CHAPTER III

ELASTIC WAVES

§22. Elastic waves in an isotropic medium

If motion occurs in a deformed body, its temperature is not in general constant, but varies in both time and space. This considerably complicates the exact equations of motion in the general case of arbitrary motions.

Usually, however, matters are simplified in that the transfer of heat from one part of the body to another (by simple thermal conduction) occurs very slowly. If the heat exchange during times of the order of the period of oscillatory motions in the body is negligible, we can regard any part of the body as thermally insulated, i.e., the motion is **adiabatic**. In adiabatic deformations, however, σ_{ik} is given in terms of u_{ik} by the usual formulae, the only difference being that the ordinary (isothermal) values of E and σ must be replaced by their adiabatic values (see §6). We shall assume in what follows that this condition is fulfilled, and accordingly E and σ in this chapter will be understood to have their adiabatic values.

In order to obtain the equations of motion for an elastic medium, we must equate the internal stress force $\dfrac{\partial \sigma_{ik}}{\partial x_k}$ to the product of the acceleration $\ddot{u}_i$ and the mass per unit volume of the body, i.e., its density ρ:

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ik}}{\partial x_k} . \qquad (22.1)$$

This is the general equation of motion.

In particular, the equations of motion for an isotropic elastic medium can be written down at once by analogy with the equation of equilibrium (7.2). We have

$$\rho \ddot{\boldsymbol{u}} = \frac{E}{2(1+\sigma)} \Delta \boldsymbol{u} + \frac{E}{2(1+\sigma)(1-2\sigma)} \boldsymbol{grad}\,\mathrm{div}\,\boldsymbol{u} . \qquad (22.2)$$

$$\boxed{\rho \ddot{\boldsymbol{u}} = \mu \Delta \boldsymbol{u} + (\lambda + \mu)\boldsymbol{grad}\,\mathrm{div}\,\boldsymbol{u}}$$

Since all deformations are supposed small, the motions considered in the theory of elasticity are small ***elastic oscillations*** or ***elastic waves***. We shall begin by discussing a plane elastic wave in an infinite isotropic medium, i.e., a wave in which the deformation $\boldsymbol{u}$ is a function only of one co-ordinate (x, say) and of the time. All derivatives with respect to y and z in equations (22.2) are then zero, and we obtain for the components of the vector $\boldsymbol{u}$ the equations

$$\left.\begin{aligned} \frac{\partial^2 u_x}{\partial x^2} - \frac{1}{c_l^2}\frac{\partial^2 u_x}{\partial t^2} = 0 \\ \frac{\partial^2 u_y}{\partial x^2} - \frac{1}{c_t^2}\frac{\partial^2 u_y}{\partial t^2} = 0 \end{aligned}\right\} \qquad (22.3)$$

(the equation for u_z is the same as that for u_y); here[1]

$$\left.\begin{aligned} c_l &= \sqrt{\frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)}} \\ c_t &= \sqrt{\frac{E}{2\rho(1+\sigma)}} \end{aligned}\right\}, \qquad (22.4)$$

$$\boxed{\begin{aligned} c_l &= \sqrt{\frac{2\mu+\lambda}{\rho}} \\ c_t &= \sqrt{\frac{\mu}{\rho}} \end{aligned}}$$

Equations (22.3) are ordinary wave equations in one dimension, and the quantities c_l and c_t which appear in them are the velocities of propagation of the wave. We see that the velocity of propagation for the component u_x is different from that for u_y and u_z.

Thus an elastic wave is essentially two waves propagated independently. In one (u_x) the displacement is in the direction of propagation; this is called the ***longitudinal wave***, and is propagated with velocity c_l. In the other wave (u_y, u_z) the displacement is in a plane perpendicular to the direction of propagation; this is called the ***transverse wave***, and is propagated with velocity c_t. It is seen from (22.4) that the velocity of longitudinal waves is always greater than that of transverse waves: we always have[2]

$$c_l > \sqrt{\frac{4}{3}}c_t. \qquad (22.5)$$

$$\boxed{\frac{c_l^2}{c_t^2} = \frac{2\mu+\lambda}{\mu} = \frac{2-2\sigma}{1-2\sigma}}$$

The velocities c_l and c_t are often called the ***longitudinal and transverse velocities of sound***.

We know that the volume change in a deformation is given by the sum of the diagonal terms in the strain tensor, i.e., by $u_{ii} \equiv \operatorname{div}\boldsymbol{u}$. In the transverse wave there is no component u_x, and, since the other components do not depend on *y* or *z*, $\operatorname{div}\boldsymbol{u} = 0$ for such a wave. Thus transverse waves do not involve any change in volume of the parts of the body. For longitudinal waves, however, $\operatorname{div}\boldsymbol{u} \neq 0$, and these waves involve compressions and expansions in the body.

The separation of the wave into two parts propagated independently with different velocities can also be effected in the general case of an arbitrary (not plane) elastic wave in an infinite medium. We rewrite

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[1] We may give also expressions for c_l and c_t in terms of the moduli of compression and rigidity and the Lamé coefficients: $c_l = \sqrt{\frac{3K+4\mu}{3\rho}} = \sqrt{\frac{\lambda+2\mu}{\rho}}$, $c_t = \sqrt{\frac{\mu}{\rho}}$.

[2] Since σ actually varies only between 0 and 1/2 (see the second footnote to §5), we

equation (22.2) in terms of the velocities c_l and c_t:

$$\ddot{\boldsymbol{u}} = c_t^2 \Delta \boldsymbol{u} + (c_l^2 - c_t^2)\,\boldsymbol{grad}\,\mathrm{div}\,\boldsymbol{u}\,. \qquad (22.6)$$

We then represent the vector $\boldsymbol{u}$ as the sum of two parts:

$$\boldsymbol{u} = \boldsymbol{u}_l + \boldsymbol{u}_t\,, \qquad (22.7)$$

of which one satisfies

$$\mathrm{div}\,\boldsymbol{u}_t = 0 \qquad (22.8)$$

and the other satisfies

$$\mathrm{curl}\,\boldsymbol{u}_l = 0\,. \qquad (22.9)$$

We know from vector analysis that this representation (i.e., the expression of a vector as the sum of the curl of a vector and the gradient of a scalar) is always possible.

Substituting $\boldsymbol{u} = \boldsymbol{u}_l + \boldsymbol{u}_t$ in (22.6), we obtain

$$\ddot{\boldsymbol{u}}_l + \ddot{\boldsymbol{u}}_t = c_t^2 \Delta(\boldsymbol{u}_l + \boldsymbol{u}_t) + (c_l^2 - c_t^2)\,\boldsymbol{grad}\,\mathrm{div}\,\boldsymbol{u}_l\,. \qquad (22.10)$$

We take the divergence of both sides. Since $\mathrm{div}\,\boldsymbol{u}_t = 0$, the result is

$$\mathrm{div}\,\ddot{\boldsymbol{u}}_l = c_t^2\,\mathrm{div}\,\Delta\boldsymbol{u}_l + (c_l^2 - c_t^2)\Delta\,\mathrm{div}\,\boldsymbol{u}_l\,,$$

or

$$\mathrm{div}(\ddot{\boldsymbol{u}}_l - c_l^2 \Delta \boldsymbol{u}_l) = 0\,.$$

The curl of the expression in parentheses is also zero, by (22.9). If the curl and divergence of a vector both vanish in all space, that vector must be zero identically. Thus

$$\frac{\partial^2 \boldsymbol{u}_l}{\partial t^2} - c_l^2 \Delta \boldsymbol{u}_l = 0\,. \qquad (22.11)$$

Similarly, taking the curl of equation (22.10) we have, since the curls of $\boldsymbol{u}_l$ and of any gradient are zero, $\boldsymbol{curl}(\ddot{\boldsymbol{u}}_t - c_t^2 \Delta \boldsymbol{u}_t) = 0$. Since the divergence of the expression in parentheses is also zero, we obtain an equation of the same form as (22.11):

$$\frac{\partial^2 \boldsymbol{u}_t}{\partial t^2} - c_t^2 \Delta \boldsymbol{u}_t = 0\,. \qquad (22.12)$$

Equations (22.11) and (22.12) are ordinary wave equations in three dimensions. Each of them represents the propagation of an elastic wave, with velocity c_l and c_t, respectively. One wave ($\boldsymbol{u}_t$) does not involve a change in volume (since $\mathrm{div}\,\boldsymbol{u}_t = 0$), while the other ($\boldsymbol{u}_l$) is accompanied by volume compressions and expansions.

In a monochromatic elastic wave, the displacement vector is

$$u = \mathrm{Re}\left\{\boldsymbol{u}_0(\boldsymbol{r})e^{-i\omega t}\right\}, \qquad (22.13)$$

where $\boldsymbol{u}_0$ is a function of the co-ordinates which satisfies the equation

$$c_l^2 \Delta \boldsymbol{u}_0 + (c_l^2 - c_t^2)\,\boldsymbol{grad}\,\mathrm{div}\,\boldsymbol{u}_0 + \omega^2 \boldsymbol{u}_0 = 0\,, \qquad (22.14)$$

obtained by substituting (22.13) in (22.6). The longitudinal and transverse

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always have $c_l > \sqrt{2}c_t$.

parts of a monochromatic wave satisfy the equations

$$\left.\begin{aligned} \Delta \boldsymbol{u}_l + k_l^2 \boldsymbol{u}_l &= 0 \\ \Delta \boldsymbol{u}_t + k_t^2 \boldsymbol{u}_t &= 0 \end{aligned}\right\}, \tag{22.15}$$

where $k_l = \dfrac{\omega}{c_l}$, $k_t = \dfrac{\omega}{c_t}$ are the wave numbers of the longitudinal and transverse waves.

Finally, let us consider the **reflection** and **refraction** of a plane monochromatic elastic wave at the boundary between two different elastic media. It must be borne in mind that the nature of the wave is in general changed when it is reflected or refracted. If a purely transverse or purely longitudinal wave is incident on a surface of separation, the result is a mixed wave containing both transverse and longitudinal parts. The nature of the wave remains unchanged (as we see from symmetry) only when it is incident normally on the surface of separation, or when a transverse wave whose oscillations are parallel to the plane of separation is incident (at any angle).

The relations giving the directions of the reflected and refracted waves can be obtained immediately from the constancy of the frequency and of the tangential components of the wave vector.[3] Let θ and θ' be the angles of incidence and reflection (or refraction) and c, c' the velocities of the two waves. Then

$$\frac{\sin\theta}{\sin\theta'} = \frac{c}{c'}. \tag{22.16}$$

For example, let the incident wave be transverse. Then $c = c_{t1}$ is the velocity of transverse waves in medium 1. For the transverse reflected wave we have $c' = c_{t1}$ also, so that (22.16) gives $\theta = \theta'$, i.e., the angle of incidence is equal to the angle of reflection. For the longitudinal reflected wave, however, $c' = c_{l1}$, and so

$$\frac{\sin\theta}{\sin\theta'} = \frac{c_{t1}}{c_{l1}}.$$

For the transverse part of the refracted wave $c' = c_{t2}$, and for a transverse incident wave

$$\frac{\sin\theta}{\sin\theta'} = \frac{c_{t1}}{c_{t2}}.$$

Similarly, for the longitudinal refracted wave

$$\frac{\sin\theta}{\sin\theta'} = \frac{c_{t1}}{c_{l2}}.$$

[3] See *Fluid Mechanics*, §65. The arguments given there are applicable in their entirety.

PROBLEMS

Problem 1. Determine the reflection coefficient for a longitudinal monochromatic wave incident at any angle on the surface of a body (with a vacuum outside).[4]

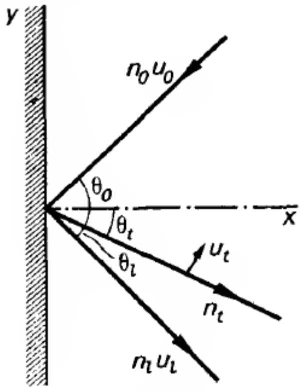

FIG. 20

Solution. When the wave is reflected, there are in general both longitudinal and transverse reflected waves. It is clear from symmetry that the displacement vector in the transverse reflected wave lies in the plane of incidence (Fig. 20, where $\boldsymbol{n}_0$, $\boldsymbol{n}_l$ and $\boldsymbol{n}_t$ are unit vectors in the direction of propagation of the incident, longitudinal reflected and transverse reflected waves, and $\boldsymbol{u}_0$, $\boldsymbol{u}_l$ and $\boldsymbol{u}_t$ the corresponding displacement vectors). The total displacement in the body is given by the sum (omitting the common factor $e^{-i\omega t}$ for brevity)

$$u = A_0 \boldsymbol{n}_0 e^{\boldsymbol{k}_0 \cdot \boldsymbol{r}} + A_l \boldsymbol{n}_l e^{\boldsymbol{k}_l \cdot \boldsymbol{r}} + A_t \boldsymbol{a} \times \boldsymbol{n}_t e^{\boldsymbol{k}_t \cdot \boldsymbol{r}},$$

where $\boldsymbol{a}$ is a unit vector perpendicular to the plane of incidence. The magnitudes of the wave vectors are $k_0 = k_l = \frac{\omega}{c_l}$, $k_t = \frac{\omega}{c_t}$, and the angles of incidence θ_0 and of reflection θ_l, θ_t are related by $\theta_l = \theta_0$, $\sin\theta_t = \frac{c_t}{c_l}\sin\theta_0$. For the components of the strain tensor at the boundary we obtain

$$\begin{cases} u_{xx} = ik_0(A_0 + A_l)\cos^2\theta_0 + iA_t k_t \cos\theta_t \sin\theta_t \\ u_{ll} = ik_0(A_0 + A_l) \\ u_{xy} = ik_0(A_0 - A_l)\sin\theta_0\cos\theta_0 + \frac{1}{2} iA_t k_t(\cos^2\theta_t - \sin^2\theta_t) \end{cases}$$

again omitting the common exponential factor. The components of the stress tensor can be calculated from the general formula (5.11), which can

[4] The more general case of the reflection of sound waves from a solid-liquid interface, and the similar problem of the reflection of a wave incident from a liquid on to a solid, are discussed by L. M. Brekhovskikh, *Waves in Layered Media*, §4, Academic Press, New York

here be conveniently written

$$\sigma_{ik} = 2\rho c_t^2 u_{ik} + \rho(c_l^2 - 2c_t^2)u_{ll}\delta_{ik} .$$

The boundary conditions at the free surface of the medium are $\sigma_{ik}n_k = 0$, whence

$$\sigma_{xx} = \sigma_{yx} = 0 ,$$

giving two equations which express A_l and A_t in terms of A_0. The result is

$$\begin{cases} A_l = A_0 \dfrac{c_t^2 \sin 2\theta_t \sin 2\theta_0 - c_l^2 \cos^2 2\theta_t}{c_t^2 \sin 2\theta_t \sin 2\theta_0 + c_l^2 \cos^2 2\theta_t} \\ A_t = -A_0 \dfrac{2c_l c_t \sin 2\theta_0 \cos 2\theta_t}{c_t^2 \sin 2\theta_t \sin 2\theta_0 + c_l^2 \cos^2 2\theta_t} \end{cases} .$$

For $\theta_0 = 0$ we have $A_l = -A_0$, $A_t = 0$, i.e., the wave is reflected as a purely longitudinal wave. The ratio of the energy flux density components normal to the surface in the reflected and incident longitudinal waves is $R_l = \left|\dfrac{A_t}{A_0}\right|^2$. The corresponding ratio for the reflected transverse wave is

$$R_t = \frac{c_t \cos\theta_t}{c_l \cos\theta_0}\left|\frac{A_t}{A_0}\right|^2 .$$

The sum of R_l and R_t is, of course, 1.

Problem 2. The same as Problem 1, but for a transverse incident wave (with the oscillations in the plane of incidence).[5]

Solution. The wave is reflected as a transverse and a longitudinal wave, with $\theta_t = \theta_0$, $c_t \sin\theta_l = c_l \sin\theta_0$. The total displacement vector is

$$\boldsymbol{u} = \boldsymbol{a} \times \boldsymbol{n}_0 A_0 e^{i\boldsymbol{k}_0\cdot\boldsymbol{r}} + \boldsymbol{n}_l A_l e^{i\boldsymbol{k}_l\cdot\boldsymbol{r}} + \boldsymbol{a} \times \boldsymbol{n}_t A_t e^{i\boldsymbol{k}_t\cdot\boldsymbol{r}} .$$

The expressions for the amplitudes of the reflected waves are

$$\begin{cases} \dfrac{A_t}{A_0} = \dfrac{c_t^2 \sin 2\theta_l \sin 2\theta_0 - c_l^2 \cos^2 2\theta_0}{c_t^2 \sin 2\theta_l \sin 2\theta_0 + c_l^2 \cos^2 2\theta_0} \\ \dfrac{A_l}{A_0} = \dfrac{2c_l c_t \sin 2\theta_0 \cos 2\theta_0}{c_t^2 \sin 2\theta_l \sin 2\theta_0 + c_l^2 \cos^2 2\theta_0} \end{cases} .$$

Problem 3. Determine the characteristic frequencies of radial vibrations of an elastic sphere of radius *R*.

Solution. We take spherical polar co-ordinates, with the origin at the centre of the sphere. For radial vibrations, ***u*** is along the radius, and is a function of ***r*** and *t* only. Hence curl***u***=0. We define the displacement "potential" ϕ by $u_r = u = \dfrac{\partial\phi}{\partial r}$. The equation of motion, expressed in

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1960.a

[5] If the oscillations are perpendicular to the plane of incidence, the wave is entirely reflected as a wave of the same kind, and so $R_t = 1$.

terms of ϕ, is just the wave equation $c_l^2 \Delta\phi = \ddot{\phi}$, or, for oscillations periodic in time ($\sim e^{-i\omega t}$),

$$\Delta\phi \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = -k^2\phi\ , \quad k = \frac{\omega}{c_l}\ .$$

The solution which is finite at the origin is $\phi = \frac{A}{r}\sin kr$ (the time factor is omitted). The radial stress is

$$\begin{aligned}\sigma_{rr} &= \rho\left\{(c_l^2 - 2c_t^2)u_{ii} + 2c_t^2 u_{rr}\right\} \\ &= \rho\left\{(c_l^2 - 2c_t^2)\Delta\phi + 2c_t^2\phi''\right\}\end{aligned},$$

or, using (1),

$$\frac{\sigma_{rr}}{\rho} = -\omega^2\phi - \frac{4c_t^2\phi'}{r}\ .$$

The boundary condition $\sigma_{rr}(R) = 0$ leads to the equation

$$\frac{\tan kR}{kR} = \frac{1}{1 - \left(\frac{kRc_l}{2c_t}\right)^2}\ ,$$

whose roots determine the characteristic frequencies $\omega = kc_l$ of the vibrations.

Problem 4. Determine the frequency of radial vibrations of a spherical cavity in an infinite elastic medium for which $c_l >> c_t$ (M. A. Isakovich 1949).

Solution. In an infinite medium, radial oscillations of the cavity are accompanied by the emission of longitudinal sound waves, leading to loss of energy and hence to damping of the oscillations. When $c_l >> c_t$ (i.e., $K >> \mu$), this emission is weak, and we can speak of the characteristic frequencies of oscillations with a small coefficient of damping.

We seek a solution of equation (1), Problem 3, in the form of an outgoing spherical wave $\phi = \frac{A}{r}e^{ikr}$, $k = \frac{\omega}{c_l}$ and, using (2), obtain from the boundary condition $\sigma_{rr}(R) = 0$ the result $\left(\frac{kRc_l}{c_t}\right)^2 = 4(1 - ikR)$. Hence, when $c_l >> c_t$,

$$\omega = \frac{2c_t}{R}\left(1 - i\frac{c_t}{c_l}\right).$$

The real part of ω gives the characteristic frequency of oscillation; the imaginary part gives the damping coefficient. In an incompressible medium ($c_l \to \infty$) there would of course be no damping. These vibrations are specifically due to the shear resistance of the medium

($\mu \neq 0$). Il should be noticed that they have $kR = 2\frac{c_t}{c_l} << 1$, i.e., the corresponding wavelength is large compared with *R*; it is interesting to compare this with the result for vibrations of an elastic sphere, where with $c_l >> c_t$ the first characteristic frequency is given by (3): $kR = \pi$.

§23. Elastic waves in crystals

§24. Surface waves

A particular kind of elastic waves are those propagated near the surface of a body without penetrating into it (***Rayleigh waves***). We write the equation of motion in the form (22.11) and (22.12):

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0\,, \tag{24.1}$$

where u is any component of the vectors $\boldsymbol{u}_l$, $\boldsymbol{u}_t$, and c is the corresponding velocity c_l or c_t, and seek solutions corresponding to these surface waves. The surface of the elastic medium is supposed plane and of infinite extent. We take this plane as the *xy*-plane; let the medium be in $z < 0$.

Let us consider a plane monochromatic surface wave propagated along the *x*-axis. Accordingly $u = e^{i(kx-\omega t)} f(z)$. Substituting this expression in (24.1), we obtain for the function *f*(*z*) the equation

$$\frac{d^2 f}{dz^2} = \left(k^2 - \frac{\omega^2}{c^2}\right) f\,.$$

If $k^2 - \frac{\omega^2}{c^2} < 0$, this equation gives a periodic function *f*, i.e., we obtain an ordinary plane wave which is not damped inside the body. We must therefore suppose that $k^2 - \frac{\omega^2}{c^2} > 0$. Then the solutions for *f* are

$$f(z) = \text{constant} \times \exp\left(\pm\sqrt{k^2 - \frac{\omega^2}{c^2}} z\right).$$

The solution with the minus sign would correspond to an unlimited increase in the deformation for $z \to -\infty$. This solution is clearly impossible, and so the plus sign must be taken.

Thus we have the following solution of the equations of motion:

$$u = \text{constant} \times e^{i(kx-\omega t)} e^{\kappa z}\,, \tag{24.2}$$

where

$$\kappa = \sqrt{k^2 - \frac{\omega^2}{ch2}}\,. \qquad (24.3)$$

It corresponds to a wave which is exponentially damped towards the interior of the medium, i.e., is propagated only near the surface. The quantity κ determines the rapidity of the damping.

The true displacement vector $\boldsymbol{u}$ in the wave is the sum of the vectors $\boldsymbol{u}_l$ and $\boldsymbol{u}_t$, the components of each of which satisfy the equation (24.1) with $c = c_l$ for $\boldsymbol{u}_l$ and c_t for $\boldsymbol{u}_t$. For **volume waves** in an infinite medium, the two parts are independently propagated waves. For **surface waves**, however, this division into two independent parts is not possible, on account of the boundary conditions. The displacement vector $\boldsymbol{u}$ must be a definite linear combination of the vectors $\boldsymbol{u}_l$ and $\boldsymbol{u}_t$. It should also be mentioned that these latter vectors have no longer the simple significance of the displacement components parallel and perpendicular to the direction of propagation.

To determine the linear combination of the vectors $\boldsymbol{u}_l$ and $\boldsymbol{u}_t$ which gives the true displacement $\boldsymbol{u}$, we must use the conditions at the boundary of the body. These give a relation between the wave vector $\boldsymbol{k}$ and the frequency ω, and therefore the velocity of propagation of the wave. At the free surface we must have $\sigma_{ik} n_k = 0$. Since the normal vector $\boldsymbol{n}$ is parallel to the z-axis, it follows that $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$, whence

$$\left.\begin{aligned} u_{xz} &= 0 \\ u_{yz} &= 0 \\ \sigma(u_{xx} + u_{yy}) + (1-\sigma) u_{zz} &= 0 \end{aligned}\right\}. \qquad (24.4)$$

Since all quantities are independent of the co-ordinate y, the second of these conditions gives

$$u_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \frac{1}{2}\frac{\partial u_y}{\partial z} = 0\,.$$

Using (24.2), we therefore have

$$u_y = 0\,. \qquad (24.5)$$

Thus the displacement vector $\boldsymbol{u}$ in a surface wave is in a plane through the direction of propagation perpendicular to the surface.

The transverse part $\boldsymbol{u}_t$ of the wave must satisfy the condition (22.8) div $\boldsymbol{u}_t = 0$, or

$$\frac{\partial u_{tx}}{\partial x} + \frac{\partial u_{tz}}{\partial z} = 0\,.$$

The dependence of u_{tx} and u_{tz} on x and z is determined by the factor $e^{ikx+\kappa_t z}$, where κ_t is given by the expression (24.3) with $c = c_t$, i.e.,

$$\kappa = \sqrt{k^2 - \frac{\omega^2}{c_t^2}} \ .$$

Hence the above condition leads to the equation

$$iku_{tx} + \kappa_t u_{tz} = 0 \ ,$$

or

$$\frac{u_{tx}}{u_{tz}} = -\frac{\kappa_t}{ik} \ .$$

Thus we can write

$$\left.\begin{aligned} u_{tx} &= \kappa_t a e^{ikx + \kappa_t z - i\omega t} \\ u_{tz} &= -ika e^{ikx + \kappa_t z - i\omega t} \end{aligned}\right\} , \qquad (24.6)$$

where a is some constant.

The longitudinal part $\boldsymbol{u}_l$ satisfies the condition (22.9) curl $\boldsymbol{u}_l = 0$, or

$$\frac{\partial u_{lx}}{\partial z} - \frac{\partial u_{lz}}{\partial x} = 0 \ ,$$

whence

$$iku_{lz} = \kappa_l u_{lx} = 0 \ ,$$

where

$$\kappa_l = \sqrt{k^2 - \frac{\omega^2}{c_l^2}} \ .$$

Thus we have

$$\left.\begin{aligned} u_{lx} &= kbe^{ikx + \kappa_l z - i\omega t} \\ u_{lz} &= -i\kappa_l b e^{ikx + \kappa_l z - i\omega t} \end{aligned}\right\} , \qquad (24.7)$$

where b is a constant.

We now use the first and third conditions (24.4). Expressing u_{ik} in terms of the derivatives of u_i, and using the velocities c_l, c_t, we can write these conditions as

$$\left.\begin{aligned} \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0 \\ c_l^2 \frac{\partial u_z}{\partial z} + (c_l^2 - 2c_t^2)\frac{\partial u_x}{\partial x} = 0 \end{aligned}\right\} . \qquad (24.8)$$

Here we must substitute $u_x = u_{lx} + u_{tx}$, $u_z = u_{lz} + u_{tz}$. The result is that the first condition (24.8) gives

$$a(k^2 + \kappa_t^2) + 2bk\kappa_l = 0 \ . \qquad (24.9)$$

The second condition leads to the equation

$$2ac_t^2 \kappa_t k + b\left[c_l^2(\kappa_l^2 - k^2) + 2c_t^2 k^2\right] = 0 \ .$$

Dividing this equation by c_t^2 and substituting

$$\kappa_l^2 - k^2 = -\frac{\omega^2}{c_l^2} = -\frac{(k^2 - \kappa_t^2)c_t^2}{c_l^2} \ ,$$

we can write it as

$$2a\kappa_t k + b(k^2 + \kappa_t{}^2) = 0\,. \qquad (24.10)$$

The condition for the two homogeneous equations (24.9) and (24.10) to be compatible is $(k^2 + \kappa_t{}^2)^2 = 4k^2\kappa_t\kappa_l$ or, squaring and substituting the values of $\kappa_t{}^2$ and $\kappa_l{}^2$,

$$\left(2k^2 - \frac{\omega^2}{c_t{}^2}\right)^4 = 16k^4\left(k^2 - \frac{\omega^2}{c_t{}^2}\right)\left(k^2 - \frac{\omega^2}{c_l{}^2}\right). \qquad (24.11)$$

From this equation we obtain the relation between ω and *k*. It is convenient to put

$$\omega = c_t k\xi\,; \qquad (24.12)$$

k^8 then cancels from both sides of the equation, and, expanding, we obtain for ξ the equation

$$\xi^6 - 8\xi^4 + 8\xi^2\left(3 - 2\frac{c_t{}^2}{c_l{}^2}\right) - 16\left(1 - \frac{c_t{}^2}{c_l{}^2}\right) = 0\,. \qquad (24.13)$$

Hence we see that ξ depends only on the ratio $\frac{c_t}{c_l}$, which is a constant characteristic of any given substance and in turn depends only on Poisson's ratio:

$$\frac{c_t}{c_l} = \sqrt{\frac{1-2\sigma}{2(1-\sigma)}}\,.$$

The quantity ξ must, of course, be real and positive, and $\xi < 1$ (so that κ_t and κ_l are real). Equation (24.13) has only one root satisfying these conditions, and so a single value of f is obtained for any given value of $\frac{c_t}{c_l}$.

Thus, for both surface waves and volume waves, the frequency is proportional to the wave number. The proportionality coefficient is the velocity of propagation of the wave,

$$U = c_t\xi\,. \qquad (24.14)$$

This gives the velocity of propagation of surface waves in terms of the velocities c_t and c_l of the transverse and longitudinal volume waves. The ratio of the amplitudes of the transverse and longitudinal parts of the wave is given in terms of ξ by the formula

$$\frac{a}{b} = -\frac{2-\xi^2}{2\sqrt{1-\xi^2}}\,. \qquad (24.15)$$

The ratio $\frac{c_t}{c_l}$ actually varies from $\frac{1}{\sqrt{2}}$ to 0 for various substances,

corresponding to the variation of σ from 0 to 1/2; ξ then varies from 0-874 to 0-955. Fig. 21 shows a graph of ξ as a function of σ.

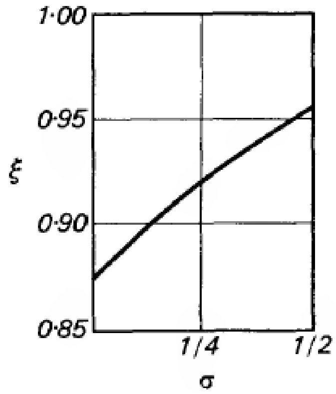

Fig. 21

PROBLEM

A plane-parallel slab of thickness h (medium 1) lies on an elastic half-space (medium 2). Determine the frequency as a function of the wave number for transverse waves in the slab whose direction of oscillation is parallel to its boundaries.

Solution. We take the plane separating the slab from the half-space as the *xy*-plane, the half-space being in $z < 0$ and the slab in $0 \le z \le h$. In the slab we have

$$u_{x1} = u_{z1} = 0\,,\;\; u_{y1} = f(z)e^{i(kx} - \omega t)\,,$$

and in medium 2 a damped wave:

$$u_{x2} = u_{z2} = 0\,,\;\; u_{y2} = Ae^{\kappa_2 z}e^{i(kx-\omega t)}\,,\;\; \kappa_2 = \sqrt{k^2 - \frac{\omega^2}{c_{t2}^{\;2}}}\,.$$

For the function $f(z)$ we have the equation

$$f'' + \kappa_1^{\;2} f = 0\,,\;\; \kappa_1 = \sqrt{\frac{\omega^2}{c_{t1}^{\;2}} - k^2}$$

(we shall see below that $\kappa_1^{\;2} > 0$), whence $f(z) = B\sin\kappa_1 z + C\cos\kappa_1 z$. At the free surface of the slab (z = h) we must have $\sigma_{xy} = 0$, i.e., $\dfrac{\partial u_{y1}}{\partial z} = 0$. At the boundary between the two media (z = 0) the conditions are $u_{y1} = u_{y2}$, $\mu_1 \dfrac{\partial u_{y1}}{\partial z} = \mu_2 \dfrac{\partial u_{y2}}{\partial z}$, μ_1 and μ_2 being the moduli of rigidity for the two media. From these conditions we find three equations

for A, B, C, and the compatibility condition is $\tan\kappa_1 h = \frac{\mu_2\kappa_2}{\mu_1\kappa_1}$. This equation gives ω as an implicit function of k; it has solutions only for real κ_1 and κ_2, and so $c_{t2} > \frac{\omega}{k} > c_{t1}$. Hence we see that such waves can be propagated only if $c_{t2} > c_{t1}$.

§25. Vibration of rods and plates

Waves propagated in thin rods and plates are fundamentally different from those propagated in a medium infinite in all directions. Here we are speaking of waves of length large compared with the thickness of the rod or plate. If the wavelength is small compared with this thickness, the rod or plate is effectively infinite in all directions as regards the propagation of the wave, and we return to the results obtained for infinite media.

Waves in which the oscillations are parallel to the axis of the rod or the plane of the plate must be distinguished from those in which they are perpendicular to it. We shall begin by studying longitudinal waves in rods.

A **longitudinal** deformation of the rod (uniform over any cross-section), with no external force on the sides of the rod, is a simple extension or compression. Thus longitudinal waves in a rod are simple extensions or compressions propagated along its length. In a simple extension, however, only the component σ_{zz} of the stress tensor (the z-axis being along the rod) is different from zero; it is related to the strain tensor by $\sigma_{zz} = Eu_{zz} = E\frac{\partial u_z}{\partial z}$ (see §5). Substituting this in the general equation of motion $\rho\ddot{u}_z = \frac{\partial\sigma_{zk}}{\partial x_k}$, we find

$$\frac{\partial^2 u_z}{\partial z^2} - \frac{\rho}{E}\frac{\partial^2 u_z}{\partial t^2} = 0\,. \tag{25.1}$$

This is the equation of **longitudinal vibrations in rods**. We see that it is an ordinary wave equation. The velocity of propagation of longitudinal waves in rods is

$$\sqrt{\frac{E}{\rho}}\,. \tag{25.2}$$

Comparing this with the expression (22.4) for c_l, we see that it is less than the velocity of propagation of longitudinal waves in an infinite medium.

Let us now consider longitudinal waves in **thin plates**. The equations

of motion for such vibrations can be written down at once by substituting $-\rho h\dfrac{\partial^2 u_x}{\partial t^2}$ and $-\rho h\dfrac{\partial^2 u_y}{\partial t^2}$ for P_x and P_y in the equilibrium equations (13.4):

$$\left.\begin{aligned}\frac{\rho}{E}\frac{\partial^2 u_x}{\partial t^2}&=\frac{1}{1-\sigma^2}\frac{\partial^2 u_x}{\partial x^2}+\frac{1}{2(1+\sigma)}\frac{\partial^2 u_x}{\partial y^2}+\frac{1}{2(1-\sigma)}\frac{\partial^2 u_y}{\partial x\partial y}\\ \frac{\rho}{E}\frac{\partial^2 u_y}{\partial t^2}&=\frac{1}{1-\sigma^2}\frac{\partial^2 u_y}{\partial y^2}+\frac{1}{2(1+\sigma)}\frac{\partial^2 u_y}{\partial x^2}+\frac{1}{2(1-\sigma)}\frac{\partial^2 u_x}{\partial x\partial y}\end{aligned}\right\}. \tag{25.3}$$

We take the case of a **plane wave** propagated along the x-axis, i.e., a wave in which the deformation depends only on the co-ordinate x, and not on y. Then equations (25.3) are much simplified, becoming

$$\left.\begin{aligned}\frac{\partial^2 u_x}{\partial t^2}-\frac{E}{\rho(1-\sigma^2)}\frac{\partial^2 u_x}{\partial x^2}=0\\ \frac{\partial^2 u_y}{\partial t^2}-\frac{E}{2\rho(1+\sigma)}\frac{\partial^2 u_y}{\partial x^2}=0\end{aligned}\right\}. \tag{25.4}$$

We thus again obtain wave equations. The coefficients are different for u_x and u_y. The velocity of propagation of a wave with oscillations parallel to the direction of propagation (u_x) is

$$\sqrt{\frac{E}{\rho(1-\sigma^2)}}. \tag{25.5}$$

The velocity for a wave (u_y) with oscillations perpendicular to the direction of propagation (but still in the plane of the plate) is equal to the velocity c_t of transverse waves in an infinite medium.

Thus we see that longitudinal waves in rods and plates are of the same nature as in an infinite medium, only the velocity being different; as before, it is independent of the frequency. Entirely different results are obtained for ***bending waves*** in rods and plates, for which the oscillations are in a direction perpendicular to the axis of the rod or the plane of the plate, i.e., involve bending.

The equations for **free oscillations** of a plate can be written down at once from the equilibrium equation (12.5). To do so, we must replace - P by the acceleration $\ddot{\zeta}$ multiplied by the mass ρh per unit area of the plate. This gives

$$\rho\frac{\partial^2\zeta}{\partial t^2}+\frac{Eh^2}{12(1-\sigma^2)}\Delta^2\zeta=0, \tag{25.6}$$

where Δ is the two-dimensional Laplacian.

Let us consider a **monochromatic** elastic wave, and accordingly seek a solution of equation (25.6) in the form

$$\zeta = \text{constant} \times e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \qquad (25.7)$$

where the wave vector $\boldsymbol{k}$ has, of course, only two components, k_x and k_y. Substituting in (25.6), we obtain the equation

$$-\rho\omega^2 + \frac{Eh^2k^4}{12(1-\sigma^2)} = 0 .$$

Hence we have the following relation between the frequency and the wave number:

$$\omega = k^2\sqrt{\frac{Eh^2}{12\rho(1-\sigma^2)}} . \qquad (25.8)$$

Thus the frequency is proportional to the square of the wave number, whereas in waves in an infinite medium it is proportional to the wave number itself.

Knowing the relation between the frequency and the wave number, we can determine the velocity of propagation of the wave from the formula

$$\boldsymbol{U} = \frac{\partial\omega}{\partial\boldsymbol{k}} .$$

The derivatives of k^2 with respect to the components k_x, k_y are, respectively, $2k_x, 2k_y$. The velocity of propagation of the wave is therefore

$$\boldsymbol{U} = \boldsymbol{k}\sqrt{\frac{Eh^2}{3\rho(1-\sigma^2)}} . \qquad (25.9)$$

It is proportional to the wave vector, and not a constant as it is for waves in a medium infinite in three dimensions.[6]

Similar results are obtained for **bending waves** in thin rods. The bending deflections of the rod are supposed small. The equations of motion are obtained by replacing $-K_x$ and $-K_y$ in the equations of equilibrium for a slightly bent rod (20.4) by the product of the acceleration $\ddot{X}$ or $\ddot{Y}$ and the mass ρS per unit length of the rod (S being its cross-sectional area). Thus

$$\left.\begin{aligned} \rho S\ddot{X} &= EI_y \frac{\partial^4 X}{\partial z^4} \\ \rho S\ddot{Y} &= EI_x \frac{\partial^4 Y}{\partial z^4} \end{aligned}\right\} . \qquad (25.10)$$

We again seek solutions of these equations in the form

$$X = \text{constant} \times e^{i(kz-\omega t)}, \quad Y = \text{constant} \times e^{i(kz-\omega t)} .$$

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[6] The wave number $k = \frac{2\pi}{\lambda}$, where λ is the wavelength. Hence the velocity of propagation should increase without limit as λ tends to zero. This physically impossible result is obtained because formula (25.9) is not valid for short waves.

Substituting in (25.10), we obtain the following relations between the frequency and the wave number:

$$\left.\begin{aligned} \omega &= k^2\sqrt{\frac{EI_y}{\rho S}} \\ \omega &= k^2\sqrt{\frac{EI_x}{\rho S}} \end{aligned}\right\}, \tag{25.11}$$

for vibrations in the x and y directions, respectively. The corresponding velocities of propagation are

$$\left.\begin{aligned} U^{(x)} &= 2k\sqrt{\frac{EI_y}{\rho S}} \\ U^{(y)} &= 2k\sqrt{\frac{EI_x}{\rho S}} \end{aligned}\right\}. \tag{25.12}$$

Finally, there is a particular case of vibration of rods called ***torsional vibration***. The corresponding equations of motion are derived by equating $C\dfrac{\partial \tau}{\partial z}$ (see §18) to the time derivative of the angular momentum of the rod per unit length. This angular momentum is $\rho I\dfrac{\partial \phi}{\partial t}$, where $\dfrac{\partial \phi}{\partial t}$ is the angular velocity (ϕ being the angle of rotation of the cross-section considered) and $I = \int (x^2 + y^2)df$ is the moment of inertia of the cross-section about its centre of mass; for pure torsional vibration each cross-section of the rod performs rotary vibrations about its centre of mass, which remains at rest. Putting $\tau = \dfrac{\partial \phi}{\partial z}$, we obtain the equation of motion in the form

$$C\frac{\partial^2 \phi}{\partial z^2} = \rho I\frac{\partial^2 \phi}{\partial t^2}. \tag{25.13}$$

Hence we see that the velocity of propagation of torsional oscillations along the rod is

$$\sqrt{\frac{C}{\rho I}}. \tag{25.14}$$

PROBLEMS

Problem 1. Determine the characteristic frequencies of longitudinal vibrations of a rod of length l, with one end fixed and the other free.

Solution. At the fixed end (z = 0) we must have $u_z = 0$, and at the free end (z = l) $\sigma_{zz} = Eu_{zz} = 0$, i.e., $\dfrac{\partial u_z}{\partial z} = 0$. We seek a solution of

equation (25.1) in the form

$$u_z = A\cos(\omega t + \alpha)\sin kz\,,$$

where $k = \omega\sqrt{\frac{\rho}{E}}$. From the condition at $z = l$ we have $\cos kl = 0$, whence the characteristic frequencies are

$$\omega = \sqrt{\frac{E}{\rho}}\frac{(2n+1)\pi}{2l}\,,$$

n being any integer.

Problem 2. The same as Problem 1, but for a rod with both ends free or both fixed.

Solution. In either case $\omega = \sqrt{\frac{E}{\rho}}\frac{n\pi}{l}$.

Problem 3. Determine the characteristic frequencies of vibration of a string of length l.

Solution. The equation of motion of the string is

$$\frac{\partial^2 X}{\partial z^2} - \frac{\rho S}{T}\frac{\partial^2 X}{\partial t^2} = 0\,;$$

cf. the equilibrium equation (20.17). The boundary conditions are that $X =$ 0 for $z = 0$ and l. The characteristic frequencies are $\omega = \sqrt{\frac{\rho S}{T}}\frac{n\pi}{l}$.

Problem 4. Determine the characteristic transverse vibrations of a rod (of length l) with clamped ends.

Solution. Equation (25.10), on substituting $X = X_0(z)\cos(\omega t + \alpha)$, becomes

$$\frac{d^4 X_0}{dz^4} = \kappa^4 X_0\,,$$

where $\kappa^4 = \frac{\omega^2 \rho S}{EI_y}$. The general integral of this equation is

$$X_0 = A\cos\kappa z + B\sin\kappa z + C\cosh\kappa z + D\sinh\kappa z\,.$$

The constants A, B, C and D are determined from the boundary conditions that $X = \frac{dX}{dz} = 0$ for $z = 0$ and l. The result is

$$X_0 = A\{(\sin\kappa l - \sinh\kappa l)(\cos\kappa z - \cosh\kappa z) - (\cos\kappa l - \cosh\kappa l)(\sin\kappa z - \sinh\kappa z)\}$$

and the equation $\cos\kappa l\cosh\kappa l = 1$, the roots of which give the characteristic frequencies. The smallest characteristic frequency is

$$\omega_{\min} = \frac{22.4}{l^2}\sqrt{\frac{EI_y}{\rho S}}\,.$$

Problem 5. The same as Problem 4, but for a rod with supported ends.

Solution. In the same way as in Problem 4, we obtain $X = A\sin\kappa z$, and

the frequencies are given by $\sin \kappa l = 0$, i.e., $\kappa = \frac{n\pi}{l}$ (n = 1, 2, ...). The smallest frequency is

$$\omega_{\min} = \frac{9.87}{l^2}\sqrt{\frac{EI_y}{\rho S}}.$$

Problem 6. The same as Problem 4, but for a rod with one end clamped and the other free.

Solution. We have for the displacement

$$X_0 = A\{(\cos\kappa l + \cosh\kappa l)(\cos\kappa z - \cosh\kappa z) + (\sin\kappa l - \sinh\kappa l)(\sin\kappa z - \sinh\kappa z)\}$$

(the clamped end being at z = 0 and the free end at z = l), and for the characteristic frequencies the equation $\cos\kappa l\cosh\kappa l + 1 = 0$. The smallest frequency is

$$\omega_{\min} = \frac{3.52}{l^2}\sqrt{\frac{EI_y}{\rho S}}.$$

Problem 7. Determine the characteristic vibrations of a rectangular plate of sides a and b, with its edges supported.

Solution. Equation (25.6), on substituting $\zeta = \zeta_0(x, y)\cos(\omega t + \alpha)$, becomes

$$\Delta^2\zeta_0 - \kappa^4\zeta_0 = 0,$$

where $\kappa^4 = \frac{12\rho(1-\sigma^2)\omega^2}{Eh^2}$. We take the co-ordinate axes along the sides of the plate. The boundary conditions (12.11) become $\zeta = \frac{\partial^2\zeta}{\partial x^2} = 0$ for x = 0 and a, $\zeta = \frac{\partial^2\zeta}{\partial y^2} = 0$ for y = 0 and b. The solution which satisfies these conditions is

$$\zeta_0 = A\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

where m and n are integers. The frequencies are given by

$$\omega = h\sqrt{\frac{E}{12\rho(1-\sigma^2)}}\pi^2\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right].$$

Problem 8. Determine the characteristic frequencies for the vibration of a rectangular membrane of sides a and b.

Solution. The equation for the vibration of a membrane is $T\Delta\zeta = \rho h\ddot{\zeta}$; cf. the equilibrium equation (14.9). The edges of the membrane must be fixed, so that $\zeta = 0$. The corresponding solution for a rectangular membrane is

$$\zeta = A\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\omega t,$$

where the characteristic frequencies are given by

$$\omega^2 = \frac{T\pi^2}{\rho h}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right),$$

m and n being integers.

Problem 9. Determine the velocity of propagation of torsional vibrations in a rod whose cross-section is a circle, an ellipse, or an equilateral triangle, and in a rod in the form of a long thin rectangular plate.

Solution. For a circular cross-section of radius R, the moment of inertia is $I = \frac{\pi}{2}R^4$; C is given in §16, Problem 1, and we find the velocity to be $\sqrt{\frac{\mu}{\rho}}$, which is the same as the velocity c_t.

Similarly (using the results of §16, Problems 2 to 4), we find for a rod of elliptical cross-section the velocity $\frac{2ab}{a^2+b^2}\sqrt{\frac{\mu}{\rho}}$, for one with an equilateral triangular cross-section $\sqrt{\frac{3\mu}{5\rho}}$, and for one which is a long rectangular plate $\frac{2h}{d}\sqrt{\frac{\mu}{\rho}}$. All these are less than c_t.

Problem 10. The surface of an incompressible fluid of infinite depth is covered by a thin elastic plate. Determine the relation between the wave number and the frequency for waves which are simultaneously propagated in the plate and near the surface of the fluid.

Solution. We take the plane of the plate as $z = 0$, and the x-axis in the direction of propagation of the wave; let the fluid be in $z < 0$. The equation of motion of the plate alone would be

$$\rho_0 h\frac{\partial^2\zeta}{\partial t^2} = -\frac{Eh^3}{12(1-\sigma^2)}\frac{\partial^4\zeta}{\partial x^4},$$

where ρ_0 is the volume density of the plate. When the fluid is present, the right-hand side of this equation must also include the force exerted by the fluid on unit area of the plate, i.e., the pressure p of the fluid. The pressure in the wave, however, can be expressed in terms of the velocity potential by $p = -\rho\frac{\partial\phi}{\partial t}$ (we neglect gravity). Hence we obtain

$$\rho_0 h\frac{\partial^2\zeta}{\partial t^2} = -\frac{Eh^3}{12(1-\sigma^2)}\frac{\partial^4\zeta}{\partial x^4} - \left[\rho\frac{\partial\phi}{\partial t}\right]_{z=0}. \qquad (1)$$

Next, the normal component of the fluid velocity at the surface must be equal to that of the plate, whence

$$\frac{\partial \zeta}{\partial t} = \left[\frac{\partial \phi}{\partial z}\right]_{z=0} . \tag{2}$$

The potential ϕ must satisfy everywhere in the fluid the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 . \tag{3}$$

We seek ζ in the form of a travelling wave $\zeta = \zeta_0 e^{i(kx-\omega t)}$; accordingly, we take as the solution of equation (3) the surface wave $\phi = \phi_0 e^{i(kx-\omega t)} e^{kz}$, which is damped in the interior of the fluid. Substituting these expressions in (1) and (2), we obtain two equations for ϕ_0 and ζ_0 and the compatibility condition is

$$\omega^2 = \frac{Eh^3}{12(1-\sigma^2)} \frac{k^5}{\rho + h\rho_0 k} .$$

§26. Anharmonic vibrations

The whole of the theory of elastic vibrations given above is approximate to the extent that any theory of elasticity is so which is based on Hooke's law. It should be recalled that the theory begins from an expansion of the elastic energy as a power series with respect to the strain tensor, which includes terms up to and including the second order. The components of the stress tensor are then linear functions of those of the strain tensor, and the equations of motion are linear.

The most characteristic property of elastic waves in this approximation is that any wave can be obtained by simple superposition (i.e., as a linear combination) of separate monochromatic waves. Each of these is propagated independently, and could exist by itself without involving any other motion. We may say that the various monochromatic waves which are simultaneously propagated in a single medium do not interact with one another.

These properties, however, no longer hold in subsequent approximations. The effects which appear in these approximations, though small, may be of importance as regards certain phenomena. They are usually called ***anharmonic effects***, since the corresponding equations of motion are non-linear and do not admit simple periodic (harmonic) solutions.

We shall consider here anharmonic effects of the third order, arising from terms in the elastic energy which are cubic in the strains. It would be too cumbersome to write out the corresponding equations of motion in their general form. However, the nature of the resulting effects can be ascertained as follows. The cubic terms in the elastic energy give

quadratic terms in the stress tensor, and therefore in the equations of motion. Let us suppose that all the linear terms in these equations are on the left-hand side, and all the quadratic terms on the right-hand side. Solving these equations by the method of successive approximations, we omit the quadratic terms in the first approximation. This leaves the ordinary linear equations, whose solution $\boldsymbol{u}_0$ can be put in the form of a superposition of monochromatic travelling waves: $\text{constant} \times e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$, with definite relations between ω and $\boldsymbol{k}$. On going to the second approximation, we must put $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1$ and retain only the terms in $\boldsymbol{u}_0$ on the right-hand sides of the equations (the quadratic terms). Since $\boldsymbol{u}_0$, by definition, satisfies the homogeneous linear equations obtained by putting the right-hand sides equal to zero, the terms in $\boldsymbol{u}_0$ on the left-hand sides will cancel. The result is a set of inhomogeneous linear equations for the components of the vector $\boldsymbol{u}_1$, where the right-hand sides contain only known functions of the co-ordinates and time. These functions, which are obtained by substituting $\boldsymbol{u}_0$ for $\boldsymbol{u}$ in the right-hand sides of the original equations, are sums of terms each of which is proportional to

$$e^{i[(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}-(\omega_1-\omega_2)t]}$$

or

$$e^{i[(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{r}-(\omega_1+\omega_2)t]},$$

where $\omega_1, \omega_2, \boldsymbol{k}_1, \boldsymbol{k}_2$ are the frequencies and wave vectors of any two monochromatic waves in the first approximation.

A particular integral of linear equations of this type is a sum of terms containing similar exponential factors to those in the free terms (the right-hand sides) of the equations, with suitably chosen coefficients. Each such term corresponds to a travelling wave with frequency $\omega_1 \pm \omega_2$ and wave vector $\boldsymbol{k}_1 \pm \boldsymbol{k}_2$. Frequencies equal to the sum or difference of the frequencies of the original waves are called ***combination frequencies***.

Thus the anharmonic effects in the third order have the result that the set of fundamental monochromatic waves (with frequencies $\omega_1, \omega_2, \ldots$ and wave vectors $\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots$) has superposed on it other "waves" of small intensity, whose frequencies are the combination frequencies such as $\omega_1 \pm \omega_2$, and whose wave vectors are such as $\boldsymbol{k}_1 \pm \boldsymbol{k}_2$. We call these "waves" in quotation marks because they are a correction effect and cannot exist alone except in certain special cases (see below). The values $\omega_1 \pm \omega_2$ and $\boldsymbol{k}_1 \pm \boldsymbol{k}_2$ do not in general satisfy the relations which hold between the frequencies and wave vectors for ordinary monochromatic waves.

It is clear, however, that there may happen to be particular values of ω_1, $\boldsymbol{k}_1$ and ω_2, $\boldsymbol{k}_2$ such that one of the relations for monochromatic waves in the medium considered also holds for $\omega_1+\omega_2$ and $\boldsymbol{k}_1+\boldsymbol{k}_2$ (for definiteness, we shall discuss sums and not differences). Putting $\omega_3=\omega_1+\omega_2$, $\boldsymbol{k}_3=\boldsymbol{k}_1+\boldsymbol{k}_2$, we can say that, mathematically, ω_3 and $\boldsymbol{k}_3$ then correspond to waves which satisfy the homogeneous linear equations of motion (with zero on the right-hand side) in the first approximation. If the right-hand sides in the second approximation contain terms proportional to $e^{i(\boldsymbol{k}_3\cdot\boldsymbol{r}-\omega_3 t)}$, then a particular integral will be a wave with the same frequency and an amplitude which increases indefinitely with time.

Thus the superposition of two monochromatic waves with values of ω_1, $\boldsymbol{k}_1$ and ω_2, $\boldsymbol{k}_2$ whose sum ω_3, $\boldsymbol{k}_3$ satisfies the above condition leads, by the anharmonic effects, to resonance: a new monochromatic wave (with parameters ω_3, $\boldsymbol{k}_3$) is formed, whose amplitude increases with time and eventually is no longer small. It is evident that, if a wave with ω_3, $\boldsymbol{k}_3$ is formed on superposition of those with ω_1, $\boldsymbol{k}_1$ and ω_2, $\boldsymbol{k}_2$, then the superposition of waves with ω_1, $\boldsymbol{k}_1$ and ω_3, $\boldsymbol{k}_3$ will also give a resonance with $\omega_2=\omega_3-\omega_1$, $\boldsymbol{k}_2=\boldsymbol{k}_3-\boldsymbol{k}_1$, and similarly ω_2, $\boldsymbol{k}_2$ and ω_3, $\boldsymbol{k}_3$ lead to ω_1, $\boldsymbol{k}_1$.

In particular, for an isotropic body ω and $\boldsymbol{k}$ are related by $\omega=c_t k$ or $\omega=c_l k$ with $c_l>c_t$. It is easy to see in which cases either of these relations can hold for each of the three combinations

$$\omega_1,\boldsymbol{k}_1;\ \ \omega_2,\boldsymbol{k}_2;\ \ \omega_3=\omega_1+\omega_2,\ \ \boldsymbol{k}_3=\boldsymbol{k}_1+\boldsymbol{k}_2$$

If $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are not in the same direction, $k_3<k_1+k_2$, and so it is clear that resonance can then occur only in the following two cases:

(1) the waves with ω_1, $\boldsymbol{k}_1$ and ω_2, $\boldsymbol{k}_2$ are transverse and that with ω_3, $\boldsymbol{k}_3$ longitudinal;

(2) one of the waves with ω_1, $\boldsymbol{k}_1$ and ω_2, $\boldsymbol{k}_2$ is transverse and the other longitudinal, and that with ω_3, $\boldsymbol{k}_3$ is longitudinal.

If the vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are in the same direction, however, resonance is possible when all three waves are longitudinal or all three are transverse.

The anharmonic effect involving resonance occurs not only when several monochromatic waves are superposed, but also when there is only one wave, with parameters ω_1, $\boldsymbol{k}_1$. In this case the right-hand sides of the equations of motion contain terms proportional to $e^{2i(\boldsymbol{k}_1\cdot\boldsymbol{r}-\omega_1 t)}$. If ω_1 and $\boldsymbol{k}_1$ satisfy the usual condition, however, then $2\omega_1$ and $2\boldsymbol{k}_1$ do so too, since this condition is homogeneous and of degree one. Thus the

anharmonic effect results in the appearance, besides the monochromatic waves with ω_1, $\boldsymbol{k}_1$ previously obtained, of waves with $2\omega_1$, $2\boldsymbol{k}_1$, i.e., with twice the frequency and twice the wave vector, and amplitude increasing with time.

Finally, we may briefly discuss how we can set up the equations of motion, allowing for the anharmonic terms. The strain tensor must now be given by the complete expression (1.3):

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right), \qquad (26.1)$$

in which the terms quadratic in u_i can not be neglected. Next, the general expression for the energy density[7] $\widetilde{E}$, in bodies having a given symmetry, must be written as a scalar formed from the components of the tensor u_{ik} and some constant tensors characteristic of the substance involved; this scalar will contain terms up to a given power of u_{ik}. Substituting the expression (26.1) for u_{ik} and omitting terms in u_i of higher orders than that power, we find the energy $\widetilde{E}$ as a function of the derivatives $\frac{\partial u_i}{\partial x_k}$ to the required accuracy.

In order to obtain the equations of motion, we notice the following result. The variation $\delta\widetilde{E}$ may be written

$$\delta\widetilde{E} = \frac{\partial \widetilde{E}}{\partial\left(\frac{\partial u_i}{\partial x_k}\right)}\delta\frac{\partial u_i}{\partial x_k},$$

or, putting

$$\sigma_{ik} = \frac{\partial \widetilde{E}}{\partial\left(\frac{\partial u_i}{\partial x_k}\right)}, \qquad (26.2)$$

$$\delta\widetilde{E} = \sigma_{ik}\frac{\partial \delta u_i}{\partial x_k} = \frac{\partial}{\partial x_k}(\sigma_{ik}\delta u_i) - \delta u_i\frac{\partial \sigma_{ik}}{\partial x_k}.$$

The coefficients of $-\delta u_i$ are the components of the force per unit volume of the body. They formally appear the same as before, and so the equations of motion can again be written

$$\rho_0 \ddot{u}_i = \frac{\partial \sigma_{ik}}{\partial x_k}, \qquad (26.3)$$

where ρ_0 is the density of the undeformed body, and the components of the tensor σ_{ik} are now given by (26.2), with $\widetilde{E}$ correct to the required

[7] We here use the internal energy $\widetilde{E}$, and not the free energy *F*, since adiabatic vibrations are involved.

accuracy. The tensor σ_{ik} is no longer symmetrical.[8]

PROBLEM

Write down the general expression for the elastic energy of an isotropic body in the third approximation.

Solution. From the components of a symmetrical tensor of rank two we can form two quadratic scalars (u_{ik}^2 and u_{ll}^2) and three cubic scalars (u_{ll}^3, $u_{ll}u_{ik}^2$ and $u_{ik}u_{il}u_{kl}$). Hence the most general scalar containing terms quadratic and cubic in u_{ik}, with scalar coefficients (since the body is isotropic), is

$$\widetilde{E} = \mu u_{ik}^2 + (\frac{1}{2}K - \frac{1}{3}\mu)u_{ll}^2 + \frac{1}{2}Au_{ik}u_{il}u_{kl} + Bu_{ik}^2 u_{ll} + \frac{1}{3}Cu_{ll}^3;$$

the coefficients of u_{ik}^2 and u_{ll}^2 have been expressed in terms of the moduli of compression and rigidity, and *A*, *B*, *C* are three new constants. Substituting the expression (26.1) for u_{ik} and retaining terms up to and including the third order, we find the elastic energy to be

$$\begin{aligned}\widetilde{E} = {} & \frac{1}{4}\mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)^2 + (\frac{1}{2}K - \frac{1}{3}\mu)\left(\frac{\partial u_l}{\partial x_l}\right)^2 \\ & + (\mu + \frac{1}{4}A)\frac{\partial u_i}{\partial x_k}\frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k} + (\frac{1}{2}B + \frac{1}{2}K - \frac{1}{3}\mu)\frac{\partial u_l}{\partial x_l}\left(\frac{\partial u_i}{\partial x_k}\right)^2. \\ & + \frac{1}{12}A\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_l}\frac{\partial u_l}{\partial x_i} + \frac{1}{2}B\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_i}\frac{\partial u_l}{\partial x_l} + \frac{1}{3}C\left(\frac{\partial u_l}{\partial x_l}\right)^3\end{aligned}$$

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[8] It should be emphasised that σ_{ik} is no longer the momentum flux density (the stress tensor). In the ordinary theory this interpretation was derived by integrating the body force density $\frac{\partial \sigma_{ik}}{\partial x_k}$ over the volume of the body. This derivation depended on the fact that, in performing the integration, we made no distinction between the co-ordinates of points in the body before and after the deformation. In subsequent approximations, however, this distinction must be made, and the surface bounding the region of integration is not the same as the actual surface of the region considered after the deformation.

It has been shown in §2 that the symmetry of the tensor σ_{ik} is due to the conservation of angular momentum. This result no longer holds, since the angular momentum density is not $x_i\dot{u}_k - x_k\dot{u}_i$ but $(x_i + u_i)\dot{u}_k - (x_k + u_k)\dot{u}_i$.