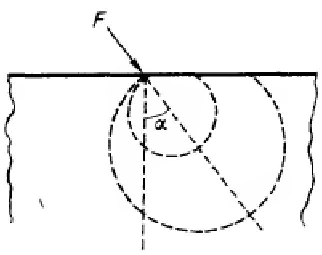

FIG. 6

have $\sigma_{\phi\phi} = \sigma_{r\phi} = 0$. Using the expressions for $\sigma_{\phi\phi}$ and $\sigma_{r\phi}$ obtained in §7, Problem 11, we find that the stress function must therefore satisfy the conditions

$$\frac{\partial \chi}{\partial r} = \text{constant}\,, \quad \frac{1}{r}\frac{\partial \chi}{\partial \phi} = \text{constant}\,, \text{ for } \phi = -\left(\frac{\pi}{2} + \alpha\right), \phi = \left(\frac{\pi}{2} - \alpha\right)$$

Both conditions are satisfied if $\chi = rf(\phi)$. With this substitution, the biharmonic equation

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial \phi^2}\right\}^2 \chi = 0$$

gives solutions for $f(\phi)$ of the forms $\sin\phi$, $\cos\phi$, $\phi\sin\phi$, $\phi\cos\phi$. The first two of these lead to stresses which are zero identically. The solution which gives the correct value for the force applied at the origin is

$$\chi = -\frac{F}{\pi} r\phi\sin\phi\,, \quad \sigma_{rr} = -\frac{2F}{\pi r}\cos\phi\,, \quad \sigma_{r\phi} = \sigma_{\phi\phi} = 0\,, \qquad (1)$$

where F is the force per unit thickness of the plate. For, projecting the internal stresses on directions parallel and perpendicular to the force $\boldsymbol{F}$, and integrating over a small semicircle centred at the origin (whose radius then tends to zero), we obtain

$$\int \sigma_{rr} r\cos\phi d\phi = -F\,,$$

$$\int \sigma_{rr} r\sin\phi d\phi = 0\,,$$

i.e., the values required to balance the external force applied at the origin.

Formulae (1) determine the required stress distribution. It is purely radial: only a radial compression force acts on any area perpendicular to the radius. The lines of equal stress are the circles $r = d\cos\phi$, which pass through the origin and whose centres lie on the line of action of the force $\boldsymbol{F}$ (Fig. 6).

The components of the strain tensor are $u_{rr} = \frac{\sigma_{rr}}{E}$, $u_{\phi\phi} = -\sigma\frac{\sigma_{rr}}{E}$, $u_{r\phi} = 0$. From these we find by integration (using the expressions (1.8) for the components u_{ik} in polar coordinates) the displacement vector

$$u_r = -\frac{2F}{\pi E}\log\frac{r}{a}\cos\phi - \frac{(1-\sigma)F}{\pi E}\phi\sin\phi\,,$$

$$u_\phi = \frac{2\sigma F}{\pi E}\sin\phi + \frac{2F}{\pi E}\log\frac{r}{a}\sin\phi + \frac{(1-\sigma)F}{\pi E}(\sin\phi - \phi\cos\phi)\,.$$

Here the constants of integration have been chosen so as to give zero displacement (translation and rotation) of the plate as a whole: an arbitrarily

chosen point at a distance a from the origin on the line of action of the force is assumed to remain fixed.

Using the solution obtained above, we can obtain the solution for any distribution of forces acting on the edge of the plate (cf. §8). It is, of course, inapplicable in the immediate neighbourhood of the origin.

Problem 3. Determine the deformation of an infinite wedge-shaped plate (of angle 2α) due to a force applied at its apex.

Solution. The stress distribution is given by formulae which differ from those of Problem 2 only in their normalisation. If the force acts along the mid-line of the wedge (F_1 in Fig. 7), we have

$$\sigma_{rr} = -\frac{F_1 \cos\phi}{r(\alpha + \frac{1}{2}\sin 2\alpha)}, \quad \sigma_{r\phi} = \sigma_{\phi\phi} = 0.$$

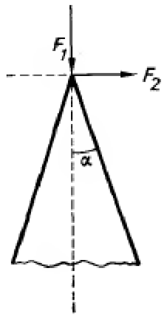

Fig. 7

If, on the other hand, the force acts perpendicular to this direction (F_2 in Fig. 7), then

$$\sigma_{rr} = -\frac{F_2 \cos\phi}{r(\alpha - \frac{1}{2}\sin 2\alpha)}.$$

In each case the angle ϕ is measured from the direction of the force.

Problem 4. Determine the deformation of a circular disc (of radius R) compressed by two equal and opposite forces Fh applied at the ends of a diameter (Fig. 8).

Solution. The solution is obtained by superposing three internal stress distributions. Two of these are

$$\sigma^{(1)}{}_{r_1 r_1} = -\frac{2F}{\pi r_1}\cos\phi_1,$$

$$\sigma^{(1)}{}_{r_1 r\phi_1} = \sigma^{(1)}{}_{\phi_1\phi_1} = 0,$$

$$\sigma^{(2)}{}_{r_2 r_2} = -\frac{2F}{\pi r_2}\cos\phi_2,$$

$$\sigma^{(2)}{}_{r_2 r\phi_2} = \sigma^{(2)}{}_{\phi_2\phi_2} = 0,$$

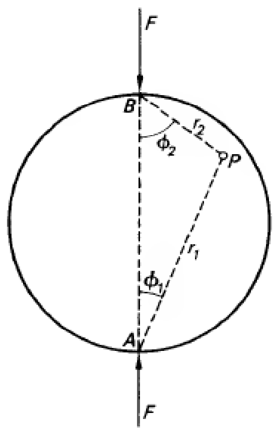

Fig. 8

where r_1, ϕ_1 and r_2, ϕ_2 are the polar co-ordinates of an arbitrary point P with origins at A and B, respectively. These are the stresses due to a normal force F applied to a point on the edge of a half-plane; see Problem 2.

The third distribution, $\sigma^{(2)}{}_{ik} = \frac{F}{\pi R}\delta_{ik}$, is a uniform extension of definite

intensity. For, if the point P is on the edge of the disc, we have $r_1 = 2R\cos\phi_1$, $r_2 = 2R\cos\phi_2$, so that $\sigma^{(1)}r_1 r_1 = \sigma^{(2)}r_2 r_2 = -\dfrac{F}{\pi R}$. Since the directions of r_1 and r_2 at this point are perpendicular, we see that the first two stress distributions give a uniform compression on the edge of the disc. These forces can be just balanced by the uniform tension given by the third system, so that the edge of the disc is free from stress, as it should be.

Problem 5. Determine the stress distribution in an infinite sheet with a circular aperture (of radius *R*) under uniform tension.

Solution. The uniform tension of a continuous sheet corresponds to stresses $\sigma^{(0)}{}_{xx} = T$, $\sigma^{(0)}{}_{yy} = \sigma^{(0)}{}_{xy} = 0$, where T is the tension force. These in turn correspond to the stress function

$$\chi^{(0)} = \frac{1}{2}Ty^2 = \frac{1}{2}Tr^2\sin^2\phi = \frac{1}{4}Tr^2(1-\cos 2\phi).$$

When there is a circular aperture (with the centre as the origin of polar co-ordinates r, ϕ), we seek the stress function in the form $\chi = \chi^{(0)} + \chi^{(1)}$, $\chi^{(1)} = f(r) + F(r)\cos 2\phi$. The integral of the biharmonic equation which is independent of ϕ is of the form $f(r) = ar^2\log r + br^2 + c\log r$, and in the integral proportional to $\cos 2\phi$ we have $F(r) = dr^2 + er^4 + \dfrac{g}{r^2}$. The constants are determined by the conditions $\sigma^{(1)}{}_{ik} = 0$ for $r = \infty$ and $\sigma_{rr} = \sigma_{r\phi} = 0$ for $r = R$. The result is

$$\chi^{(1)} = \frac{1}{2}TR^2\left\{-\log r + \left(1 - \frac{R^2}{2r^2}\right)\cos 2\phi\right\},$$

and the stress distribution is given by

$$\sigma_{rr} = \frac{1}{2}T\left(1-\frac{R^2}{r^2}\right)\left\{1+\left(1-\frac{3R^2}{r^2}\right)\cos 2\phi\right\},$$

$$\sigma_{\phi\phi} = \frac{1}{2}T\left\{1+\frac{R^2}{r^2} - \left(1+\frac{3R^4}{r^4}\right)\cos 2\phi\right\},$$

$$\sigma_{r\phi} = -\frac{1}{2}T\left(1+\frac{2R^2}{r^2}-\frac{3R^4}{r^4}\right)\sin 2\phi.$$

In particular, at the edge of the aperture we have $\sigma_{\phi\phi} = T(1-2\cos 2\phi)$, and for $\phi = \pm\pi/2$, $\delta_{\phi\phi} = 3T$, i.e., three times the stress at infinity (cf. §7, Problem 12)

§14. Large deflections of plates

The theory of the bending of thin plates given in §§11-13 is applicable only to fairly small deflections. Anticipating the result given below, it may be mentioned here that the condition for that theory to be applicable is that the deflection ζ is small compared with the thickness *h* of the plate. Let us now derive the equations of equilibrium for a plate undergoing large deflections. The deflection ζ is not now supposed small compared with *h*. It should be emphasised, however, that the deformation itself must still be small, in the sense that the components of the strain tensor must be small. In practice, this usually implies the condition $\zeta << l$, i.e., the deflection must be small compared with the dimension *l* of the plate.

The bending of a plate in general involves a stretching of it.[6] For small deflections this stretching can be neglected. For large deflections, however, this is not possible ; there is therefore no neutral surface in a plate undergoing large deflections. The existence of a stretching which accompanies the bending is peculiar to plates, and distinguishes them from thin rods, which can undergo large deflections without any general stretching. This property of plates is a purely geometrical one. For example, let a flat circular plate be bent into a segment of a spherical surface. If the bending is such that the circumference of the plate remains constant, its diameter must increase. If the diameter is constant, on the other hand, the circumference must be reduced.

The energy (11.6), which may be called the ***pure bending energy***, is only the part of the total energy which arises from the non-uniformity of the tension and compression through the thickness of the plate, in the absence of any general stretching. The total energy includes also a part due to this general stretching; this may be called the ***stretching energy***.

Deformations consisting of pure bending and pure stretching have been considered in §§11-13. We can therefore use the results obtained in these sections. It is not necessary to consider the structure of the plate across its thickness, and we can regard it as a two-dimensional surface of negligible thickness.

We first derive an expression for the strain tensor pertaining to the stretching of a plate (regarded as a surface) which is simultaneously bent and stretched in its plane. Let ***u*** be the two-dimensional displacement vector (with components u_x, u_y) for pure stretching; ζ, as before, denotes the transverse displacement in bending. Then the element of length

[6] An exception is, for instance, the bending of a flat plate into a cylindrical surface.

$dl = \sqrt{dx^2 + dy^2}$ of the undeformed plate is transformed by the deformation into an element dl', whose square is given by $dl' = (dx + du_x)^2 + (dy + du_y)^2 + d\zeta^2$. Putting here $du_x = \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy$, and similarly for du_y and $d\zeta$, we obtain to within higher-order terms $dl'^2 = dl^2 + 2u_{\alpha\beta}dx_\alpha dx_\beta$, where the two-dimensional strain tensor is defined as

$$u_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right) + \frac{1}{2}\frac{\partial \zeta}{\partial x_\alpha}\frac{\partial \zeta}{\partial x_\beta}. \qquad (14.1)$$

(In this and the following sections, Greek suffixes take the two values *x* and *y*; as usual, summation over repeated suffixes is understood.) The terms quadratic in the derivatives of u_α are here omitted; the same cannot, of course, be done with the derivatives of ζ, since there are no corresponding first-order terms.

The stress tensor $\sigma_{\alpha\beta}$ due to the stretching of the plate is given by formula (13.2), in which $u_{\alpha\beta}$ must be replaced by the total strain tensor given by formula (14.1). The pure bending energy is given by formula (11.6), and can be written $\int \Psi_1(\zeta)dxdy$, where $\Psi_1(\zeta)$ denotes the integrand in (11.6). The stretching energy per unit volume of the plate is, by the general formulae, $\frac{1}{2}u_{\alpha\beta}\sigma_{\alpha\beta}$. The energy per unit surface area is obtained by multiplying by *h*, so that the total stretching energy can be written $\int \Psi_2(u_{\alpha\beta})df$, where

$$\Psi_2 = \frac{1}{2}hu_{\alpha\beta}\sigma_{\alpha\beta}. \qquad (14.2)$$

Thus the total free energy of a plate undergoing large deflections is

$$F_{pl} = \int \{\Psi_1(\zeta) + \Psi_2(u_{\alpha\beta})\}df. \qquad (14.3)$$

Before deriving the equations of equilibrium, let us estimate the relative magnitude of the two parts of the energy. The first derivatives of ζ are of the order of ζ / l, where *l* is the dimension of the plate, and the second derivatives are of the order of ζ / l^2. Hence we see from (11.6) that $\Psi_1 \sim \frac{Eh^3\zeta^2}{l^4}$. The order of magnitude of the tensor components $u_{\alpha\beta}$ is

ζ^2/l^2, and so $\Psi_2 \sim \dfrac{Eh\zeta^4}{l^4}$. A comparison shows that the neglect of Ψ_2 in the approximate theory of the bending of plates is valid only if $\zeta^2 << h^2$.

The condition of minimum energy is $\delta F + \delta U = 0$, where U is the potential energy in the field of the external forces. We shall suppose that the external stretching forces, if any, can be neglected in comparison with the bending forces. (This is always valid unless the stretching forces are very large, since a thin plate is much more easily bent than stretched.) Then we have for δU the same expression as in §12: $\delta U = -\int P\delta\zeta df$, where P is the external force per unit area of the plate. The variation of the integral $\int \Psi_1 df$ has already been calculated in §12, and is

$$\delta\int \Psi_1 df = \frac{Eh^3}{12(1-\sigma^2)}\int \Delta^2\zeta\delta\zeta df .$$

The contour integrals in (12.3) are omitted, since they give only the boundary conditions on the equation of equilibrium, and not that equation itself, which is of interest here.

Finally, let us calculate the variation of the integral $\int \Psi_2 df$. The variation must be taken both with respect to the components of the vector $\boldsymbol{u}$ and with respect to ζ. We have

$$\delta\int \Psi_2 df = \int \frac{\partial \Psi_2}{\partial u_{\alpha\beta}}\delta u_{\alpha\beta} df .$$

The derivatives of the free energy per unit volume with respect to $u_{\alpha\beta}$ are $\sigma_{\alpha\beta}$; hence $\dfrac{\partial \Psi_2}{\partial u_{\alpha\beta}} = h\sigma_{\alpha\beta}$. Substituting also for $u_{\alpha\beta}$ the expression (14.1), we obtain

$$\begin{aligned}\delta\int \Psi_2 df &= h\int \sigma_{\alpha\beta}\delta u_{\alpha\beta} df \\ &= \frac{1}{2}h\int \sigma_{\alpha\beta}\left\{\frac{\partial\delta u_\alpha}{\partial x_\beta} + \frac{\partial\delta u_\beta}{\partial x_\alpha} + \frac{\partial\zeta}{\partial x_\alpha}\frac{\partial\delta\zeta}{\partial x_\beta} + \frac{\partial\delta\zeta}{\partial x_\alpha}\frac{\partial\zeta}{\partial x_\beta}\right\}df ,\end{aligned}$$

or, by the symmetry of $\sigma_{\alpha\beta}$,

$$\delta\int \Psi_2 df = h\int \sigma_{\alpha\beta}\left\{\frac{\partial\delta u_\alpha}{\partial x_\beta} + \frac{\partial\delta\zeta}{\partial x_\beta}\frac{\partial\zeta}{\partial x_\alpha}\right\}df .$$

Integrating by parts, we obtain

$$\delta\int \Psi_2 df = -h\int\left\{\frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta}\delta u_\alpha + \frac{\partial}{\partial x_\beta}\left(\sigma_{\alpha\beta}\frac{\partial\zeta}{\partial x_\alpha}\right)\delta\zeta\right\}df .$$

The contour integrals along the circumference of the plate are again omitted.

Collecting the above results, we have

$$\delta F_{pl} + \delta U = \iint \left[\left\{ \frac{Eh^3}{12(1-\sigma^2)} \Delta^2 \zeta - h \frac{\partial}{\partial x_\beta} \left(\sigma_{\alpha\beta} \frac{\partial \zeta}{\partial x_\alpha} \right) - P \right\} \delta\zeta - h \frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} \delta u_\alpha \right] df = 0$$

In order that this relation should be satisfied identically, the coefficients of $\delta\zeta$ and δu_α must each be zero. Thus we obtain the equations

$$\frac{Eh^3}{12(1-\sigma^2)} \Delta^2 \zeta - h \frac{\partial}{\partial x_\beta} \left(\sigma_{\alpha\beta} \frac{\partial \zeta}{\partial x_\alpha} \right) = P \,, \qquad (14.4)$$

$$\frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} = 0 \,. \qquad (14.5)$$

The unknown functions here are the two components u_x, u_y of the vector $\boldsymbol{u}$ and the transverse displacement ζ. The solution of the equations gives both the form of the bent plate (i.e., the function $\zeta(x, y)$) and the extension resulting from the bending. Equations (14.4) and (14.5) can be somewhat simplified by introducing the function χ related to $\sigma_{\alpha\beta}$ by (13.7). Equation (14.4) then becomes

$$\frac{Eh^3}{12(1-\sigma^2)} \Delta^2 \zeta - h \left(\frac{\partial^2 \chi}{\partial y^2} \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \chi}{\partial x^2} \frac{\partial^2 \zeta}{\partial y^2} - 2 \frac{\partial^2 \chi}{\partial x \partial y} \frac{\partial^2 \zeta}{\partial x \partial y} \right) = P \,. \qquad (14.6)$$

Equations (14.5) are satisfied automatically by the expressions (13.7). Hence another equation is needed ; this can be obtained by eliminating u_α from the relations (13.7) and (13.2).

To do this, we proceed as follows. We express $u_{\alpha\beta}$ in terms of $\sigma_{\alpha\beta}$, obtaining from (13.2)

$$\begin{cases} u_{xx} = \dfrac{\sigma_{xx} - \sigma\sigma_{yy}}{E} \\ u_{yy} = \dfrac{\sigma_{yy} - \sigma\sigma_{xx}}{E} \\ u_{xy} = \dfrac{(1+\sigma)\sigma_{xy}}{E} \end{cases} .$$

Substituting here the expression (14.1) for $u_{\alpha\beta}$, and (13.7) for $\sigma_{\alpha\beta}$, we find the equations

$$\begin{cases} \dfrac{\partial u_x}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial \zeta}{\partial x}\right)^2 = \dfrac{1}{E}\left(\dfrac{\partial^2 \chi}{\partial y^2} - \sigma \dfrac{\partial^2 \chi}{\partial x^2}\right) \\ \dfrac{\partial u_y}{\partial y} + \dfrac{1}{2}\left(\dfrac{\partial \zeta}{\partial y}\right)^2 = \dfrac{1}{E}\left(\dfrac{\partial^2 \chi}{\partial x^2} - \sigma \dfrac{\partial^2 \chi}{\partial y^2}\right) \\ \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} + \dfrac{\partial \zeta}{\partial x}\dfrac{\partial \zeta}{\partial y} = -\dfrac{2(1+\sigma)}{E}\dfrac{\partial^2 \chi}{\partial x \partial y} \end{cases} .$$

We take $\dfrac{\partial^2}{\partial y^2}$ of the first, $\dfrac{\partial^2}{\partial x^2}$ of the second, $-\dfrac{\partial^2}{\partial x \partial y}$ of the third, and add. The terms in u_x and u_y then cancel, and we have the equation

$$\Delta^2 \chi + E\left\{\frac{\partial^2 \chi}{\partial x^2}\frac{\partial^2 \zeta}{\partial y^2} - \left(\frac{\partial^2 \zeta}{\partial x \partial y}\right)^2\right\} = 0 . \qquad (14.7)$$

Equations (14.6) and (14.7) form a complete system of equations for large deflections of thin plates (A. Föppl 1907). These equations are very complicated, and cannot be solved exactly, even in very simple cases. It should be noticed that they are non-linear.

We may mention briefly a particular case of deformations of thin plates, that of membranes. A ***membrane*** is a thin plate subject to large external stretching forces applied at its circumference. In this case we can neglect the additional longitudinal stresses caused by bending of the plate, and therefore suppose that the components of the tensor $\sigma_{\alpha\beta}$ are simply equal to the constant external stretching forces. In equation (14.4) we can then neglect the first term in comparison with the second, and we obtain the equation of equilibrium

$$h\sigma_{\alpha\beta}\frac{\partial^2 \zeta}{\partial x_\alpha \partial x_\beta} + P = 0 , \qquad (14.8)$$

with the boundary condition that $\zeta = 0$ at the edge of the membrane. This is a linear equation. The case of isotropic stretching, when the extension of the membrane is the same in all directions, is particularly simple. Let T be the absolute magnitude of the stretching force per unit length of the edge of the membrane. Then $h\sigma_{\alpha\beta} = T\delta_{\alpha\beta}$, and we obtain the equation of equilibrium in the form

$$T\Delta\zeta + P = 0 . \qquad (14.9)$$

PROBLEMS

Problem 1. Determine the deflection of a plate as a function of the force on it when $\zeta >> h$.

Solution. An estimate of the terms in equation (14.7) shows that $\chi \sim E\zeta^2$.

For $\zeta >> h$, the first term in (14.6) is small compared with the second, which is of the order of magnitude $\frac{h\zeta\chi}{l^4} \sim \frac{Eh\zeta^3}{l^4}$ (*l* being the dimension of the plate). If this is comparable with the external force *P*, we have $\zeta \sim \left(\frac{l^4 P}{Eh}\right)^{1/2}$. Hence, in particular, we see that ζ is proportional to the cube root of the force.

Problem 2. Determine the deformation of a circular membrane (of radius *R*) placed horizontally in a gravitational field.

Solution. We have $P = \rho g h$; in polar co-ordinates, (14.9) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\zeta}{dr}\right) = -\frac{\rho g h}{T}.$$

The solution finite for $r = 0$ and zero for $r = R$ is

$$\zeta = \frac{\rho g h(R^2 - r^2)}{4T}.$$

§15. Deformations of shells

In discussing hitherto the deformations of thin plates, we have always assumed that the plate is flat in its undeformed state. However, deformations of plates which are curved in the undeformed state (called ***shells***) have properties which are fundamentally different from those of the deformations of flat plates.

The stretching which accompanies the bending of a flat plate is a second-order effect in comparison with the bending deflection itself. This is seen, for example, from the fact that the strain tensor (14.1), which gives this stretching, is quadratic in ζ. The situation is entirely different in the deformation of shells: here the stretching is a first-order effect, and therefore is important even for small bending deflections. This property is most easily seen from a simple example, that of the uniform stretching of a spherical shell. If every point undergoes the same radial displacement ζ, the length of the equator increases by $2\pi\zeta$,. The relative extension is $\frac{2\pi\zeta}{2\pi R} = \frac{\zeta}{R}$, and hence the strain tensor also is proportional to the first power of ζ. This effect tends to zero as $R \to \infty$, i.e., as the curvature tends to zero, and is therefore due to the curvature of the shell.

Let *R* be the order of magnitude of the radius of curvature of the shell, which is usually of the same order as its dimension. Then the strain tensor for

the stretching which accompanies the bending is of the order of $\frac{\zeta}{R}$, the corresponding stress tensor is $\sim\frac{E\zeta}{R}$, and the deformation energy per unit area is, by (14.2), of the order of $Eh\left(\frac{\zeta}{R}\right)^2$. The pure bending energy, on the other hand, is of the order of $\frac{Eh^3\zeta^2}{R^4}$, as before. We see that the ratio of the two is of the order of $\left(\frac{R}{h}\right)^2$, i.e., it is very large. It should be emphasized that this is true whatever the ratio of the bending deflection ζ to the thickness h, whereas in the bending of flat plates the stretching was important only for $\zeta \geq h$.

In some cases there may be a special type of bending of the shell in which no stretching occurs. For example, a cylindrical shell (open at both ends) can be deformed without stretching if all the generators remain parallel (i.e., if the shell is, as it were, compressed along some generator). Such deformations without stretching are geometrically possible if the shell has free edges (i.e., is not closed) or if it is closed but its curvature has opposite signs at different points. For example, a closed spherical shell cannot be bent without being stretched, but if a hole is cut in it (the edge of the hole not being fixed), then such a deformation becomes possible. Since the pure bending energy is small compared with the stretching energy, it is clear that, if any given shell permits deformation without stretching, then such deformations will, in general, actually occur when arbitrary external forces act on the shell. The requirement that the bending is unaccompanied by stretching places considerable restrictions on the possible displacements u_α. These restrictions are purely geometrical, and can be expressed as differential equations, which must be contained in the complete system of equilibrium equations for such deformations. We shall not pause to discuss this question further.

If, however, the deformation of the shell involves stretching, then the tensile stresses are in general large compared with the bending stresses, which may be neglected. Shells for which this is done are called ***membranes***.

The stretching energy of a shell can be calculated as the integral

$$F_{pl} = \frac{1}{2} h \int u_{\alpha\beta} \sigma_{\alpha\beta} df, \qquad (15.1)$$

taken over the surface. Here $u_{\alpha\beta}$ ($\alpha, \beta = 1,2$) is the two-dimensional strain

tensor in the appropriate curvilinear co-ordinates, and the stress tensor $\sigma_{\alpha\beta}$ is related to $u_{\alpha\beta}$ by formulae (13.2), which can be written, in two-dimensional tensor notation, as

$$\sigma_{\alpha\beta} = \frac{E\left[(1-\sigma)u_{\alpha\beta} + \sigma\delta_{\alpha\beta}u_{\gamma\gamma}\right]}{1-\sigma^2}. \qquad (15.2)$$

A case requiring special consideration is that where the shell is subjected to the action of forces applied to points or lines on the surface and directed through the shell. These may be, in particular, the reaction forces on the shell at points (or lines) where it is fixed. The concentrated forces result in a bending of the shell in small regions near the points where they are applied; let *d* be the dimension of such a region for a force *f* applied at a point (so that its area is of the order of d^2). Since the deflection ζ varies considerably over a distance *d*, the bending energy per unit area is of the order of $\frac{Eh^3\zeta^2}{d^4}$, and the total bending energy (over an area $\sim d^2$) is of the order of $\frac{Eh^3\zeta^2}{d^2}$. The strain tensor for the stretching is again $\sim \frac{\zeta}{R}$, and the total stretching energy due to the concentrated forces is $\sim \frac{Eh\zeta^2 d^2}{R^2}$. Since the bending energy increases and the stretching energy decreases with decreasing *d*, it is clear that both energies must be taken into account in determining the deformation near the point of application of the forces. The size *d* of the region of bending is given in order of magnitude by the condition that the sum of these energies is a minimum, whence

$$d \sim \sqrt{hR}\,. \qquad (15.3)$$

The energy $\sim \frac{Eh^2\zeta^2}{R}$. Varying this with respect to ζ and equating the result to the work done by the force *f*, we find the deflection $\zeta \sim \frac{fR}{Eh^2}$.

However, if the forces acting on the shell are sufficiently large, the shape of the shell may be considerably changed by bulges which form in it. The determination of the deformation as a function of the applied loads requires special investigation in this unusual case.[7]

Let a convex shell (with edges fixed in such a way that it is geometrically rigid) be subjected to the action of a large concentrated force *f* along the

[7] The results given below are due to A. V. Pogorelov (1960). A more precise analysis of the problem together with some similar ones is given in his book Teoriya obolochek pri zakriticheskikh deformatsiyakh (Theory of Shells at Supercritical Deformations), Moscow 1965.

inward normal. For simplicity we shall assume that the shell is part of a sphere of radius *R*. The region of the bulge will be a spherical cap which is almost a mirror image of its original shape (Fig. 9 shows a meridional section of the shell). The problem is to determine the size of the bulge as a function of the force.

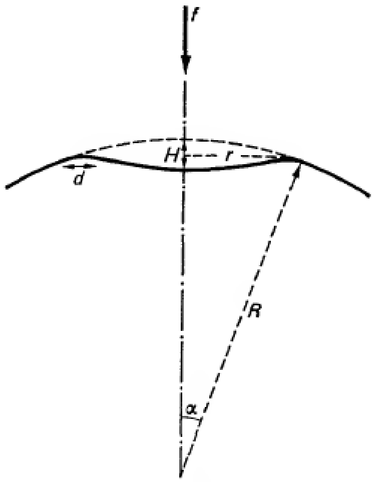

FIG. 9

The major part of the elastic energy is concentrated in a narrow strip near the edge of the bulge, where the bending of the shell is relatively large ; we shall call this the ***bending strip*** and denote its width by *d*. This energy may be estimated, assuming that the radius *r* of the bulge region is much less than *R*, so that the angle $\alpha << 1$ (Fig. 9). Then $r = R\sin\alpha \sim R\alpha$, and the depth of the bulge $H = 2R(1-\cos\alpha) \sim R\alpha^2$. Let ζ denote the displacement of points on the shell in the bending strip. Just as previously, we find that the energies of bending along the meridian and of stretching along the circle of latitude[8] per unit surface area are, respectively, in order of magnitude, $\frac{Eh^3\zeta^2}{d^4}$ and $\frac{Eh\zeta^2}{R^2}$. The order of magnitude of the displacement ζ is in this case determined geometrically: the direction of the meridian changes by an angle $\sim\alpha$ over the width *d*, and so $\zeta \sim \alpha d \sim \frac{rd}{R}$. Multiplying by the area of the bending strip ($\sim rd$), we obtain the energies $\frac{Eh^3r^3}{R^2d}$ and $\frac{Ehd^3r^3}{R^4}$. The condition for their sum to be a minimum again gives

[8] The curvature of the shell does not affect the bending along the meridian in the first approximation, so that this bending occurs without any general stretching along the meridian, as in the cylindrical bending of a flat plate.

$d \sim \sqrt{hR}$, and the total elastic energy is then $\sim Er^3\left(\frac{h}{R}\right)^{5/2}$, or[9]

$$\text{constant} \times \frac{Eh^{5/2}H^{3/2}}{R}. \qquad (15.4)$$

In this derivation it has been assumed that $d << r$; formula (15.4) is therefore valid if the condition[10]

$$\frac{Rh}{r^2} << 1 \qquad (15.5)$$

holds.

The required relation between the depth of the bulge H and the applied force f is obtained by equating f to the derivative of the energy (15.4) with respect to H. Thus we find

$$H \sim \frac{f^2R^2}{E^2h^5}. \qquad (15.6)$$

It should be noticed that this relation is non-linear.

Finally, let the deformation (bulge) of the shell occur under a uniform external pressure p). In this case the work done is $p\Delta V$, where $\Delta V \sim Hr^2 \sim H^2R$ is the change in the volume within the shell when the bulge occurs. Equating to zero the derivative with respect to H of the total free energy (the difference between the elastic energy (15.4) and this work), we obtain

$$H \sim \frac{h^5E^2}{R^4p^2}. \qquad (15.7)$$

The inverse variation (H increasing when p decreases) shows that in this case the bulge is unstable. The value of H given by formula (15.7) corresponds to unstable equilibrium for a given p: bulges with larger values of H grow of their own accord, while smaller ones shrink (it is easy to verify that (15.7) corresponds to a maximum and not a minimum of the total free energy). There is a critical value p_{cr} of the external load beyond which even small changes in the shape of the shell increases in size spontaneously. This value may be defined as that which gives $H \sim h$ in (15.7):

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[9] A more accurate calculation shows that the constant coefficient is $1.2(1-\sigma^2)^{-3/4}$.

[10] When a bulge is formed, the outer layers of a spherical segment become the inner ones and are therefore compressed, while the inner layers become the outer ones and are stretched. The relative extension (or compression) $\sim \frac{h}{R}$, and so the corresponding total energy in the region of the bulge $\sim E\left(\frac{h}{R}\right)^2 hr^2$. With the condition (15.5) it is in fact small in comparison with the energy (15.4) in the bending strip.

$$p_{cr} \sim \frac{Eh^2}{R^2}\,. \tag{15.8}$$

We shall add to the above brief account of shell theory only a few simple examples in the following Problems.

PROBLEMS

Problem 1. Derive the equations of equilibrium for a spherical shell (of radius R) deformed symmetrically about an axis through its centre.

Solution. We take as two-dimensional co-ordinates on the surface of the shell the angles θ, ϕ in a system of spherical polar co-ordinates, whose origin is at the centre of the sphere and polar axis along the axis of symmetry of the deformed shell.

Let P_r be the external radial force per unit surface area. This force must be balanced by a radial resultant of internal stresses acting tangentially on an element of the shell. The condition is

$$\frac{h(\sigma_{\phi\phi} + \sigma_{\theta\theta})}{R} = P_r\,. \tag{1}$$

This equation is exactly analogous to Laplace's equation for the pressure difference between two media caused by surface tension at the surface of separation.

Next, let $Q_z(\theta)$ be the resultant of all external forces on the part of the shell lying above the co-latitude θ; this resultant is along the polar axis. The force $Q_z(\theta)$ must be balanced by the projection on the polar axis of the stresses $2\pi R h \sigma_{\theta\theta} \sin\theta$ acting on the cross-section $2\pi R h \sin\theta$ of the shell at that latitude. Hence

$$2\pi R h \sigma_{\theta\theta} \sin^2\theta = Q_z(\theta)\,. \tag{2}$$

Equations (1) and (2) determine the stress distribution, and the strain tensor is then given by the formulae

$$\left.\begin{aligned} u_{\theta\theta} &= \frac{\sigma_{\theta\theta} - \sigma\sigma_{\phi\phi}}{E} \\ u_{\phi\phi} &= \frac{\sigma_{\phi\phi} - \sigma\sigma_{\theta\theta}}{E} \\ u_{\theta\phi} &= 0 \end{aligned}\right\}. \tag{3}$$

Finally, the displacement vector is obtained from the equations

$$\left.\begin{aligned} u_{\theta\theta} &= \frac{1}{R}\left(\frac{du_\theta}{d\theta} + u_r\right) \\ u_{\theta\phi} &= \frac{1}{R}\left(u_\theta \cot\theta + u_r\right) \end{aligned}\right\}. \tag{4}$$

Problem 2. Determine the deformation under its own weight of a hemispherical shell convex upwards, the edge of which moves freely on a horizontal support (Fig. 10).

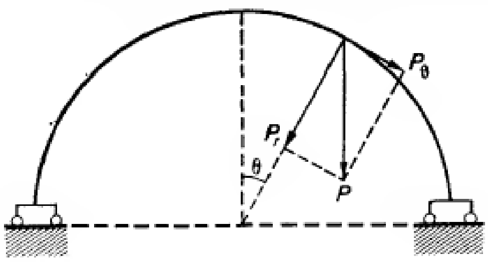

Fig. 10

Solution. We have $P_r = -\rho g h \cos\theta$, $Q_z = -2\pi R^2 \rho g h(1-\cos\theta)$; Q_z is the total weight of the shell above the circle of co-latitude θ. From (1) and (2) of Problem 1 we find

$$\begin{cases} \sigma_{\theta\theta} = -\dfrac{R\rho g}{1+\cos\theta} \\ \sigma_{rr} = R\rho g\left(\dfrac{1}{1+\cos\theta} - \cos\theta\right) \end{cases}.$$

From (3) we calculate $u_{\phi\phi}$ and $u_{\theta\theta}$, and then obtain u_θ and u_r from (4); the constant in the integration of the first equation (4) is chosen so that for $\theta = \pi/2$ we have $u_\theta = 0$. The result is

$$\begin{cases} u_\theta = \dfrac{R^2\rho g(1+\sigma)}{E}\left\{\dfrac{\cos\theta}{1+\cos\theta} + \log(1+\cos\theta)\right\}\sin\theta \\ u_r = \dfrac{R^2\rho g(1+\sigma)}{E}\left\{1 - \dfrac{2+\sigma}{1+\sigma}\cos\theta - \cos\theta\log(1+\cos\theta)\right\} \end{cases}.$$

The value of u_r for $\theta = \pi/2$ gives the horizontal displacement of the support.

Problem 3. Determine the deformation of a hemispherical shell with clamped edges, convex downwards and filled with liquid (Fig. 11); the weight of the shell itself can be neglected in comparison with that of the liquid.

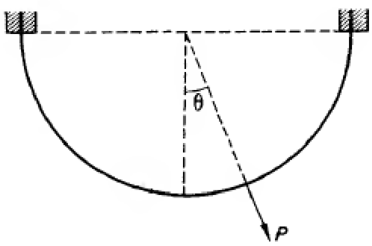

Fig. 11

Solution. We have

$$P_r = \rho_0 gR\cos\theta\,,\quad P_\theta = 0\,,$$

$$Q_z = 2\pi R^2\int_0^\theta P_r\cos\theta\sin\theta d\theta = \frac{2}{3}\pi R^3\rho_0 g(1-\cos^3\theta)\,,$$

where ρ_0 is the density of the liquid. We find from (1) and (2) of Problem 1

$$\begin{cases}\sigma_{\theta\theta} = -\dfrac{R^2\rho_0 g}{3h}\cdot\dfrac{1-\cos^3\theta}{\sin^2\theta}\\[2ex] \sigma_{\phi\phi} = \dfrac{R^2\rho_0 g}{3h}\cdot\dfrac{-1+3\cos\theta-2\cos^3\theta}{\sin^2\theta}\end{cases}.$$

The displacements are

$$\begin{cases}u_\theta = -\dfrac{R^3\rho_0 g(1+\sigma)}{3Eh}\left\{\dfrac{\cos\theta}{1+\cos\theta}+\log(1+\cos\theta)\right\}\sin\theta\\[2ex] u_r = -\dfrac{R^3\rho_0 g(1+\sigma)}{3Eh}\left\{1-\dfrac{3}{1+\sigma}\cos\theta-\cos\theta\log(1+\cos\theta)\right\}\end{cases}.$$

For $\theta=\pi/2$, u_r is not zero as it should be. This means that the shell is actually so severely bent near the clamped edge that the above solution is invalid.

Problem 4. A shell in the form of a spherical cap rests on a fixed support (Fig. 12). Determine the bending resulting from the weight Q of the shell.

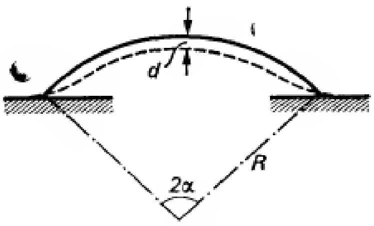

FIG. 12

Solution. The main deformation occurs near the edge, which is bent as shown by the dashed line in Fig. 12. The displacement u_θ is small compared with the radial displacement $u_r \equiv \zeta$. Since ζ decreases rapidly as we move away from the supported edge, the deformation can be regarded as that of a long flat plate (of length $2\pi R\sin\alpha$). This deformation is composed of a bending and a stretching of the plate. The relative extension at each point is ζ/R (R being the radius of the shell), and therefore the stretching energy is $\dfrac{E\zeta^2}{2R^2}$ per unit volume. Using as the independent variable the distance x from the line of support, we have for the total stretching energy

$$F_{1,pl} = 2\pi R \sin\alpha \frac{Eh}{2R^2} \int \zeta^2 dx .$$

The bending energy is

$$F_{2,pl} = 2\pi R \sin\alpha \frac{Eh^3}{24(1-\sigma^2)} \int \left(\frac{d^2\zeta}{dx^2} \right)^2 dx .$$

Varying the sum $F_{pl} = F_{1,pl} + F_{2,pl}$ with respect to ζ, we obtain

$$\frac{d^4\zeta}{dx^4} + \frac{12(1-\sigma^2)}{h^2R^2}\zeta = 0 .$$

For $x \to \infty$, ζ must tend to zero, and for $x = 0$ we must have the boundary conditions of zero moment of the forces ($\zeta'' = 0$) and equality of the normal force and the corresponding component of the force of gravity:

$$2\pi R \sin\alpha \frac{Eh^3}{12(1-\sigma^2)} \zeta''' = Q\cos\alpha .$$

The solution which satisfies these conditions is $\zeta = Ae^{-\kappa x} \cos\kappa x$, where

$$\kappa = \left[\frac{3(1-\sigma^2)}{h^2R^2} \right]^{1/4}, \quad A = \frac{Q\cot\alpha}{Eh} \left[\frac{3R^2(1-\sigma^2)}{8\pi h^2} \right]^{1/4} .$$

The bending of the shell is

$$d = \zeta(0)\cos\alpha = A\cos\alpha .$$

§16. Torsion of rods

Let us now consider the deformation of thin rods. This differs from all the cases hitherto considered, in that the displacement vector ***u*** may be large even for small strains, i.e., when the tensor u_{ik} is small.[11] For example, when a long thin rod is slightly bent, its ends may move a considerable distance, even though the relative displacements of neighbouring points in the rod are small.

There are **two types** of deformation of a rod which may be accompanied by a large displacement of certain parts of it. One of these consists in **bending** the rod, and the other in **twisting** it. We shall begin by considering the latter case.

A ***torsional deformation*** is one in which, although the rod remains straight, each transverse section is rotated through some angle relative to those below it. If the rod is long, even a slight torsion causes sufficiently distant cross-sections to turn through large angles. The generators on the

[11] The only exception is a simple extension of a rod without change of shape, in which case the vector ***u*** is always small if the tensor u_{ik} is small, i.e., if the extension is small.

sides of the rod, which are parallel to its axis, become helical in form under torsion.

Let us consider a thin straight rod of arbitrary cross-section. We take a co-ordinate system with the z-axis along the axis of the rod and the origin somewhere inside the rod. We use also the ***torsion angle*** τ, which is the angle of rotation per unit length of the rod. This means that two neighbouring cross-sections at a distance dz will rotate through a relative angle $d\phi = \tau dz$ (so that $\tau = \frac{d\phi}{dz}$). The torsional deformation itself, i.e., the relative displacement of adjoining parts of the rod, is assumed small. The condition for this to be so is that the relative angle turned through by cross-sections of the rod at a distance apart of the order of its transverse dimension R is small, i.e.,

$$\tau R << 1 . \tag{16.1}$$

Let us examine a small portion of the length of the rod near the origin, and determine the displacements $\boldsymbol{u}$ of the points of the rod in that portion. As the undisplaced cross-section we take that given by the xy-plane. When a radius vector $\boldsymbol{r}$ turns through a small angle $\delta\phi$, the displacement of its end is given by

$$\delta\boldsymbol{r} = \boldsymbol{\delta\varphi} \times \boldsymbol{r} , \tag{16.2}$$

where $\boldsymbol{\delta\varphi}$ is a vector whose magnitude is the angle of rotation and whose direction is that of the axis of rotation. In the present case, the rotation is about the z-axis, and for points of co-ordinate z the angle of rotation relative to the xy-plane is τz (since τ can be regarded as a constant in some region near the origin). Then formula (16.2) gives for the components u_x, u_y of the displacement vector

$$u_x = -\tau z y , \quad u_y = \tau z x . \tag{16.3}$$

When the rod is twisted, the points in it in general undergo a displacement along the z-axis also. Since for $\tau = 0$ this displacement is zero, it may be supposed proportional to τ when τ is small. Thus

$$u_z = \tau\psi(x, y) , \tag{16.4}$$

where $\psi(x, y)$ is some function of x and y, called the ***torsion function***. As a result of the deformation described by formulae (16.3) and (16.4), each cross-section of the rod rotates about the z-axis, and also becomes curved instead of plane. It should be noted that, by taking the origin at a particular point in the xy-plane, we "fix" a certain point in the cross-section of the rod in such a way that it cannot move in that plane (but it can move in the z-direction). A different choice of origin would not, of course, affect the

torsional deformation itself, but would give only an unimportant displacement of the rod as a whole.

Knowing ***u***, we can find the components of the strain tensor. Since ***u*** is small in the region under consideration, we can use the formula

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right).$$

The result is

$$u_{xx} = u_{yy} = u_{xy} = u_{zz} = 0,$$

$$u_{xz} = \frac{\tau}{2}\left(\frac{\partial \psi}{\partial x} - y\right), \quad u_{yz} = \frac{\tau}{2}\left(\frac{\partial \psi}{\partial y} + x\right). \quad (16.5)$$

It should be noticed that $u_{ll} = 0$; in other words, torsion does not result in a change in volume, i.e., it is a pure shear deformation.

For the components of the stress tensor we find

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0,$$

$$\sigma_{xz} = 2\mu u_{xz} = \mu\tau\left(\frac{\partial \psi}{\partial x} - y\right), \quad \sigma_{yz} = 2\mu u_{yz} = \mu\tau\left(\frac{\partial \psi}{\partial y} + x\right).$$

(16.6)

Here it is more convenient to use the ***modulus of rigidity*** μ in place of E and σ. Since only σ_{xz} and σ_{yz} are different from zero, the general equations of equilibrium $\frac{\partial \sigma_{ik}}{\partial x_k} = 0$ reduce to

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0. \quad (16.7)$$

Substituting (16.6), we find that the torsion function must satisfy the equation

$$\Delta\psi = 0, \quad (16.8)$$

where Δ is the two-dimensional Laplacian.

It is rather more convenient, however, to use a different auxiliary function $\chi(x, y)$, defined by

$$\sigma_{xz} = 2\mu\tau\frac{\partial \chi}{\partial y}, \quad \sigma_{yz} = -2\mu\tau\frac{\partial \chi}{\partial x}; \quad (16.9)$$

this function satisfies more convenient boundary conditions on the circumference of the rod (see below). Comparing (16.9) and (16.6), we obtain

$$\frac{\partial \psi}{\partial x} = y + 2\frac{\partial \chi}{\partial y}, \quad \frac{\partial \psi}{\partial y} = -x - 2\frac{\partial \chi}{\partial x}. \quad (16.10)$$

Differentiating the first of these with respect to y, the second with respect to x, and subtracting, we obtain for the function χ the equation

$$\Delta\chi = -1. \quad (16.11)$$

To determine the boundary conditions on the surface of the rod, we note that, since the rod is thin, the external forces on its sides must be small compared with the internal stresses in the rod, and can therefore be put equal to zero in seeking the boundary conditions. This fact is exactly analogous to what we found in discussing the bending of thin plates. Thus we must have $\sigma_{ik} n_k = 0$ on the sides of the rod; since the z-direction is along the axis, $n_z = 0$, and this equation becomes

$$\sigma_{zx} n_x + \sigma_{zy} n_y = 0 .$$

Substituting (16.9), we obtain

$$\frac{\partial \chi}{\partial y} n_x - \frac{\partial \chi}{\partial x} n_y = 0 .$$

The components of the vector normal to a plane contour (the circumference of the rod) are $n_x = -\dfrac{dy}{dl}$, $n_y = \dfrac{dx}{dl}$, where x and y are co-ordinates of points on the contour and dl is an element of arc. Thus we have

$$\frac{\partial \chi}{\partial x} dx + \frac{\partial \chi}{\partial y} dy = d\chi = 0 ,$$

whence $\chi = \text{constant}$, i.e., χ is constant on the circumference. Since only the derivatives of the function χ appear in the definitions (16.9), it is clear that any constant may be added to χ. If the cross-section is singly connected, we can therefore use, without loss of generality, the boundary condition

$$\chi = 0 \qquad (16.12)$$

on equation (16.11).[12]

For a multiply connected cross-section, however, χ will have different constant values on each of the closed curves bounding the cross-section. Hence we can put $\chi = 0$ on only one of these curves, for instance the outermost (C_0 in Fig. 13). The values of χ on the remaining

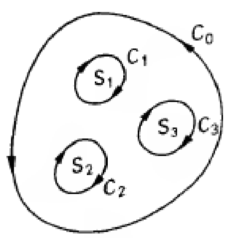

Fig. 13

[12] The problem of determining the torsion deformation from equation (16.11) with the boundary condition (16.12) is formally identical with that of determining the bending of a uniformly loaded plane membrane from equation (14.9).

It is useful to note also an analogy with fluid mechanics: an equation of the form (16.11) determines the velocity distribution $v(x, y)$ for a viscous fluid in a pipe, and the boundary condition (16.12) corresponds to the condition $\mathbf{v} = 0$ at the fixed walls of the pipe (see *Fluid Mechanics*, §17).

bounding curves are found from conditions which are a consequence of the one-valuedness of the displacement $u_z = \tau\psi(x,y)$ as a function of the co-ordinates. For, since the torsion function $\psi(x,y)$ is one-valued, the integral of its differential $d\psi$ round a closed contour must be zero. Using the relations (16.10), we therefore have

$$\begin{aligned}\oint d\psi &= \oint\left(\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy\right)\\ &= -2\oint\left(\frac{\partial\chi}{\partial x}dy - \frac{\partial\chi}{\partial y}dx\right) - 2\oint(xdy - ydx),\\ &= 0\end{aligned}$$

or

$$\oint\frac{\partial\chi}{\partial n}dl = -S\,, \qquad (16.13)$$

where $\frac{\partial\chi}{\partial n}$ is the derivative of the function χ along the outward normal to the curve, and S the area enclosed by the curve. Applying (16.13) to each of the closed curves $C_1, C_2, \ldots$, we obtain the required conditions.

Let us determine the free **energy** of a rod under torsion. The energy per unit volume is

$$F = \frac{1}{2}\sigma_{ik}u_{ik} = \sigma_{xz}u_{xz} + \sigma_{yz}u_{yz} = \frac{\sigma_{xz}{}^2 + \sigma_{yz}{}^2}{2\mu}$$

or, substituting (16.9),

$$F = 2\mu\tau^2\left[\left(\frac{\partial\chi}{\partial x}\right)^2 + \left(\frac{\partial\chi}{\partial y}\right)^2\right] \equiv 2\mu\tau^2(\boldsymbol{grad}\chi)^2\,,$$

where ***grad*** denotes the two-dimensional gradient. The torsional energy per unit length of the rod is obtained by integrating over the cross-section of the rod, i.e., it is $\frac{1}{2}C\tau^2$, where the constant $C = 4\mu\int(\boldsymbol{grad}\chi)^2 df$, and is called the ***torsional rigidity*** of the rod. The total elastic energy of the rod is equal to the integral

$$F_{rod} = \frac{1}{2}\int C\tau^2 dz\,, \qquad (16.14)$$

taken along its length.

Putting

$$(\boldsymbol{grad}\chi)^2 = \mathrm{div}(\chi\boldsymbol{grad}\chi) - \chi\Delta\chi = \mathrm{div}(\chi\boldsymbol{grad}\chi) + \chi$$

and transforming the integral of the first term into one along the circumference of the rod, we obtain

$$C = 4\mu \oint \chi \frac{\partial \chi}{\partial n} dl + 4\mu \int \chi df \, . \qquad (16.15)$$

If the cross-section is singly connected, the first term vanishes by the boundary condition $\chi = 0$, leaving

$$C = 4\mu \int \chi dxdy \, . \qquad (16.16)$$

For a multiply connected cross-section (Fig. 13), we put $\chi = 0$ on the outer boundary C_0 and denote by χ_k the constant values of χ on the inner boundaries C_k, obtaining by (16.13)

$$C = 4\mu \sum_k \chi_k S_k + 4\mu \int \chi dxdy \, ; \qquad (16.17)$$

it should be remembered that, in integrating in the first term in (16.15), we go anti-clockwise round the contour C_0 and clockwise round all the others.

Let us consider now a more usual case of torsion, where one of the ends of the rod is held fixed and the external forces are applied only to the other end. These forces are such that they cause only a twisting of the rod, and no other deformation such as bending. In other words, they form a couple which twists the rod about its axis. The moment of this couple will be denoted by *M*.

We should expect that, in such a case, the torsion angle τ is constant along the rod. This can be seen, for example, from the condition that the free energy of the rod is a minimum in equilibrium. The total energy of a deformed rod is equal to the sum $F_{rod} + U$, where *U* is the potential energy due to the action of the external forces. Substituting in (16.14) $\tau = \frac{d\phi}{dz}$ and varying with respect to the angle ϕ, we find

$$\delta \frac{1}{2} \int C \left(\frac{d\phi}{dz} \right)^2 dz + \delta U = \delta \int C \frac{d\phi}{dz} \frac{d\delta\phi}{dz} dz + \delta U = 0 \, ,$$

or, integrating by parts,

$$-\int C \frac{d\tau}{dz} \delta\phi dz + \delta U + [C\tau\delta\phi] = 0 \, .$$

The last term on the left is the difference of the values at the limits of integration, i.e., at the ends of the rod. One of these ends, say the lower one, is fixed, so that $\delta\phi = 0$ there. The variation δU of the potential energy is minus the work done by the external forces in rotation through an angle $\delta\phi$. As we know from mechanics, the work done by a couple in such a rotation is equal to the product $M\delta\phi$ of the angle of rotation and the moment of the couple. Since there are no other external forces, $\delta U = -M\delta\phi$, and we have

$$\int C\frac{d\tau}{dz}\delta\phi dz+[\delta\phi(-M+C\tau]=0\,. \qquad (16.18)$$

The second term on the left has its value at the upper end of the rod. In the integral over *z*, the variation $\delta\phi$ is arbitrary, and so we must have

$$C\frac{d\tau}{dz}=0\,,$$

i.e.,

$$\tau=\text{constant}\,. \qquad (16.19)$$

Thus the torsion angle is constant along the rod. The total angle of rotation of the upper end of the rod relative to the lower end is τl, where *l* is the length of the rod.

In equation (16.18), the second term also must be zero, and we obtain the following expression for the constant torsion angle:

$$\tau=\frac{M}{C}\,. \qquad (16.20)$$

PROBLEMS

Problem 1. Determine the torsional rigidity of a rod whose cross-section is a circle of radius *R*.

Solution. The solutions of Problems 1-4 are formally identical with those of problems of the motion of a viscous fluid in a pipe of corresponding cross-section (see the last footnote to this section). The discharge *Q* is here represented by *C*.

For a rod of circular cross-section we have, taking the origin at the centre of the circle, $\chi=\frac{1}{4}(R^2-x^2-y^2)$, and the torsional rigidity is $C=\frac{1}{2}\mu\pi R^4$. For the function ψ we have, from (16.10), $\psi=\text{constant}$. A constant ψ, however, corresponds by (16.4) to a simple displacement of the whole rod along the *z*-axis, and so we can suppose that $\psi=0$. Thus the transverse sections of a circular rod undergoing torsion remain plane.

Problem 2. The same as Problem 1, but for an elliptical cross-section of semi-axes *a* and *b*.

Solution. The torsional rigidity is $C=\dfrac{\pi\mu a^3b^3}{a^2+b^2}$. The distribution of longitudinal displacements is given by the torsion function $\psi=\dfrac{(b^2-a^2)xy}{b^2+a^2}$, where the co-ordinate axes coincide with those of the ellipse.

Problem 3. The same as Problem 1, but for an equilateral triangular cross-section of side a.

Solution. The torsional rigidity is $C = \frac{\sqrt{3}\mu a^4}{80}$. The torsion function is

$$\psi = \frac{y(x\sqrt{3} + y)(x\sqrt{3} - y)}{6a}$$

the origin being at the centre of the triangle and the x-axis along an altitude.

Problem 4. The same as Problem 1, but for a rod in the form of a long thin plate (of width d and thickness $h << d$).

Solution. The problem is equivalent to that of viscous fluid flow between plane parallel walls. The result is that $C = \frac{1}{3}\mu d h^3$.

Problem 5. The same as Problem 1, but for a cylindrical pipe of internal and external radii R_1 and R_2, respectively.

Solution. The function $\chi \frac{1}{4}(R_2{}^2 - r^2)$ (in polar co-ordinates) satisfies the condition (16.13) at both boundaries of the annular cross-section of the pipe. From formula (16.17) we then find $C = \frac{1}{2}\mu\pi(R_2{}^4 - R_1{}^4)$.

Problem 6. The same as Problem 1, but for a thin-walled pipe of arbitrary cross-section.

Solution. Since the walls are thin, we can assume that χ varies through the wall thickness h, from zero on one side to χ_1 on the other, according to the linear law $\chi = \frac{\chi_1 y}{h}$ (y being a co-ordinate measured through the wall). Then the condition (16.13) gives $\frac{\chi_1 L}{h} = S$, where L is the perimeter of the pipe cross-section and S the area which it encloses. The second term in the expression (16.17) is small compared with the first, and we obtain $C = \frac{4hS^2\mu}{L}$. If the pipe is cut longitudinally along a generator, the torsional rigidity falls sharply, becoming (by the result of Problem 4) $C = \frac{1}{2}\mu L h^3$.

§17. Bending of rods

A bent rod is stretched at some points and compressed at others. Lines on the convex side of the bent rod are extended, and those on the concave side are compressed. As with plates, there is a **neutral surface** in the rod, which

undergoes neither extension nor compression. It separates the region of compression from the region of extension.

Let us begin by investigating a bending deformation in a small portion of the length of the rod, where the bending may be supposed slight; by this we here mean that not only the strain tensor but also the magnitudes of the displacements of points in the rod are small. We take a co-ordinate system with the origin on the neutral surface in the portion considered, and the *z*-axis parallel to the axis of the undeformed rod. Let the bending occur in the *zx*-plane.[13]

As in the bending of plates and the twisting of rods, the external forces on the sides of a thin bent rod are small compared with the internal stresses, and can be taken as zero in determining the boundary conditions at the sides of the rod. Thus we have everywhere on the sides of the rod $\sigma_{ik} n_k = 0$, or, since $n_z = 0$, $\sigma_{xx} n_x + \sigma_{xy} n_y = 0$, and similarly for $i = y, z$. We take a point on the circumference of a cross-section for which the normal $\boldsymbol{n}$ is parallel to the *x*-axis. There will be another such point somewhere on the opposite side of the rod. At both these points $n_y = 0$, and the above equation gives $\sigma_{xx} = 0$. Since the rod is thin, however, σ_{xx} must be small everywhere in the cross-section if it vanishes on either side. We can therefore put $\sigma_{xx} = 0$ everywhere in the rod. In a similar manner, it can be seen that all the components of the stress tensor except σ_{zz} must be zero. That is, in the bending of a thin rod only the extension (or compression) component of the internal stress tensor is large. A deformation in which only the component σ_{zz} of the stress tensor is non-zero is just a simple extension or compression (§5). Thus there is a simple extension or compression in every volume element of a bent rod. The amount of this varies, of course, from point to point in every cross-section, and so the whole rod is bent.

It is easy to determine the relative extension at any point in the rod. Let us consider an element of length *dz* parallel to the axis of the rod and near the origin. On bending, the length of this element becomes *dz*'. The only elements which remain unchanged are those which lie in the neutral surface. Let *R* be the radius of curvature of the neutral surface near the origin. The lengths *dz* and *dz*' can be regarded as elements of arcs of circles whose radii are, respectively, *R* and *R* + *x*, *x* being the co-ordinate of the point where *dz*' lies. Hence

[13] In a rod undergoing only small deflections we can suppose that the bending occurs in a single plane. This follows from the result of differential geometry that the deviation of a slightly bent curve from a plane (its torsion) is of a higher order of smallness than its curvature.

$$dz' = \frac{R+x}{R}dz = \left(1+\frac{x}{R}\right)dz .$$

The relative extension is therefore $\frac{dz' - dz}{dz} = \frac{x}{R}$.

The relative extension of the element dz, however, is equal to the component u_{zz} of the strain tensor. Thus

$$u_{zz} = \frac{x}{R} . \qquad (17.1)$$

We can now find σ_{zz} by using the relation $\sigma_{zz} = Eu_{zz}$ which holds for a simple extension. This gives

$$\sigma_{zz} = \frac{Ex}{R} . \qquad (17.2)$$

The position of the **neutral surface** in a bent rod has now to be determined. This can be done from the condition that the deformation considered must be pure bending, with no general extension or compression of the rod. The total internal stress force on a cross-section of the rod must therefore be zero, i.e., the integral $\int \sigma_{zz} df$, taken over a cross-section, must vanish. Using the expression (17.2) for σ_{zz} , we obtain the condition

$$\int xdf = 0 . \qquad (17.3)$$

We can now bring in the centre of mass of the cross-section, which is that of a uniform flat disc of the same shape. The co-ordinates of the centre of mass are, as we know, given by the integrals $\frac{\int xdf}{\int df}$, $\frac{\int ydf}{\int df}$. Thus the condition (17.3) signifies that, in a co-ordinate system with the origin in the neutral surface, the x co-ordinate of the centre of mass of any cross-section is zero. The neutral surface therefore passes through the centres of mass of the cross-sections of the rod.

Two components of the strain tensor besides σ_{zz} are non-zero, since for a simple extension we have $u_{xx} = u_{yy} = -\sigma u_{zz}$. Knowing the strain tensor, we can easily find the displacement also:

$$u_{zz} = \frac{\partial u_z}{\partial z} = \frac{x}{R} , \quad \frac{\partial u_x}{\partial x} = \frac{\partial u_y}{\partial y} = -\frac{\sigma x}{R} ,$$

$$\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} = 0 , \quad \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 0 , \quad \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 0 .$$

Integration of these equations gives the following expressions for the

components of the displacement:

$$\left.\begin{aligned} u_x &= -\frac{1}{2R}\left\{z^2 + \sigma(x^2 - y^2)\right\} \\ u_y &= -\frac{\sigma xy}{R} \\ u_z &= \frac{xz}{R} \end{aligned}\right\} . \tag{17.4}$$

The constants of integration have been put equal to zero; this means that we "fix" the origin.

It is seen from formulae (17.4) that the points initially on a cross-section $z = \text{constant} \equiv z_0$ will be found, after the deformation, on the surface $z = z_0 + u_z = z_0(1 + \frac{x}{R})$. We see that, in the approximation used, the cross-sections remain plane but are turned through an angle relative to their initial positions. The shape of the cross-section changes, however; for example, when a rod of rectangular cross-section (sides a, b) is bent, the sides $y = \pm b/2$ of the cross-section become $y = \pm b/2 + u_y = \pm\frac{b}{2}(1 - \frac{\sigma x}{R})$, i.e., no longer parallel but still straight. The sides $x = \pm a/2$, however, are bent into the parabolic curves (Fig. 14):

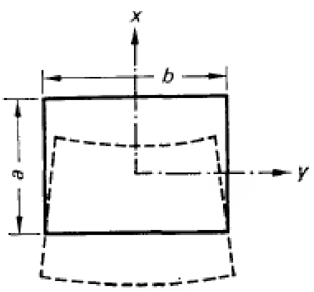

FIG. 14

$$x = \pm\frac{1}{2}a + u_x = \pm\frac{1}{2}a - \frac{1}{2R}\left[z_0{}^2 + \sigma(\frac{1}{4}a^2 - y^2)\right].$$

The **free energy** per unit volume of the rod is

$$\frac{1}{2}\sigma_{ik}u_{ik} = \frac{1}{2}\sigma_{zz}u_{zz} = \frac{1}{2}\frac{Ex^2}{R^2}.$$

Integrating over the cross-section of the rod, we have

$$\frac{1}{2}\frac{E}{R^2}\int x^2 df . \tag{17.5}$$

This is the free energy per unit length of a bent rod. The radius of curvature R is that of the neutral surface. However, since the rod is thin, R can here be regarded, to the same approximation, as the radius of curvature of the bent

rod itself, regarded as a line (often called an "***elastic line***").

In the expression (17.5) it is convenient to introduce the **moment of inertia of the cross-section**. The moment of inertia about the *y*-axis in its plane is defined as

$$I_y = \int x^2 df\,, \tag{17.6}$$

analogously to the ordinary moment of inertia, but with the surface element *df* instead of the mass element. Then the free energy per unit length of the rod can be written

$$\frac{1}{2}\frac{EI_y}{R^2}\,. \tag{17.7}$$

We can also determine the **moment of the internal stress forces** on a given cross-section of the rod (the ***bending moment***). A force $\sigma_{zz} df = \frac{xE}{R} df$ acts in the *z*-direction on the surface element *df* of the cross-section. Its moment about the *y*-axis is $x\sigma_{zz} df$. Hence the total moment of the forces about this axis is

$$M_y = \frac{E}{R}\int x^2 df = \frac{EI_y}{R}\,. \tag{17.8}$$

Thus the curvature $1 / R$ of the elastic line is proportional to the bending moment on the cross-section concerned.

The magnitude of I_y depends on the direction of the *y*-axis in the cross-sectional plane. It is convenient to express I_y in terms of the **principal moments of inertia**. If θ is the angle between the *y*-axis and one of the principal axes of inertia in the cross-section, we know from mechanics that

$$I_y = I_1\cos^2\theta + I_2\sin^2\theta\,, \tag{17.9}$$

where I_1 and I_2 are the principal moments of inertia. The planes through the *z*-axis and the principal axes of inertia are called the ***principal planes of bending***.

If, for example, the cross-section is rectangular (with sides *a*, *b*), its centre of mass is at the centre of the rectangle, and the principal axes of inertia are parallel to the sides. The principal moments of inertia are

$$I_1 = \frac{a^3 b}{12}\,,\quad I_2 = \frac{ab^3}{12}\,. \tag{17.10}$$

For a circular cross-section of radius *R*, the centre of mass is at the centre of the circle, and the principal axes are arbitrary. The moment of inertia about

any axis lying in the cross-section and passing through the centre is

$$I = \frac{\pi}{4} R^4 . \qquad (17.11)$$

§18. The energy of a deformed rod

In §17 we have discussed only a small portion of the length of a bent rod. In going on to investigate the deformation throughout the rod, we must begin by finding a suitable method of describing this deformation. It is important to note that, when a rod undergoes large bending deflections,[14] there is in general a twisting of it as well, so that the resulting deformation is a combination of pure bending and torsion.

To describe the deformation, it is convenient to proceed as follows. We divide the rod into infinitesimal elements, each of which is bounded by two adjacent cross-sections. For each such element we use a co-ordinate system ξ, η, ζ, so chosen that all the systems are parallel in the undeformed state, and their ζ-axes are parallel to the axis of the rod. When the rod is bent, the co-ordinate system in each element is rotated, and in general differently in different elements. Any two adjacent systems are rotated through an infinitesimal relative angle.

Let $d\boldsymbol{\varphi}$ be the vector of the angle of relative rotation of two systems at a distance dl apart along the rod (we know that an infinitesimal angle of rotation can be regarded as a vector parallel to the axis of rotation; its components are the angles of rotation about each of the three axes of co-ordinates).

To describe the deformation, we use the vector

$$\boldsymbol{\Omega} = \frac{d\boldsymbol{\varphi}}{dl} , \qquad (18.1)$$

which gives the "rate" of rotation of the co-ordinate axes along the rod. If the deformation is a pure torsion, the co-ordinate system rotates only about the axis of the rod, i.e., about the ζ-axis. In this case, therefore, the vector $\boldsymbol{\Omega}$ is parallel to the axis of the rod, and is just the torsion angle τ used in §16. Correspondingly, in the general case of an arbitrary deformation we can call the component Ω_ζ of the vector $\boldsymbol{\Omega}$ the ***torsion angle***. For a pure bending of the rod in a single plane, on the other hand, the vector $\boldsymbol{\Omega}$ has no component Ω_ζ, i.e., it lies in the $\xi\eta$-plane at each point. If we take the plane of bending as the $\xi\zeta$-plane, then the rotation is about the η-axis at

[14] By this, it should be remembered, we mean that the vector $\boldsymbol{u}$ is not small, but the strain tensor is still small.

every point, i.e., $\boldsymbol{\Omega}$ is parallel to the η-axis.

We take a unit vector $\boldsymbol{t}$ tangential to the rod (regarded as an elastic line). The derivative $\dfrac{d\boldsymbol{t}}{dl}$ is the curvature vector of the line; its magnitude is $1/R$, where R is the radius of curvature,[15] and its direction is that of the principal normal to the curve. The change in a vector due to an infinitesimal rotation is equal to the vector product of the rotation vector and the vector itself. Hence the change in the vector $\boldsymbol{t}$ between two neighbouring points of the elastic line is given by $d\boldsymbol{t} = d\boldsymbol{\varphi} \times \boldsymbol{t}$, or, dividing by dl,

$$\frac{d\boldsymbol{t}}{dl} = \boldsymbol{\Omega} \times \boldsymbol{t}. \tag{18.2}$$

Multiplying this equation vectorially by $\boldsymbol{t}$, we have

$$\boldsymbol{\Omega} = \boldsymbol{t} \times \frac{d\boldsymbol{t}}{dl} + \boldsymbol{t}(\boldsymbol{t} \cdot \boldsymbol{\Omega}). \tag{18.3}$$

The direction of the tangent vector at any point is the same as that of the ζ-axis at that point. Hence $\boldsymbol{t} \cdot \boldsymbol{\Omega} = \Omega_\zeta$. Using the unit vector $\boldsymbol{n}$ along the principal normal ($\boldsymbol{n} = R\dfrac{d\boldsymbol{t}}{dl}$), we can therefore put

$$\boldsymbol{\Omega} = \boldsymbol{t} \times \frac{\boldsymbol{n}}{R} + \boldsymbol{t}\Omega_\zeta. \tag{18.4}$$

The first term on the right is a vector with two components Ω_ξ, Ω_η. The unit vector $\boldsymbol{t} \times \boldsymbol{n}$ is the binormal unit vector. Thus the components Ω_ξ, Ω_η form a vector along the binormal to the rod, whose magnitude equals the curvature $1/R$.

By using the vector $\boldsymbol{\Omega}$ to characterise the deformation and ascertaining its properties, we can derive an expression for the **elastic free energy** of a bent rod. The elastic energy per unit length of the rod is a quadratic function of the deformation, i.e., in this case, a quadratic function of the components of the vector $\boldsymbol{\Omega}$. It is easy to see that there can be no terms in this quadratic form proportional to $\Omega_\xi\Omega_\zeta$ and $\Omega_\eta\Omega_\zeta$. For, since the rod is uniform along its length, all quantities, and in particular the energy, must remain constant when the direction of the positive ζ-axis is reversed, i.e., when ζ is replaced by $-\zeta$, whereas the products mentioned change sign.

For $\Omega_\xi = \Omega_\eta = 0$ we have a pure torsion, and the expression for the energy must be that obtained in §16. Thus the term in $\Omega_\zeta^{\,2}$ in the free

[15] It may be recalled that any curve in space is characterised at each point by a **curvature** and a **torsion**. This torsion (which we shall not use) should not be confused with the torsional deformation, which is a twisting of a rod about its axis.

energy is $\frac{1}{2}C\Omega_\zeta{}^2$.

Finally, the terms quadratic in Ω_ξ and Ω_η can be obtained by starting from the expression (17.7) for the energy of a slightly bent short section of the rod. Let us suppose that the rod is only slightly bent. We take the $\xi\zeta$-plane as the plane of bending, so that the component Ω_ξ is zero; there is also no torsion in a slight bending. The expression for the energy must then be that given by (17.7), i.e., $\frac{1}{2}\frac{EI_\eta}{R^2}$. We have seen, however, that $\frac{1}{R^2}$ is the square of the two-dimensional vector $(\Omega_\xi, \Omega_\eta)$. Hence the energy must be of the form $\frac{1}{2}EI_\eta \Omega_\eta{}^2$. For an arbitrary choice of the ξ and η axes this expression becomes, as we know from mechanics,

$$\frac{1}{2}E(I_{\eta\eta}\Omega_\eta{}^2 + 2I_{\eta\xi}\Omega_\eta\Omega_\xi + I_{\xi\xi}\Omega_\xi{}^2),$$

where $I_{\eta\eta}, I_{\eta\xi}, I_{\xi\xi}$ are the components of the inertia tensor for the cross-section of the rod. It is convenient to take the ξ and η axes to coincide with the principal axes of inertia. We then have simply $\frac{1}{2}E(I_1\Omega_\xi{}^2 + I_2\Omega_\eta{}^2)$, where I_1, I_2 are the principal moments of inertia.

Since the coefficients of $\Omega_\xi{}^2$ and $\Omega_\eta{}^2$ are constants, the resulting expression must be valid for large deflections also.

Finally, integrating over the length of the rod, we obtain the following expression for the **elastic free energy** of a bent rod

$$F_{rod} = \int \left\{ \frac{1}{2}I_1 E\Omega_\xi{}^2 + \frac{1}{2}I_2 E\Omega_\eta{}^2 + \frac{1}{2}CE\Omega_\xi{}^2 \right\} dl. \quad (18.5)$$

Next, we can express in terms of $\boldsymbol{\Omega}$ the moment of the forces acting on a cross-section of the rod. This is easily done by again using the results previously obtained for pure torsion and pure bending. In pure torsion, the moment of the forces about the axis of the rod is $C\tau$. Hence we conclude that, in the general case, the moment M_ζ about the ζ-axis must be $C\Omega_\zeta$. Next, in a slight deflection in the $\xi\zeta$-plane, the moment about the η-axis is $\frac{EI_2}{R}$. In such a bending, however, the vector $\boldsymbol{\Omega}$ is along the η-axis, so that $1/R$ is just the magnitude of $\boldsymbol{\Omega}$, and $\frac{EI_2}{R} = EI_2\Omega$. Hence we conclude that, in the general case, we must have $M_\xi = EI_1\Omega_\xi$, $M_\eta = EI_2\Omega_\eta$ (the

ξ and η axes being along the principal axes of inertia in the cross-section). Thus the components of the moment vector $\boldsymbol{M}$ are

$$M_\xi = EI_1\Omega_\xi , \quad M_\eta = EI_2\Omega_\eta , \quad M_\zeta = C\Omega_\zeta . \qquad (18.6)$$

The **elastic energy** (18.5), expressed in terms of the moment of the forces, is

$$F_{rod} = \int \left\{ \frac{M_\xi{}^2}{2I_1E} + \frac{M_\eta{}^2}{2I_2E} + \frac{M_\zeta{}^2}{2C} \right\} dl . \qquad (18.7)$$

An important case of the bending of rods is that of a **slight bending**, in which the deviation from the initial position is everywhere small compared with the length of the rod. In this case torsion can be supposed absent, and we can put $\Omega_\zeta = 0$, so that (18.4) gives simply

$$\boldsymbol{\Omega} = \boldsymbol{t} \times \frac{\boldsymbol{n}}{R} \equiv \boldsymbol{t} \times \frac{d\boldsymbol{t}}{dl} . \qquad (18.8)$$

We take a co-ordinate system *x*, *y*, *z* fixed in space, with the *z*-axis along the axis of the undeformed rod (instead of the system ξ, η, ζ for each point in the rod), and denote by *X*, *Y* the co-ordinates *x*, *y* for points on the elastic line; *X* and *Y* give the displacement of points on the line from their positions before the deformation.

Since the bending is only slight, the tangent vector $\boldsymbol{t}$ is almost parallel to the .*z*-axis, and the difference in direction can be approximately neglected. The unit tangent vector is the derivative $\boldsymbol{t} = \frac{d\boldsymbol{r}}{dl}$ of the radius vector $\boldsymbol{r}$ of a point on the curve with respect to its length. Hence

$$\frac{dr}{dl} = \frac{d^2r}{dl^2} \approx \frac{d^2r}{dz^2} ;$$

the derivative with respect to the length can be approximately replaced by the derivative with respect to *z*. In particular, the *x* and *y* components of this vector are, respectively, $\frac{d^2X}{dz^2}$ and $\frac{d^2Y}{dz^2}$. The components Ω_ξ, Ω_η are, to the same accuracy, equal to Ω_x, Ω_y, and we have from (18.8)

$$\Omega_\xi = -\frac{d^2Y}{dz^2} , \quad \Omega_\eta = \frac{d^2X}{dz^2} . \qquad (18.9)$$

Substituting these expressions in (18.5), we obtain the **elastic energy of a slightly bent rod** in the form

$$F_{rod} = \frac{1}{2} E \int \left\{ I_1 \left(\frac{d^2Y}{dz^2} \right) + I_2 \left(\frac{d^2X}{dz^2} \right) \right\} dz . \qquad (18.10)$$

Here I_1 and I_2 are the moments of inertia about the axes of *x* and *y* ,

respectively, which are the principal axes of inertia.

In particular, for a rod of circular cross-section, $I_1 = I_2 \equiv I$, and the integrand is just the sum of the squared second derivatives, which in the approximation considered is the square of the curvature:

$$\left(\frac{d^2X}{dz^2}\right)^2 + \left(\frac{d^2Y}{dz^2}\right)^2 \approx \frac{1}{R^2}.$$

Hence formula (18.10) can be plausibly generalised to the case of slight bending of a circular rod having any shape (not necessarily straight) in its undeformed state. To do so, we must write the bending energy as

$$F_{rod} = \frac{1}{2} EI \int \left(\frac{1}{R} - \frac{1}{R_0}\right)^2 dz, \qquad (18.11)$$

where R_0 is the radius of curvature at any point of the undeformed rod. This expression has a minimum, as it should, in the undeformed state ($R = R_0$), and for $R_0 \to \infty$ it becomes formula (18.10).

§19. The equations of equilibrium of rods

We can now derive the equations of equilibrium for a bent rod. We again consider an infinitesimal element bounded by two adjoining cross-sections of the rod, and calculate the total force acting on it. We denote by $\boldsymbol{F}$ the resultant internal stress on a cross-section.[16] The components of this vector are the integrals of $\sigma_{i\zeta}$ over the cross-section:

$$F_i = \int \sigma_{i\zeta} df. \qquad (19.1)$$

If we regard the two adjoining cross-sections as the ends of the element, a force $\boldsymbol{F} + d\boldsymbol{F}$ acts on the upper end, and $-\boldsymbol{F}$ on the lower end; the sum of these is the differential $d\boldsymbol{F}$. Next, let $\boldsymbol{K}$ be the external force on the rod per unit length. Then an external force $\boldsymbol{K}dl$ acts on the element of length dl. The resultant of the forces on the element is therefore $d\boldsymbol{F} + \boldsymbol{K}dl$. **This must be zero in equilibrium**. Thus we have

$$\frac{d\boldsymbol{F}}{dl} = -\boldsymbol{K}. \qquad (19.2)$$

A second equation is obtained from the condition that the **total moment of the forces on the element is zero**. Let $\boldsymbol{M}$ be the moment of the internal stresses on the cross-section. This is the moment about a point (the origin) which lies in the plane of the cross-section; its components are given by

[16] This notation will not lead to any confusion with the free energy, which does not appear in

formulae (18.6). We shall calculate the total moment, on the element considered, about a point O lying in the plane of its upper end. Then the internal stresses on this end give a moment $\boldsymbol{M} + d\boldsymbol{M}$. The moment about O of the internal stresses on the lower end of the element is composed of the moment $-\boldsymbol{M}$ of those forces about the origin O' in the plane of the lower end and the moment about O of the total force $-\boldsymbol{F}$ on that end. This latter moment is $-d\boldsymbol{l}\times(-\boldsymbol{F})$, where $d\boldsymbol{l}$ is the vector of the element of length of the rod between O' and O. The moment due to the external forces $\boldsymbol{K}$ is of a higher order of smallness. Thus the total moment acting on the element considered is $d\boldsymbol{M} + d\boldsymbol{l}\times\boldsymbol{F}$. In equilibrium, this must be zero:

$$d\boldsymbol{M} + d\boldsymbol{l}\times\boldsymbol{F} = 0 .$$

Dividing this equation by dl and using the fact that $\frac{d\boldsymbol{l}}{dl} = \boldsymbol{t}$ is the unit vector tangential to the rod (regarded as a line), we have

$$\frac{d\boldsymbol{M}}{dl} = \boldsymbol{F}\times\boldsymbol{t} . \tag{19.3}$$

Equations (19.2) and (19.3) form a **complete set of equilibrium equations** for a rod bent in any manner.

If the external forces on the rod are concentrated, i.e., applied only at isolated points of the rod, the equilibrium equations at all other points are much simplified. For $\boldsymbol{K} = 0$ we have from (19.2)

$$\boldsymbol{F} = \text{constant}, \tag{19.4}$$

i.e., the stress resultant is constant along any portion of the rod between points where forces are applied. The values of the constant are found from the fact that the difference $\boldsymbol{F}_2 - \boldsymbol{F}_1$ of the forces at two points 1 and 2 is

$$\boldsymbol{F}_2 - \boldsymbol{F}_1 = -\sum \boldsymbol{K} , \tag{19.5}$$

where the sum is over all forces applied to the segment of the rod between the two points. It should be noticed that, in the difference $\boldsymbol{F}_2 - \boldsymbol{F}_1$, the point 2 is further from the point from which l is measured than is the point 1; this is important in determining the signs in equation (19.5). In particular, if only one concentrated force $\boldsymbol{f}$ acts on the rod, and is applied at its free end, then $\boldsymbol{F}$ = constant = $\boldsymbol{f}$ at all points of the rod.

The second equilibrium equation (19.3) is also simplified. Putting $\boldsymbol{t} = \frac{d\boldsymbol{l}}{dl} = \frac{d\boldsymbol{r}}{dl}$ (where $\boldsymbol{r}$ is the radius vector from any fixed point to the point considered) and integrating, we obtain

§§19-21.

$$\boldsymbol{M} = \boldsymbol{F} \times \boldsymbol{r} + \boldsymbol{constant}\,, \qquad (19.6)$$

since $\boldsymbol{F}$ is constant.

If concentrated forces also are absent, and the rod is bent by the application of concentrated moments, i.e., of concentrated couples, then $\boldsymbol{F}$ = constant at all points of the rod, while $\boldsymbol{M}$ is discontinuous at points where couples are applied, the discontinuity being equal to the moment of the couple.

Let us consider also the **boundary conditions** at the ends of a bent rod. Various cases are possible.

The end of the rod is said to be ***clamped*** (Fig. 4a, §12) if it cannot move either longitudinally or transversely, and moreover its direction (i.e., the direction of the tangent to the rod) cannot change. In this case the boundary conditions are that the co-ordinates of the end of the rod and the unit tangential vector $\boldsymbol{t}$ there are given. The reaction force and moment exerted on the rod by the clamp are determined by solving the equations.

The opposite case is that of a ***free end***, whose position and direction are arbitrary. In this case the boundary conditions are that the force $\boldsymbol{F}$ and moment $\boldsymbol{M}$ must be zero at the end of the rod.[17]

If the end of the rod is ***fixed to a hinge***, it cannot be displaced, but its direction can vary. In this case the moment of the forces on the freely turning end must be zero.

Finally, if the rod is ***supported*** (Fig. 4b), it can slide at the point of support but cannot undergo transverse displacements. In this case the direction $\boldsymbol{t}$ of the rod at the support and the point on the rod at which it is supported are unknown. The moment of the forces at the point of support must be zero, since the rod can turn freely, and the force $\boldsymbol{F}$ at that point must be perpendicular to the rod; a longitudinal force would cause a further sliding of the rod at this point.

The boundary conditions for other modes of fixing the rod can easily be established in a similar manner. We shall not pause to add to the typical examples already given.

It was mentioned at the beginning of §18 that a rod of arbitrary cross-section undergoing large deflections is in general twisted also, even if no external twisting moment is applied to the rod. An exception occurs when a rod is bent in one of its principal planes, in which case there is no torsion. For a rod of circular cross-section no torsion results for any bending (if there is no external twisting moment, of course). This can be seen as follows. The

[17] If a concentrated force $\boldsymbol{f}$ is applied to the free end of the rod, the boundary condition is $\boldsymbol{F} = \boldsymbol{f}$

twisting is given by the component $\Omega_\zeta = \boldsymbol{\Omega} \cdot \boldsymbol{t}$ of the vector $\boldsymbol{\Omega}$. Let us calculate the derivative of this along the rod. To do so, we use the fact that $\Omega_\zeta = \dfrac{M_\zeta}{C}$:

$$\frac{d}{dl}(\boldsymbol{M} \cdot \boldsymbol{t}) = C\frac{d\Omega_\zeta}{dl} = \frac{d\boldsymbol{M}}{dl} \cdot \boldsymbol{t} + \boldsymbol{M} \cdot \frac{d\boldsymbol{t}}{dl} .$$

Substituting (19.3), we see that the first term is zero, so that

$$C\frac{d\Omega_\zeta}{dl} = \boldsymbol{M} \cdot \frac{d\boldsymbol{t}}{dl} .$$

For a rod of circular cross-section, $I_1 = I_2 \equiv I$; by (18.3) and (18.6), we can therefore write $\boldsymbol{M}$ in the form

$$\boldsymbol{M} = EI\boldsymbol{t} \times \frac{d\boldsymbol{t}}{dl} + \boldsymbol{t}C\Omega_\zeta . \qquad (19.7)$$

Multiplying by $\dfrac{d\boldsymbol{t}}{dl}$, we have zero on the right-hand side, so that

$$\frac{d\Omega_\zeta}{dl} = 0 ,$$

whence

$$\Omega_\zeta = \text{constant} , \qquad (19.8)$$

i.e., the torsion angle is constant along the rod. If no twisting moments are applied to the ends of the rod, then Ω_ζ is zero at the ends, and there is no torsion anywhere in the rod.

For a rod of circular cross-section, we can therefore put for pure bending

$$\boldsymbol{M} = EI\boldsymbol{t} \times \frac{d\boldsymbol{t}}{dl} = EI\frac{d\boldsymbol{r}}{dl} \times \frac{d^2\boldsymbol{r}}{dl^2} . \qquad (19.9)$$

Substituting this in (19.3), we obtain the **equation for pure bending of a circular rod**:

$$EI\frac{d\boldsymbol{r}}{dl} \times \frac{d^3\boldsymbol{r}}{dl^3} = \boldsymbol{F} \times \frac{d\boldsymbol{r}}{dl} . \qquad (19.10)$$

PROBLEMS

Problem 1. Reduce to quadratures the problem of determining the shape of a rod of circular cross-section bent in one plane by concentrated forces.

Solution. Let us consider a portion of the rod lying between points where the forces are applied; on such a portion $\boldsymbol{F}$ is constant. We take the plane of the bent rod as the *xy*-plane, with the *y*-axis parallel to the force $\boldsymbol{F}$, and introduce

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not $\boldsymbol{F} = 0$.

the angle θ between the tangent to the rod and the y-axis. Then $\frac{dx}{dl} = \sin\theta$, $\frac{dy}{dl} = \cos\theta$, where x, y are the co-ordinates of a point on the rod. Expanding the vector products in (19.10), we obtain the following equation for θ as a function of the arc length l: $EI\frac{d^2\theta}{dl^2} - F\sin\theta = 0$. A first integration gives $\frac{1}{2}EI\left(\frac{d\theta}{dl}\right)^2 + F\cos\theta = c_1$ and

$$l = \pm\sqrt{\frac{EI}{2}}\int\frac{d\theta}{\sqrt{c_1 - F\cos\theta}} + c_2. \qquad (1)$$

The function $\theta(l)$ can be obtained in terms of elliptic functions. The co-ordinates

$$x = \int \sin\theta dl\,, \quad y = \int \cos\theta dl$$

are

$$\left.\begin{aligned} x &= \pm\sqrt{\frac{2EI(c_1 - F\cos\theta)}{F^2}} + \text{constant} \\ y &= \pm\sqrt{\frac{EI}{2}}\int\frac{\cos\theta d\theta}{\sqrt{c_1 - F\cos\theta}} + \text{constant} \end{aligned}\right\}. \qquad (2)$$

The moment $\boldsymbol{M}$ (19.9) is parallel to the z-axis, and its magnitude is

$$M = EI\frac{d\theta}{dl}.$$

Problem 2. Determine the shape of a bent rod with one end clamped and the other under a force $\boldsymbol{f}$ perpendicular to the original direction of the rod (Fig. 15).

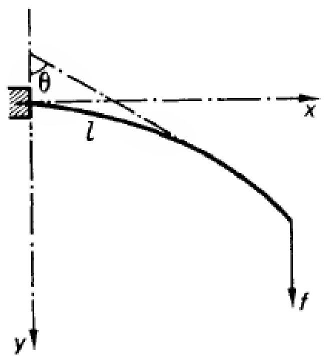

FIG. 15

Solution. We have $\boldsymbol{F}$ = constant = $\boldsymbol{f}$ everywhere on the rod. At the clamped end ($l = 0$), $\theta = \pi/2$, and at the free end ($l = L$, the length of the rod) $\boldsymbol{M} = 0$, i.e., $\theta' = 0$. Putting $\theta(L) \equiv \theta_0$, we have in (1), Problem 1, $c_1 = f\cos\theta_0$

and

$$l=\sqrt{\frac{EI}{2f}}\int_{\theta}^{\pi/2}\frac{d\theta}{\sqrt{\cos\theta_0-\cos\theta}}.$$

Hence we obtain the equation for θ_0:

$$L=\sqrt{\frac{EI}{2f}}\int_{\theta_0}^{\pi/2}\frac{d\theta}{\sqrt{\cos\theta_0-\cos\theta}}.$$

The shape of the rod is given by

$$\left.\begin{aligned}x&=\sqrt{\frac{2EI}{f}}\left[\sqrt{\cos\theta_0}-\sqrt{\cos\theta_0-\cos\theta}\right]\\ y&=\sqrt{\frac{EI}{2f}}\int_{\theta}^{\pi/2}\frac{\cos\theta d\theta}{\sqrt{\cos\theta_0-\cos\theta}}\end{aligned}\right\}.$$

Problem 3. The same as Problem 2, but for a force ***f*** parallel to the original direction of the rod.

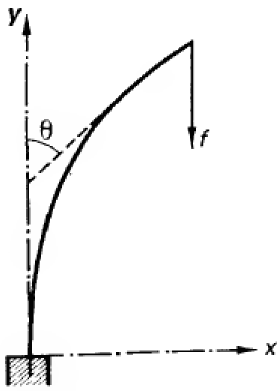

FIG. 16

Solution. We have ***F*** = - ***f***; the co-ordinate axes are taken as shown in Fig. 16. The boundary conditions are $\theta=0$ for l = 0, $\theta'=0$ for $l=L$. Then

$$l=\sqrt{\frac{EI}{2f}}\int_{0}^{\theta}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}},$$

where

$$\theta_0=\theta(L)$$

is given by

$$L=\sqrt{\frac{EI}{2f}}\int_{0}^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}}.$$

For x and y we obtain

$$\left.\begin{aligned}x&=\sqrt{\frac{2EI}{f}}\left[\sqrt{1-\cos\theta_0}-\sqrt{\cos\theta-\cos\theta_0}\right]\\ y&=\sqrt{\frac{EI}{2f}}\int_{0}^{\theta}\frac{\cos\theta d\theta}{\sqrt{\cos\theta-\cos\theta_0}}\end{aligned}\right\}.$$

For a small deflection, $\theta_0 << 1$, and we can write

$$L \approx \sqrt{\frac{EI}{f}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{{\theta_0}^2 - \theta^2}} = \frac{\pi}{2}\sqrt{\frac{EI}{f}},$$

i.e., θ_0 does not appear. This shows that, in accordance with the result of §21, Problem 3, the solution in question exists only for $f \geq \frac{\pi^2 EI}{4L^2}$, i.e., when the rectilinear shape ceases to be stable.

Problem 4. The same as Problem 2, but for the case where both ends of the rod are **supported** and a force ***f*** is applied at its centre. The distance between the supports is *L*.

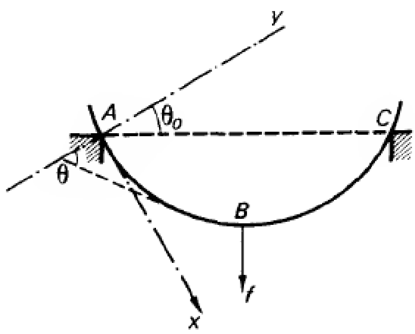

FIG. 17

Solution. We take the co-ordinate axes as shown in Fig. 17. The force ***F*** is constant on each of the segments *AB* and *BC*, and on each is perpendicular to the direction of the rod at the point of support *A* or *C*. The difference between the values of ***F*** on *AB* and *BC* is ***f***, and so we conclude that, on *AB*, $F\sin\theta_0 = -\frac{1}{2}f$, where θ_0 is the angle between the *y*-axis and the line *AC*.

At the point *A* (l = 0) we have the conditions $\theta = \pi/2$ and *M* = 0, i.e., $\theta' = 0$, so that on *AB*

$$l = \sqrt{\frac{EI\sin\theta_0}{f}} \int_\theta^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}},$$

$$x = 2\sqrt{\frac{EI\sin\theta_0\cos\theta}{f}},$$

$$y = \sqrt{\frac{EI\sin\theta_0}{f}} \int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta.$$

The angle θ_0 is determined from the condition that the projection of *AB* on the straight line *AC* must be $L_0/2$, whence

$$\frac{L_0}{2} = \sqrt{\frac{EI\sin\theta_0}{f}} \int_{\theta_0}^{\pi/2} \frac{\cos(\theta - \theta_0)}{\sqrt{\sin\theta}} d\theta.$$

For some value θ_0 lying between 0 and $\pi/2$ the derivative $\frac{df}{d\theta_0}$ (f being regarded as a function of θ_0) passes through zero to positive values. A further decrease in θ_0 , i.e., increase in the deflection, would mean a decrease in f. This means that the solution found here becomes unstable, the rod collapsing between the supports.

Problem 5. Reduce to quadratures the problem of three-dimensional bending of a rod under the action of concentrated forces.

Solution. Let us consider a segment of the rod between points where forces are applied, on which $\boldsymbol{F}$ = constant. Integrating (19.10), we obtain

$$EI\frac{d\boldsymbol{r}}{dl}\times\frac{d^2\boldsymbol{r}}{dl^2}=\boldsymbol{F}\times\boldsymbol{r}+c\boldsymbol{F}\ ; \tag{1}$$

the constant of integration has been written as a vector $c\boldsymbol{F}$ parallel to $\boldsymbol{F}$, since, by appropriately choosing the origin, i.e., by adding a constant vector to $\boldsymbol{r}$, we can eliminate any vector perpendicular to $\boldsymbol{F}$. Multiplying (1) scalarly and vectorially by $\boldsymbol{r}'$ (the prime denoting differentiation with respect to l), and using the fact that $\boldsymbol{r}'\cdot\boldsymbol{r}''=0$ (since $\boldsymbol{r}'^2=1$), we obtain $\boldsymbol{F}\cdot\boldsymbol{r}\times\boldsymbol{r}'+c\boldsymbol{F}\cdot\boldsymbol{r}'=0$, $EI\boldsymbol{r}''=(\boldsymbol{F}\times\boldsymbol{r})\times\boldsymbol{r}'+c\boldsymbol{F}\times\boldsymbol{r}'$. In components (with the z-axis parallel to $\boldsymbol{F}$) we obtain $(xy'-yx')+cz'=0$, $EIz''=-F(xx'+yy')$. Using cylindrical polar coordinates r,ϕ,z we have

$$r^2\phi'+cz'=0\ ,\quad EIz''=-Frr'\ . \tag{2}$$

The second of these gives

$$z'=\frac{F(A-r^2)}{2EI}\ , \tag{3}$$

where A is a constant. Combining (2) and (3) with the identity $r'^2+r^2\phi'^2+z'^2=1$, we find

$$dl=\frac{rdr}{\sqrt{r^2-\dfrac{(r^2+c^2)(A-r^2)^2F^2}{4E^2I^2}}}\ ,$$

and then (2) and (3) give

$$z=\frac{F}{2EI}\int\frac{(A-r^2)rdr}{\sqrt{r^2-\dfrac{(r^2+c^2)(A-r^2)^2F^2}{4E^2I^2}}}\ ,$$

$$\phi=-\frac{cF}{2EI}\int\frac{(A-r^2)dr}{r\sqrt{r^2-\dfrac{(r^2+c^2)(A-r^2)^2F^2}{4E^2I^2}}}$$

which gives the shape of the bent rod.

Problem 6. A rod of circular cross-section is subjected to torsion (with torsion angle τ) and twisted into a spiral. Determine the force and moment which must be applied to the ends of the rod to keep it in this state.

Solution. Let R be the radius of the cylinder on whose surface the spiral lies (and along whose axis we take the z-direction) and α the angle between the tangent to the spiral and a plane perpendicular to the z-axis; the pitch h of the spiral is related to α and R by $h = 2\pi R \tan\alpha$. The equation of the spiral is $x = R\cos\phi$, $y = R\sin\phi$, $z = \phi R\tan\alpha$, where ϕ is the angle of rotation about the z-axis. The element of length is $dl = \frac{R}{\cos\alpha} d\phi$. Substituting these expressions in (19.7), we calculate the components of the vector $\boldsymbol{M}$, and then the force $\boldsymbol{F}$ from formula (19.3); $\boldsymbol{F}$ is constant everywhere on the rod. The result is that the force $\boldsymbol{F}$ is parallel to the z-axis and its magnitude is $F = F_z = \frac{C\tau}{R}\sin\alpha - \frac{EI}{R^2}\cos^2\alpha\sin\alpha$. The moment $\boldsymbol{M}$ has a z-component $M_z = C\tau\sin\alpha + \frac{EI}{R}\cos^3\alpha$ a and a ϕ-component, along the tangent to the cross-section of the cylinder, $M_\phi = FR$.

Problem 7. Determine the form of a flexible wire (whose resistance to bending can be neglected in comparison with its resistance to stretching) suspended at two points and in a gravitational field.

Solution. We take the plane of the wire as the xy-plane, with the y-axis vertically downwards. In equation (19.3) we can neglect the term $\frac{d\boldsymbol{M}}{dl}$, since $\boldsymbol{M}$ is proportional to EI. Then $\boldsymbol{F}\times\boldsymbol{t} = 0$, i.e., $\boldsymbol{F}$ is parallel to $\boldsymbol{t}$ at every point, and we can put $\boldsymbol{F} = F\boldsymbol{t}$. Equation (19.2) then gives

$$\frac{d}{dl}\left(F\frac{dx}{dl}\right) = 0, \quad \frac{d}{dl}\left(F\frac{dy}{dl}\right) = q,$$

where q is the weight of the wire per unit length; hence $F\frac{dx}{dl} = c$, $F\frac{dy}{dl} = ql$, and so $F = \sqrt{c^2 + q^2 l^2}$, so that

$$\frac{dx}{dl} = \frac{A}{\sqrt{A^2 + l^2}}, \quad \frac{dy}{dl} = \frac{l}{\sqrt{A^2 + l^2}},$$

where $A = \frac{c}{q}$. Integration gives $x = A\sinh^{-1}\frac{l}{A}$, $y = \sqrt{A^2 + l^2}$, whence

$$y = A\cosh\frac{x}{A},$$

i.e., the wire takes the form of a **catenary**. The choice of origin and the constant A are determined by the fact that the curve must pass through the two given points and have a given length.

§20. Small deflections of rods

The equations of equilibrium are considerably simplified in the important case of small deflections of rods. This case holds if the direction of the vector $\boldsymbol{t}$ tangential to the rod varies only slowly along its length, i.e., the derivative $\frac{d\boldsymbol{t}}{dl}$ is small. In other words, the radius of curvature of the bent rod is everywhere large compared with the length of the rod. In practice, this condition amounts to requiring that the transverse deflection of the rod is small compared with its length. It should be emphasised that the deflection need not be small compared with the thickness of the rod, as it had to be in the approximate theory of small deflections of plates given in §§11-12.[18]

Differentiating (19.3) with respect to the length, we have

$$\frac{d^2\boldsymbol{M}}{dl^2} = \frac{d\boldsymbol{F}}{dl}\times\boldsymbol{t} + \boldsymbol{F}\times\frac{d\boldsymbol{t}}{dl}. \qquad (20.1)$$

The second term contains the small quantity $\frac{d\boldsymbol{t}}{dl}$, and so can usually be neglected (some exceptional cases are discussed below). Substituting in the first term $\frac{d\boldsymbol{F}}{dl} = -\boldsymbol{K}$, we obtain the equation of equilibrium in the form

$$\frac{d^2\boldsymbol{M}}{dl^2} = \boldsymbol{t}\times\boldsymbol{K}. \qquad (20.2)$$

We write this equation in components, substituting in it from (18.6) and (18.9)

$$M_x = -EI_1Y'', \quad M_y = EI_2X'', \quad M_z = 0, \qquad (20.3)$$

where the prime denotes differentiation with respect to z. The unit vector $\boldsymbol{t}$ may be supposed to be parallel to the z-axis. Then (20.2) gives

$$EI_2X^{(IV)} - K_x = 0, \quad EI_1Y^{(IV)} - K_y = 0. \qquad (20.4)$$

These equations give the deflections X and Y as functions of z, i.e., the shape of a slightly bent rod.

The stress resultant $\boldsymbol{F}$ on a cross-section of the rod can also be expressed in terms of the derivatives of X and Y. Substituting (20.3) in (19.3), we obtain

[18] We shall not give the complex theory of the bending of rods which are not straight when

$$F_x = -EI_2 X''' , \quad F_y = -EI_1 Y''' . \qquad (20.5)$$

We see that the second derivatives give the moment of the internal stresses, while the third derivatives give the stress resultant. The force (20.5) is called the ***shearing force***. If the bending is due to concentrated forces, the shearing force is constant along each segment of the rod between points where forces are applied, and has a discontinuity at each of these points equal to the force applied there.

The quantities EI_2 and EI_1 are called the ***flexural rigidities*** of the rod in the *xz* and *yz* planes, respectively.[19]

If the external forces applied to the rod act in one plane, the bending takes place in one plane, though not in general the same plane. The angle between the two planes is easily found. If α is the angle between the plane of action of the forces and the first principal plane of bending (the *xz*-plane), the equations of equilibrium become $X^{(IV)} = \dfrac{K}{EI_2}\cos\alpha$, $Y^{(IV)} = \dfrac{K}{EI_1}\sin\alpha$. The two equations differ only in the coefficient of *K*. Hence *X* and *Y* are proportional, and $Y = \dfrac{XI_2}{I_1}\tan\alpha$. The angle θ between the plane of bending and the *xz*-plane is given by

$$\tan\theta = \frac{I_2}{I_1}\tan\alpha . \qquad (20.6)$$

For a rod of circular cross-section $I_1 = I_2$ and $\alpha = \theta$, i.e., the bending occurs in the plane of action of the forces. The same is true for a rod of any cross-section when $\alpha = 0$, i.e., when the forces act in a principal plane. The magnitude of the deflection $\zeta = \sqrt{X^2 + Y^2}$ satisfies the equation

$$EI^{(IV)} = K , \quad \frac{I_1 I_2}{\sqrt{I_1^2\cos^2\alpha + I_2^2\sin^2\alpha}} . \qquad (20.7)$$

The shearing force ***F*** is in the same plane as ***K***, and its magnitude is

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undeformed, but only consider one simple example (see Problems 8 and 9).

[19] An equation of the form

$$DX^{(IV)} - K_x = 0 \quad (20.4a)$$

also describes the bending of a thin plate in certain limiting cases. Let a rectangular plate (with sides *a*, *b* and thickness *h*) be fixed along its sides *a* (parallel to the *y*-axis) and bent along its sides *b* (parallel to the *z*-axis) by a load uniform in the *y*-direction. In the general case of arbitrary *a* and *b*, the two-dimensional equation (12.5), with the appropriate boundary conditions at the fixed and free edges, must be used to determine the bending. In the limiting case *a* >> *b*, however, the deformation may be regarded as uniform in the *y*-direction, and then the two-dimensional equilibrium equation becomes of the form (20.4a), with the flexural rigidity replaced by $D = Eh^3 a/12(1-\sigma^2)$. Equation (20.4a) is also applicable to the opposite limiting case *a* << *b*, when the plate can be regarded as a rod of length *b* with a narrow rectangular cross-section (a rectangle of sides *a* and *h*); in this case, however, the flexural rigidity is $D = EI_2 = Eh^3 a/12$ .a

$$F = -EI\zeta''' . \qquad (20.8)$$

Here I is the "effective" moment of inertia of the cross-section of the rod.

We can write down explicitly the boundary conditions on the equations of equilibrium for a slightly bent rod. If the end of the rod is clamped, we must have $X = Y = 0$ there, and also $X' = Y' = 0$, since its direction cannot change. Thus the conditions at a clamped end are

$$X = Y = 0 , \quad X' = Y' = 0 . \qquad (20.9)$$

The reaction force and moment at the point of support are determined from the known solution by formulae (20.3) and (20.5).

When the bending is sufficiently slight, the hinging and supporting of a point on the rod are equivalent as regards the boundary conditions. The reason is that, in the latter case, the longitudinal displacement of the rod at its point of support is of the second order of smallness compared with the transverse deflection, and can therefore be neglected. The boundary conditions of zero transverse displacement and moment give

$$X = Y = 0 , \quad X'' = Y'' = 0 . \qquad (20.10)$$

The direction of the end of the rod and the reaction force at the point of support are obtained by solving the equations.

Finally, at a free end, the force $\boldsymbol{F}$ and moment $\boldsymbol{M}$ must be zero. According to (20.3) and (20.5), this gives the conditions

$$X'' = Y'' = 0 , \quad X''' = Y''' = 0 . \qquad (20.11)$$

If a concentrated force is applied at the free end, then $\boldsymbol{F}$ must be equal to this force, and not to zero.

It is not difficult to generalise equations (20.4) to the case of a rod of variable cross-section. For such a rod the moments of inertia I_1 and I_2 are functions of z. Formulae (20.3), which determine the moment at any cross-section, are still valid. Substitution in (20.2) now gives

$$E\frac{d^2}{dz^2}\left(I_1 \frac{d^2 Y}{dz^2} \right) = K_y , \quad E\frac{d^2}{dz^2}\left(I_2 \frac{d^2 X}{dz^2} \right) = K_x , \qquad (20.12)$$

in which I_1 and I_2 must be differentiated. The shearing force is

$$F_x = -E\frac{d}{dz}\left(I_2 \frac{d^2 X}{dz^2} \right) , \quad F_y = -E\frac{d}{dz}\left(I_1 \frac{d^2 Y}{dz^2} \right) . \qquad (20.13)$$

Let us return to equations (20.1). Our neglect of the second term on the right-hand side may in some cases be illegitimate, even if the bending is slight. The cases involved are those in which a large internal stress resultant acts along the rod, i.e., F_z is very large. Such a force is usually caused by a strong tension of the rod by external stretching forces applied to its ends. We denote by T the constant lengthwise stress F_z . If the rod is strongly

compressed instead of being extended, T will be negative. In expanding the vector product $\boldsymbol{F} \times \frac{d\boldsymbol{t}}{dl}$ we must now retain the terms in T, but those in F_x and F_y can again be neglected. Substituting X'', Y'', 1 for the components of the vector $\frac{d\boldsymbol{t}}{dl}$, we obtain the equations of equilibrium in the form

$$\left.\begin{aligned} EI_2 X^{(IV)} - TX'' - K_x = 0 \\ EI_1 Y^{(IV)} - TY'' - K_y = 0 \end{aligned}\right\}. \qquad (20.14)$$

The expressions (20.5) for the shearing force will now contain additional terms giving the projections of the force T (along the vector $\boldsymbol{t}$) on the x and y axes:

$$\left.\begin{aligned} F_x = -EI_2 X''' + TX' \\ F_y = -EI_1 Y''' + TY' \end{aligned}\right\}. \qquad (20.15)$$

These formulae can also, of course, be obtained directly from (19.3).

In some cases a large force T can result from the bending itself, even if no stretching forces are applied. Let us consider a rod with both ends **clamped** or **hinged** to fixed supports, so that no longitudinal displacement is possible. Then the bending of the rod must result in an extension of it, which leads to a force T in the rod. It is easy to estimate the magnitude of the deflection for which this force becomes important. The length $L + \Delta L$ of the bent rod is given by

$$L + \Delta L = \int_0^L \sqrt{1 + X'^2 + Y'^2}\, dz,$$

taken along the straight line joining the points of support. For slight bending the square root can be expanded in series, and we find

$$\Delta L = \frac{1}{2}\int_0^L (X'^2 + Y'^2)\, dz.$$

The stress force in simple stretching is equal to the relative extension multiplied by Young's modulus and by the area S of the cross-section of the rod. Thus the force T is

$$T = \frac{ES}{2L}\int_0^L (X'^2 + Y'^2)\, dz. \qquad (20.16)$$

If δ is the order of magnitude of the transverse bending, the derivatives X' and Y' are of the order of $\frac{\delta}{L}$, so that the integral in (20.16) is of the order of $\frac{\delta^2}{L}$, and $T \sim ES\left(\frac{\delta}{L}\right)^2$. The orders of magnitude of the first and second

terms in (20.14) are, respectively, $\frac{EI\delta}{L^4}$ and $\frac{T\delta}{L^2} \sim \frac{ES\delta^3}{L^4}$. The moment of inertia I is of the order of h^4, and $S \sim h^2$, where h is the thickness of the rod. Substituting, we easily find that the first and second terms in (20.14) are comparable in magnitude if $\delta \sim h$. Thus, when a rod with fixed ends is bent, the equations of equilibrium can be used in the form (20.4) only if the deflection is small in comparison with the thickness of the rod. If δ is not small compared with h (but still, of course, small compared with L), equations (20.14) must be used. The force T in these equations is not known *a priori*. It must first be regarded as a parameter in the solution, and then determined by formula (20.16) from the solution obtained; this gives the relation between T and the bending forces applied to the rod.

The opposite limiting case is that where the resistance of the rod to bending is small compared with its resistance to stretching, so that the first terms in equations (20.14) can be neglected in comparison with the second terms. Physically this case can be realized either by a very strong tension force T or by a small value of EI, which can result from a small thickness h. Rods under strong tension are called ***strings***. In such cases the equations of equilibrium are

$$TX''+K_x=0\,,\quad TY''=K_y=0\,. \qquad (20.17)$$

The ends of the string are fixed, in the sense that their co-ordinates are given, i.e.,

$$X=Y=0\,. \qquad (20.18)$$

The direction of the ends cannot be decided arbitrarily, but is given by the solution of the equations.

In conclusion, we may show how the equations of equilibrium of a slightly bent rod may be obtained from the **variational principle**, using the expression (18.10) for the elastic energy:

$$F_{rod}=\frac{1}{2}E\int\left\{I_1Y''^2+I_2X''^2\right\}dz\,.$$

In equilibrium the sum of this energy and the potential energy due to the external forces ***K*** acting on the rod must be a minimum, i.e., we must have

$$\delta F_{rod}-\int(K_x\delta X+K_y\delta Y)dz=0\,,$$

where the second term is the work done by the external forces in an infinitesimal displacement of the rod. In varying F_{rod}, we effect a repeated integration by parts:

$$\frac{1}{2}\delta\int X''^2\,dz = \int X''\delta X''dz$$
$$= [X''\delta X'\} - \int X'''\delta X'dz \qquad ,$$
$$= [X''\delta X'\} - [X'''\delta X] + \int X^{(IV)}\delta X dz$$

and similarly for the integral of Y''^2. Collecting terms, we obtain

$$\int [(EI_1 Y^{(IV)} - K_y)\delta Y + (EI_2 X^{(IV)} - K_x)\delta X]dz$$
$$+ EI_1\left[(Y''\delta Y' - Y'''\delta Y)\right] + EI_2\left[(X''\delta X' - X'''\delta X)\right] = 0$$

The integral gives the equilibrium equations (20.4), since the variations δX and δY are arbitrary. The integrated terms give the boundary conditions on these equations; for example, at a free end the variations δX, δY, $\delta X'$, $\delta Y'$ are arbitrary, and the corresponding conditions (20.11) are obtained. Also, the coefficients of δX and δY in these terms give the expressions (20.5) for the components of the shearing force, and those of $\delta X'$ and $\delta Y'$ give the expressions (20.3) for the components of the bending moment.

Finally, the equations of equilibrium (20.14) in the presence of a tension force T can be obtained by the same method if we include in the energy a term

$$T\Delta L = \frac{1}{2}T\int (X'^2 + Y'^2)dz \ ,$$

which is the work done by the force T over a distance ΔL equal to the extension of the rod.

PROBLEMS

Problem 1. Determine the shape of a rod (of length l) bent by its own weight, for various modes of support at the ends.

Solution. The required shape is given by a solution of the equation $\zeta^{(IV)} = \frac{q}{EI}$, where q is the weight per unit length, with the appropriate boundary conditions at its ends, as shown in the text. The following shapes and maximum displacements are obtained for various modes of support at the ends of the rod. The origin is at one end of the rod in each case.

(a) Both ends clamped:

$$\zeta = \frac{qz^2(z-l)^2}{24EI}, \qquad \zeta(l/2) = \frac{ql^4}{384EI} .$$

(b) Both ends supported:

$$\zeta = \frac{qz(z^3 - 2lz^2 + l^3)}{24EI}, \qquad \zeta(l/2) = \frac{5ql^4}{384EI}.$$

(c) One end ($z = l$) clamped, the other supported:

$$\zeta = \frac{qz(2z^3 - 3lz^2 + l^3)}{48EI}, \qquad \zeta(0.42l) = \frac{0.0054ql^4}{EI}.$$

(d) One end (z = 0) clamped, the other free:

$$\zeta = \frac{qz^2(z^2 - 4lz + 6l^2)}{24EI}, \qquad \zeta(l) = \frac{ql^4}{8EI}.$$

Problem 2. Determine the shape of a rod bent by a force f applied to its mid-point.

Solution. We have $\zeta^{(IV)} = 0$ everywhere except at $z = l / 2$. The boundary conditions at the ends of the rod ($z = 0$ and $z = l$) are determined by the mode of support; at $z = l / 2$, ζ, ζ' and ζ'' must be continuous, and the discontinuity in the shearing force $F = -EI\zeta'''$ must be equal to f.

The shape of the rod (for $0 \le z \le l/2$) and the maximum displacement are given by the following formulae:

(a) Both ends clamped:

$$\zeta = \frac{fz^2(3l - 4z)}{48EI}, \qquad \zeta(l/2) = \frac{fl^3}{192EI}.$$

(b) Both ends supported:

$$\zeta = \frac{fz(3l^2 - 4z^2)}{48EI}, \qquad \zeta(l/2) = \frac{fl^3}{48EI}.$$

The rod is symmetrical about its mid-point, so that the functions $\zeta(z)$ in $l/2 \le z \le l$ are obtained simply by replacing z by l - z.

Problem 3. The same as Problem 2, but for a rod clamped at one end ($z = 0$) and free at the other end ($z = l$), to which a force f is applied.

Solution. At all points of the rod F = constant = f, so that $\zeta''' = -\frac{f}{EI}$. Using the conditions $\zeta = 0$, $\zeta' = 0$ for $z = 0$, $\zeta'' = 0$ for $z = l$, we obtain

$$\zeta = \frac{fz^2(3l - z)}{6EI}, \qquad \zeta(l) = \frac{fl^3}{3EI}.$$

Problem 4. Determine the shape of a rod with fixed ends, bent by a couple at its mid-point.

Solution. At all points of the rod $\zeta^{(IV)} = 0$, and at $z = l / 2$ the moment $M = EI\zeta''$ has a discontinuity equal to the moment m of the applied couple. The results are:

(a) Both ends clamped:

$$\zeta = \frac{mz^2(l-2z)}{8EIl} \quad \text{for} \quad 0 \le z \le l/2,$$

$$\zeta = -\frac{m(l-z)^2[l-2(l-z)]}{8EIl} \quad \text{for} \quad l/2 \le z \le l.$$

(b) Both ends hinged:

$$\zeta = \frac{mz(l^2-4z^2)}{24EIl} \quad \text{for} \quad 0 \le z \le l/2,$$

$$\zeta = -\frac{m(l-z)[l^2-4(l-z)^2]}{24EIl} \quad \text{for} \quad l/2 \le z \le l.$$

The rod is bent in opposite directions on the two sides of $z = l/2$.

Problem 5. The same as Problem 4, but for the case where one end is clamped and the other end free, the couple being applied at the latter end.

Solution. At all points of the rod $M = EI\zeta'' = m$, and at $z = 0$ we have $\zeta = 0$, $\zeta' = 0$. The shape is given by

$$\zeta = \frac{mz^2}{EI}.$$

Problem 6. Determine the shape of a circular rod with hinged ends stretched by a force T and bent by a force f applied at its mid-point.

Solution. On the segment $0 \le z \le l/2$ the shearing force is $f/2$, so that (20.15) gives the equation

$$\zeta''' - \frac{T}{EI}\zeta' = -\frac{f}{2EI}.$$

The boundary conditions are $\zeta = \zeta'' = 0$ for $z = 0$ and l; $\zeta' = 0$ for $z = l/2$ (since ζ' is continuous). The shape of the rod (in the segment $0 \le z \le l/2$) is given by

$$\zeta = \frac{f}{2T}\left(z - \frac{\sinh kz}{k\cosh\frac{kl}{2}}\right), \quad k = \sqrt{\frac{T}{EI}}.$$

For small k this gives the result obtained in Problem 2 (b). For large k it becomes $\zeta = \dfrac{fz}{2T}$, i.e., in accordance with equations (20.17), a flexible wire under a force f takes the form of two straight pieces intersecting at $z = l/2$.

If the force T is due to the stretching of the rod by the transverse force, it

must be determined by formula (20.16). Substituting the above result, we obtain the equation

$$\frac{1}{k^6}\left[\frac{3}{2}+\frac{1}{2}\tanh^2\frac{kl}{2}-\frac{3}{kl}\tanh\frac{kl}{2}\right]=\frac{8E^2I^3}{f^2S},$$

which determines *T* as an implicit function of *f*.

Problem 7. A circular rod of infinite length lies in an elastic substance, i.e., when it is bent a force $K=-\alpha\zeta$ proportional to the deflection acts on it. Determine the shape of the rod when a concentrated force *f* acts on it.

Solution. We take the origin at the point where the force *f* is applied. The equation $EI\zeta^{(IV)}=-\alpha\zeta$ holds everywhere except at *z* = 0. The solution must satisfy the condition $\zeta=0$ at $z=\pm\infty$, and at *z* = 0 ζ' and ζ'' must be continuous; the difference between the shearing forces $F=-EI\zeta'''$ for $z\to 0+$ and $z\to 0-$ must be *f*. The required solution is

$$\zeta=\frac{f}{8\beta^3 EI}e^{-\beta|z|}[\cos\beta\,|\,z\,|+son\beta\,|\,z\,|],\ \beta=\left(\frac{\alpha}{4EI}\right)^{1/4}.$$

Problem 8. Derive the equation of equilibrium for a slightly bent thin circular rod which, in its undeformed state, is an arc of a circle and is bent in its plane by radial forces.

Solution. Taking the origin of polar co-ordinates r,ϕ at the centre of the circle, we write the equation of the deformed rod as $r=a+\zeta(\phi)$, where *a* is the radius of the arc and ζ a small radial displacement. Using the expression for the radius of curvature in polar co-ordinates, we find as far as the first order in ζ

$$\frac{1}{R}=\frac{r^2-rr'+2r'^2}{(r^2+r'^2)^{3/2}}\approx\frac{1}{a}-\frac{\zeta+\zeta''}{a^2},$$

where the prime denotes differentiation with respect to ϕ. According to (18.11), the elastic bending energy is

$$F_{rod}=\frac{EI}{2}\int_0^{\phi_0}\left(\frac{1}{R}-\frac{1}{a}\right)^2 ad\phi=\frac{EI}{2a^3}\int_0^{\phi_0}(\zeta+\zeta'')^2d\phi,$$

ϕ_0 being the angle subtended by the arc at its centre. The equation of equilibrium is obtained from the variational principle

$$\delta F_{rod}-\int_0^{\phi_0}\delta\zeta K_r ad\phi=0,$$

where K_r is the external radial force per unit length, with the auxiliary condition

$$\int_0^{\phi_0}\zeta d\phi=0,$$

which is, in this approximation, the statement of the fact that the total length

of the rod is unchanged, i.e., it undergoes no general extension. Using Lagrange's method, we put

$$\delta F_{rod} - \int_0^{\phi_0} aK_r \delta\zeta d\phi + a\alpha \int_0^{\phi_0} \delta\zeta d\phi = 0\,,$$

where α is a constant. Varying the integrand in F_{rod} and integrating the $\delta\zeta''$ term twice by parts, we obtain

$$\int \left\{ \frac{EI}{a^3}(\zeta + 2\zeta'' + \zeta^{(IV)}) - aK_r + \alpha a \right\} \delta\zeta d\phi + \frac{EI}{a^3}[(\zeta + \zeta'')\delta\zeta'] - \frac{EI}{a^3}[(\zeta' + \zeta''')\delta\zeta] = 0$$

Hence we find the equation of equilibrium[20]

$$\frac{EI}{a^4}(\zeta^{(IV)} + 2\zeta'' + \zeta) - K_r + \alpha = 0\,, \qquad (1)$$

the shearing force $F = -\frac{EI}{a^3}(\zeta' + \zeta''')$, and the bending moment $M = \frac{EI}{a^2}(\zeta + \zeta'')$; cf. the end of §20. The constant α is determined from the condition that the rod as a whole is not stretched.

Problem 9. Determine the deformation of a circular ring bent by two forces *f* applied along a diameter (Fig. 18).

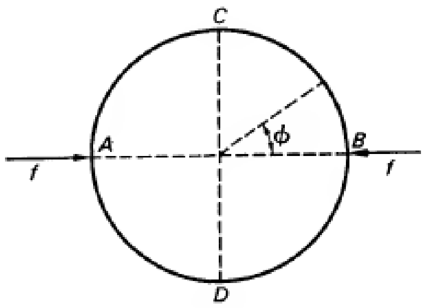

Fig. 18

Solution. Integrating equation (1), Problem 8, along the circumference of the ring, we have $2\pi\alpha a = \int K_r a d\phi = 2f$. We have equation (1) with $K_r = 0$ everywhere except at $\phi = 0$ and $\phi = \pi$:

$$\zeta^{(IV)} + 2\zeta'' + \zeta + \frac{fa^3}{\pi EI} = 0\,.$$

The required deformation of the ring is symmetrical about the diameters *AB* and *CD*, and so we must have $\zeta' = 0$ at *A*, *B*, *C* and *D*. The difference in the shearing forces for $\phi \to 0\pm$ must be *f*. The solution of the equation of equilibrium which satisfies these conditions is

[20] In the absence of external forces, $K_r = 0$ and $\alpha = 0$; the non-zero solutions of the resulting homogeneous equation correspond to a simple rotation or translation of the whole rod.

$$\zeta = \frac{fa^3}{EI}\left(\frac{1}{\pi} + \frac{1}{4}\phi\cos\phi - \frac{\pi}{8}\cos\phi - \frac{1}{4}\sin\phi\right), \quad 0 \le \phi \le \pi .$$

In particular, the points A and B approach through a distance

$$\left|\zeta(0) + \zeta(\pi)\right| = \frac{fa^3}{EI}\left(\frac{\pi}{4} - \frac{2}{\pi}\right).$$

§21. The stability of elastic systems

The behaviour of a rod subject to longitudinal compressing forces is the simplest example of the important phenomenon of elastic instability, first discovered by L. Euler.

In the absence of transverse bending forces K_x, K_y, the equations of equilibrium (20.14) for a compressed rod have the evident solution $X = Y = 0$, which corresponds to the rod's remaining straight under a longitudinal force $|T|$. This solution, however, gives a stable equilibrium of the rod only if the compressing force $|T|$ is less than a certain critical value T_{cr}. For $|T| < T_{cr}$, the straight rod is stable with respect to any small perturbation. In other words, if the rod is slightly bent by some small force, it will tend to return to its original position when that force ceases to act.

If, on the other hand, $|T| > T_{cr}$, the straight rod is in unstable equilibrium. An infinitesimal bending suffices to destroy the equilibrium, and a large bending of the rod results. It is clear that, if this is so, the compressed rod cannot actually remain straight.

The behaviour of the rod after it ceases to be stable must satisfy the equations for bending with large deflections. The value T_{cr} of the critical load, however, can be obtained from the equations for small deflections. For $|T| = T_{cr}$, the straight rod is in neutral equilibrium. This means that, besides the solution $X = Y = 0$, there must also be states where the rod is slightly bent but still in equilibrium. Hence the critical value of T_{cr} is the value of $|T|$ for which the equations

$$\left.\begin{aligned} EI_2 X^{(IV)} + |T|\,X'' = 0 \\ EI_1 Y^{(IV)} + |T|\,Y'' = 0 \end{aligned}\right\} \qquad (21.1)$$

have a non-zero solution. This solution gives also the nature of the deformation of the rod immediately after it ceases to be stable.

The following Problems give some typical cases of the loss of stability in various elastic systems.

PROBLEMS

Problem 1. Determine the critical compression force for a rod with hinged

ends.

Solution. Since we are seeking the smallest value of $|T|$ for which equations (21.1) have a non-zero solution, it is sufficient to consider only the equation which contains the smaller of I_1 and I_2. Let $I_2 < I_1$. Then we seek a solution of the equation $EI_2 X^{(IV)} + |T| X'' = 0$ in the form $X = A + Bz + C\sin kz + D\cos kz$, where $k = \sqrt{\frac{|T|}{EI_2}}$. The non-zero solution which satisfies the conditions $X = X'' = 0$ for $z = 0$ and $z = l$ is $X = C\sin kz$, with $\sin kl = 0$. Hence we find the required critical force to be $T_{cr} = \frac{\pi^2 EI_2}{l^2}$. On ceasing to be stable, the rod takes the form shown in Fig. 19a.

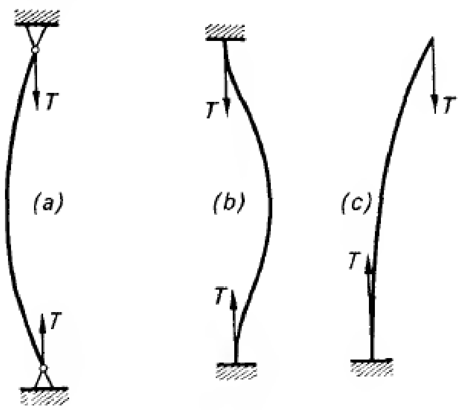

Fig. 19

Problem 2. The same as Problem 1, but for a rod with clamped ends (Fig. 19b).

Solution. $T_{cr} = \frac{4\pi^2 EI_2}{l^2}$.

Problem 3. The same as Problem 1, but for a rod with one end clamped and the other free (Fig. 19c).

Solution. $T_{cr} = \frac{\pi^2 EI_2}{4l^2}$.

Problem 4. Determine the critical compression force for a circular rod with hinged ends in an elastic medium (see §20, Problem 7).

Solution. The equations (21.1) must now be replaced by

$$EIX^{(IV)} + |T| X'' + \alpha X = 0.$$

A similar treatment gives the solution $X = A\sin\frac{n\pi z}{l}$,

$$T_{cr} = \frac{\pi^2 EI}{l^2}\left(n^2 + \frac{\alpha l^4}{n^2 \pi^4 EI}\right),$$

where n is the integer for which T_{cr} is least. When α is large, $n > 1$, i.e., the rod exhibits several undulations as soon as it ceases to be stable.

Problem 5. A circular rod is subjected to torsion, its ends being clamped. Determine the critical torsion beyond which the straight rod becomes unstable.

Solution. The critical value of the torsion angle is determined by the appearance of non-zero solutions of the equations for slight bending of a twisted rod. To derive these equations, we substitute the expression (19.7) $\boldsymbol{M} = EI\boldsymbol{t} \times \frac{d\boldsymbol{t}}{dl} + C\tau\boldsymbol{t}$, where τ is the constant torsion angle, in equation (19.3). This gives

$$EI\boldsymbol{t} \times \frac{d^2\boldsymbol{t}}{dl^2} + C\tau\frac{d\boldsymbol{t}}{dl} - \boldsymbol{F} \times \boldsymbol{t} = 0 .$$

We differentiate; since the bending is not large, $\boldsymbol{t}$ may be regarded as a constant vector $\boldsymbol{t}_0$ along the axis of the rod (the z-axis) in differentiating the first and third terms. Since also $\frac{d\boldsymbol{F}}{dl} = 0$ (there being no external forces except at the ends of the rod), we obtain

$$EI\boldsymbol{t}_0 \times \frac{d^3\boldsymbol{t}}{dl^3} + C\tau\frac{d^2\boldsymbol{t}}{dl^2} = 0 ,$$

or, in components,

$$\left.\begin{aligned} Y^{(IV)} - \kappa X''' = 0 \\ X^{(IV)} + \kappa Y''' = 0 \end{aligned}\right\} ,$$

where $\kappa = \frac{C\tau}{EI}$. Taking as the unknown function $\xi = X + iY$, we obtain $\xi^{(IV)} - i\kappa\xi''' = 0$. We seek a solution which satisfies the conditions $\xi = 0$, $\xi' = 0$ for $z = 0$ and $z = l$, in the form $\xi = a(1 + i\kappa z - e^{i\kappa z}) + bz^2$, and obtain as the compatibility condition of the equations for a and b the relation $e^{ikl} = \frac{2 + i\kappa l}{2 - i\kappa l}$, whence $\frac{\kappa l}{2} = \tan\frac{\kappa l}{2}$. The smallest root of this equation is $\frac{\kappa l}{2} = 4.49$, so that $\tau_{cr} = 8.98\frac{EI}{Cl}$.

Problem 7. Determine the limit of stability of a vertical rod under its own weight, the other end being clamped.

Solution. If the longitudinal stress $F_z \equiv T$ varies along the rod, $\frac{dF_z}{dl} \neq 0$ in the first term of (20.1), and equations (20.14) are replaced by

$$\left.\begin{aligned} EI_2 X^{(IV)} - (TX')' - K_x = 0 \\ EI_1 Y^{(IV)} - (TY')' - K_y = 0 \end{aligned}\right\}.$$

In the case considered, there are no transverse bending forces anywhere in the rod, and $T = -q(l-z)$, where q is the weight of the rod per unit length and z is measured from the lower end. Assuming that $I_2 < I_1$, we consider the equation

$$EI_2 X''' = TX' = -q(l-z)X';$$

for $z = l$, X''' = 0 automatically. Yhe general integral of this equation for the function $u = X'$ is

$$u = \eta^{1/3}[aJ_{-1/3}(\eta) + bJ_{1/3}(\eta)],$$

where

$$\eta = \frac{2}{3}\sqrt{\frac{q(l-z)^3}{EI_2}}.$$

The boundary conditions X' = 0 for z = 0 and X'' = 0 for $z = l$ give for the function $u(\eta)$ the conditions u = 0 for $\eta = \eta_0 \equiv \frac{2}{3}\sqrt{\frac{ql^3}{EI_2}}$, $u'\eta^{1/3} = 0$ for $\eta = 0$. In order to satisfy these conditions we must put b = 0 and $J_{-1/2}(\eta_0) = 0$. The smallest root of this equation is $\eta_0 = 1.87$, and so the critical length is $l_{cr} = 1.98\left(\frac{EI_2}{q}\right)^{1/3}$.

Problem 8. A rod has an elongated cross-section, so that $I_2 >> I_1$. One end is clamped and a force f is applied to the other end, which is free, so as to bend it in the principal *xz*-plane (in which the flexural rigidity is EI_2). Determine the critical force f_{cr} at which the rod bent in a plane becomes unstable and the rod is bent sideways (in the *yz*-plane), at the same time undergoing torsion.

Solution. Since the rigidity EI_2 is large compared with EI_1 (and with the torsional rigidity C),[21] the instability as regards sideways bending occurs while the deflection in the *xz*-plane is still small. To determine the point where instability sets in, we must form the equations for slight sideways bending of the rod, retaining the terms proportional to the products of the force f in the *xz*-plane and the small displacements. Since there is a concentrated force only at the free end of the rod, we have $\boldsymbol{F} = \boldsymbol{f}$ at all points, and at the free end ($z = l$) the moment $\boldsymbol{M}$ = 0; from formula (19.6) we find the components of the moment relative to a fixed system of co-ordinates x, y, z:

[21] For example, for a narrow rectangular cross-section of sides b and h ($b >> h$), we have $EI_1 = bh^3E/12$, $EI_2 = b^3hE/12$, $C = bh^3\mu/3$.

$M_x = 0$, $M_y = (l-z)f$, $M_z = (Y = Y_0)f$, where $Y_0 = Y(l)$. Taking the components along co-ordinate axes ξ, η, ζ fixed at each point to the rod, we obtain as far as the first-order terms in the displacements $M_\xi = \phi(l-z)f$, $M_\eta = (l-z)f$, $M_\zeta = (l-z)f\dfrac{dY}{dz} + f(Y - Y_0)$, where ϕ is the total angle of rotation of a cross-section of the rod under torsion; the torsion angle $\tau = \dfrac{d\phi}{dz}$ is not constant along the rod. According to (18.6) and (18.9), however, we have for a small deflection

$$\left.\begin{aligned} M_\xi &= -EI_1 Y'' \\ M_\eta &= EI_2 X'' \\ M_\zeta &= C\phi' \end{aligned}\right\};$$

comparing, we obtain the equations of equilibrium

$$\left.\begin{aligned} EI_2 X'' &= (l-z)f \\ EI_1 Y'' &= -\phi(l-z)f \\ C\phi' &= (l-z)fY' + (Y - Y_0)f \end{aligned}\right\}.$$

The first of these equations gives the main bending of the rod, in the *xz*-plane; we require the value of *f* for which non-zero solutions of the second and third equations appear. Eliminating *Y*, we find

$$\phi'' + k^2 (l-z)^2 \phi = 0, \quad k^2 = \frac{f^2}{EI_1 C}.$$

The general integral of this equation is

$$\phi = a\sqrt{l-z}\,J_{1/4}\left[\frac{k}{2}(l-z)^2\right] + b\sqrt{l-z}\,J_{-1/4}\left[\frac{k}{2}(l-z)^2\right].$$

At the clamped end ($z = 0$) we must have $\phi = 0$, and at the free end the twisting moment $C\phi' = 0$. From the second condition we have $a = 0$, and then the first gives $J_{-1/4}(\frac{k}{2}l^2) = 0$. The smallest root of this equation is $\frac{k}{2}l^2 = 2.006$, whence $f_{cr} = 4.01\dfrac{\sqrt{EI_1 C}}{l^2}$.