ON THE PROBLEM OF TURBULENCE

by

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Although the turbulent motion has been extensively discussed in literature from different points of view, the very essence of this phenomenon is still lacking sufficient clearness. To the author's opinion, the problem may appear in a new light if the process of initiation of turbulence is examined thoroughly.

In the case of incompressible fluids the unsteadiness of the laminar motion is known to be determined as follows. Upon the principal motion with a velocity distribution $v_0(x, y, z)$ there is superimposed a small disturbance $v_1(x, y, z, t)$; the substitution of $v = v_0 + v_1$ in the equation of motion of a viscous fluid and the neglect of terms of the second order of smallness lead to a linear differential equation for the perturbation v_1. Further, v_1 is sought in the form

$$v_1 = A(t) f(x, y, z) , \tag{1}$$

where the time function $A(t)$ may be represented as

$$A(t) = const \cdot e^{-i\Omega t} . \tag{2}$$

The problem of determining the possible values of the "frequencies" Ω where the boundary conditions of motion are given, is an "Eigenvalue" problem. By solving it one will obtain a spectrum of proper frequencies Ω (which are complex values in the general case). This spectrum, generally speaking, contains separate, isolated, values ("discrete spectrum") and also contains frequencies continuously filling whole intervals of values ("continuous spectrum"). It may be supposed that the frequencies of the continuous spectrum correspond to such motions v_1 as are not damped at infinity, while the frequencies of the discrete spectrum correspond to motions which are damped at infinity rather rapidly (as is the case in many other Eigenvalue problems).

For the problem of steadiness of the principal motion those of the frequencies $\Omega = \omega + i\gamma$ (ω , γ are real) are relevant in which the imaginary part is negative ($\gamma < 0$). The presence of such proper frequencies in the spectrum indicates the unsteadiness of the principal motion with respect to infinitely small perturbations. Such values of Ω are only possible among the frequencies of the discrete spectrum. In fact, the principal motion presents at infinity a plane-parallel homogeneous flow (we mean a

flow past a body of finite dimensions. In so far as a plane-parallel flow is in no case steady, it will be evident that any perturbation that fails to disappear at infinity must necessarily be clamped in time, or, in other words, correspond to frequencies Ω with $\gamma > 0$. Accordingly, only the Ω frequencies of the discrete spectrum can be considered below.

In the case of sufficiently small velocities the principal motion is a steady one (inasmuch as a resting fluid is in any case steady). On the other hand, with sufficiently large Reynolds numbers the laminar flow past a body is unsteady at any rate. In fact, with large Reynolds numbers the motion faraway from the body is not appreciably different from a plane-parallel flow unless in the region of the narrow “track”. Now it follows from Lord Rayleigh’s work that no motion with a two-dimensional velocity distribution of such a type is steady, and one may expect that the same is true of the three-dimensional track.

If the values of the proper frequencies Ω are taken to be functions of the Reynolds number of the principal motion, then the critical value Re_{cr} is determined by the fact that for $\mathrm{Re} = \mathrm{Re}_{cr}$ the imaginary part of one of the frequencies Ω will vanish; suppose this frequency to be $\Omega_1 = \omega_1 + i\gamma_1$. For $\mathrm{Re} > \mathrm{Re}_{cr}$ we have $\gamma_1 > 0$; for such Reynolds numbers as are near to the critical value Re_{cr}, γ_1 is small in comparison with ω_1. However, the expression (1-2) for the respective function $v_1(x, y, z, t)$ (with $\Omega = \Omega_1$) is only true for a very brief interval of time, as measured from the instant at which the stationary regime is broken. This is owing to the fact that the factor $e^{\gamma_2 t}$ grows rapidly with time. As a. matter of fact, the modulus $|A|$ of the amplitude of nonstationary motion does not increase infinitely, but rather tends to a certain limit. With Re near to Re_{cr} (Re is always supposed to be greater than Re_{cr}), this limit is yet very small, too, and for determining it one may proceed as follows.

For very small times, when (2) is still applicable, we have

$$\frac{d\,|A|^2}{dt} = 2\gamma_1\,|A|^2 .$$

In substance this expression is but the first term of a series of powers of A and A*. With the increase of the modulus $|A|$ the subsequent three terms of this series must be taken into account. The next terms are terms of the third order. We, however, are interested not in the exact value of the differential quotient $\frac{d\,|A|^2}{dt}$, but in its mean value with respect to time, the averaging being made over time intervals that are large in comparison with the period

$\frac{2\pi}{\omega_1}$ of the periodic spectrum $e^{-i\omega_1 t}$ (as $\omega_1 >> \gamma_1$, this period is small compared to the time $1/\gamma_1$ during which the modulus $|A|$ changes appreciably). But the terms of the third order involve a. periodic spectrum, and so they are eliminated upon averaging. (Strictly speaking, they do not vanish altogether, but yield quantities of order four; these quantities are supposed to be included into the terms of the fourth order). The terms of the fourth order include a term proportional to $A^2 A^{*2} = |A|^4$; this term is not eliminated by averaging. Thus, up to terms of the fourth order we have

$$\overline{\frac{d|A|^2}{dt}} = 2\gamma_1 |A|^2 - \alpha |A|^4 . \tag{3}$$

Here α is a positive constant (the case of negative α is considered below).

There are no signs of averaging over $|A|^2$ and $|A|^4$, because this operation is carried out over such time intervals as are small in comparison with $1/\gamma_1$. For the same reason in solving this equation we must disregard the bar over the derivative in the left band member. The solution of the equation (3) has the form

$$\frac{1}{|A|^2} = \frac{\alpha}{2\gamma_1} + const \cdot e^{-2\gamma_1 t} ,$$

i.e., $|A|^2$ tends asymptotically to a limit

$$|A|^2_{\max} = \frac{2\gamma_1}{\alpha} ; \tag{4}$$

γ_1 is a function of Reynolds number; it vanishes with $\mathrm{Re} = \mathrm{Re}_{cr}$. Therefore for small $\mathrm{Re} - \mathrm{Re}_{cr}$ we have $\gamma_1 = const \cdot (\mathrm{Re} - \mathrm{Re}_{cr})$. Substituting this in (4) we shall see that

$$|A|_{\max} \sim \sqrt{\mathrm{Re} - \mathrm{Re}_{cr}} . \tag{5}$$

Thus, the unsteadiness of the laminar motion for $\mathrm{Re} > \mathrm{Re}_{cr}$ leads to the appearance of a non-stationary periodic motion. When Re is close to Re_{cr} this motion can be represented as a superposition of a stationary motion $v_0(x, y, z)$ over a periodic motion $v_1(x, y, z, t)$, having a small but finite amplitude which varies with Re directly as $\sqrt{\mathrm{Re} - \mathrm{Re}_{cr}}$. The velocity distribution in this motion has therefore the form

$$v_1 = f(x, y, z) e^{-i(\omega_1 t / \beta_1)} , \tag{6}$$

where β_1 is a constant initial phase. When the differences $\mathrm{Re} - \mathrm{Re}_{cr}$ are large, there is no longer any sense in separating the velocities into two parts

v_0 and $vl1$. Here we have to deal simply with a periodic motion of frequency ω_1. If instead of the time the phase $\phi_1 \equiv \omega_1 t + \beta_1$ is used as the independent variable, the function $v(x,y,z,\phi_1)$ may be said to be a periodic function of ϕ_1 with a period 2π, but no simple trigonometric function. It can be represented as a Fourier series

$$v = \sum A_p(x,y,z)e^{-i\phi_1 p} \tag{7}$$

(the summation is carried out over all positive and negative integers p).

The essential fact is that only the absolute value of the factor, but not its phase are determined by the equation (3). The phase ϕ_1 remains in substance indefinite and depends upon the initial conditions which are a matter of change and may cause β_1 to take any value. It will be obvious that the periodic motion under consideration is not determined uniquely by the given stationary boundary conditions of motion; one quantity, the phase, remains arbitrary. This motion may be said to have one degree of freedom, whereas stationary motion is completely determined by the given boundary conditions, and enjoys not a single degree of freedom.

As Re is further increased, this periodic motion, too, eventually becomes unsteady. The investigation of its unsteadiness should be conducted in a manner analogous to that described above- The role of the principal motion is now played by the periodic motion $v_0(x,y,z,t)$ of frequency ω_1. Substituting $v = v_0 + v_2$ with small v_2 into the equation of motion, we shall again obtain for v_2 a linear equation, but this time the coefficients of this equation are not only functions of the co-ordinates, but of time also; with respect to time, they are periodic functions with a period $\frac{2\pi}{\omega_1}$. The solution of such an equation should be sought in the form $v_2 = \Pi(x,y,z,t)e^{-\Omega t}$ where $\Pi(x,y,z,t)$ is a periodical function of time (with a period $\frac{2\pi}{\omega_1}$). Unsteadiness sets in again when the frequency $\Omega_2 = \omega_2 + i\gamma_2$ turns up whose imaginary part γ_2 is positive and the corresponding real part ω_2 determines then the newly appearing frequency.

The result is a quasi-periodic motion characterized by two different periods. It involves two arbitrary quantities (phases), i.e., has two degrees of freedom.

In the course of a further increase of the Reynolds number **more and more new periods appear in succession**, and the motion assumes an involved character typical of a developed turbulence. For every value of Re

the motion has a definite number of degrees of freedom; in the limit as Re tends to infinity, the number of degrees of freedom becomes likewise infinitely large. With *n* degrees of freedom the velocity distribution is described by an expression of the type

$$v(x,y,z,t)=\sum_{p_1,\ldots,p_n} A_{p_1,\ldots,p_n}(x,y,z)e^{-i\sum_{i=1}^{n} p_i\phi_i} \tag{8}$$

(summation over all integral numbers $p_1, p_2, \ldots, p_n$) where the phases are $\phi_i = \omega_i t + \beta_i$; it contains *n* arbitrary initial phases β_i. The frequencies ω_i being incommensurable, it will be apparent that during a sufficiently long interval of time the fluid will pass through the states which are as close as we will it to a state set beforehand by choosing freely a set of simultaneous values for the phases ϕ_i. It should, of course, be borne in mind that the states whose phases differ only by a multiple of 2π are identical physically. So a **turbulent motion is to a certain extent a quasi-periodical motion**.

The setting-up of a turbulent regime has a somewhat different character in those exceptional cases (the Poiseuille motion and others) where the laminar motion remains stable with respect to infinitesimal perturbations, no matter how large are the Reynolds numbers. If the latter are sufficiently small, no non-stationary motion is possible here at all; a steady non-stationary motion becomes possible only after a certain value of Re is reached, which is here in the nature of a critical value. With very large Reynolds numbers, the stationary motion may, notoriously, be materialized only if one is careful enough in eliminating the perturbations superimposed upon the motion. Contrary to this, if Re is close to Re_{cr}, the non-stationary motion is difficult to materialize. It may be thought therefore that the true value of Re_{cr}, say, in the case of Poiseuille motion, lies in any case below the value generally adopted at present. As for the properties of the turbulent motion that appears here with $\mathrm{Re} > \mathrm{Re}_{cr}$, it should, contrary to the preceding case, enjoy from the outset a large number of degrees of freedom.

Finally, in principle, there is one more possible type of the loss of steadiness by the laminar motion; this corresponds to the case where the coefficient before $|A|^4$ in (3) is positive, so that

$$\frac{d|A|^2}{dt} = 2\gamma|A|^2 + \alpha|A|^4$$

with positive α. If Re is somewhat smaller than Re_{cr}, the term of the second order is negative (since $\gamma_1 < 0$ for $\mathrm{Re} < \mathrm{Re}_{cr}$). But, the term of the

fourth order being positive, the derivative $\overline{\frac{d|A|^2}{dt}}$ will become positive when the amplitude of $|A|$ is sufficiently large. This means that the motion becomes steady with respect to sufficiently large perturbations even for $\mathrm{Re} > \mathrm{Re}_{cr}$. Thus, this type of unsteadiness is characterized by the fact that for a certain value, Re_{cr}, of the Reynolds number the motion becomes unsteady with respect to infinitesimal disturbances, but even with $\mathrm{Re} > \mathrm{Re}_{cr}$ there is unsteadiness in response to perturbations of a finite magnitude. In this case along with the above-mentioned critical Reynolds number there should exist another, "lower" number which determines the instant of appearance of stable non-stationery solutions of the equations of motion.