ON SHOCK WAVES AT LARGE DISTANCES
FROM THE PLACE OF THEIR ORIGIN

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It is shown that at large distances from the body, moving with a. velocity exceeding that of the sound, there exists not one (as is usually assumed) but two subsequent shock waves. The shape of these waves and the law of the decrease of their intensity with distance is determined. The propagation of a spherical shock (explosive) wave at large distances from the place of explosion is examined.

At large distances from the place of their origin shock waves are faint and have, therefore, the character of sound waves. However, for our purpose the ordinary linear approximation is insufficient and it is necessary to consider the properties of **sound waves** with a small amplitude in the **second approximation**. We shall have to do below with cylindrical and spherical waves; since, however, at large distances cylindrical or spherical waves can be considered in every small region as a plane one, we shall dwell preliminarily on some properties of plane waves.

As is well known, a one-dimensional progressive wave with an arbitrary amplitude is described by the so-called **Riemann solution** of the equations of motion

$$x = t[v + c(v)] + f(v),$$

where $f(v)$ is an arbitrary function of the velocity v of the gas, while c – the local velocity of sound, connected with v by the relation

$$v = \int \frac{dp}{\rho c} = \int \left(-\frac{\partial V}{\partial p}\right)^{1/2} dp$$

(ρ - density, V - specific volume of the gas). These two formulae determine, implicitly, the velocity v (and along with it the other quantities characterizing the wave) as a function of :x and t, i.e., the shape of the wave at any given instant. For $t = 0$ we have $x = f(v)$, i.e., the function inverse to $f(v)$ determines the shape of the wave at the initial moment.

The quantity

$$u = v + c(v) \tag{1}$$

is the velocity at which the points of the wave profile are travelling. This

velocity is different for different points of the profile, and that is why it does not remain constant and varies its shape with time. Expressing u as function, say, of the pressure p in the wave, we have for the derivative $\frac{du}{dp}$:

$$\frac{du}{dp} = \frac{dc}{dp} + \frac{1}{\rho c},$$

but $c = \sqrt{\frac{\partial p}{\partial \rho}} = V\sqrt{-\frac{\partial p}{\partial V}}$ so that calculation yields

$$\frac{du}{dp} = \frac{1}{2}\rho^2 c^3 \left(\frac{\partial^2 V}{\partial p^2}\right)_S.$$

The adiabatic derivative $\left(\frac{\partial^2 V}{\partial p^2}\right)_S$ (S - entropy) is positive for all gases, so that $\frac{dV}{dp} > 0$. The velocity of the displacement of a. given point of the wave profile is thus the larger the larger is the pressure at this point; the compressions are, therefore, gradually advancing with respect to the rarefactions.

For a wave with a small amplitude the displacement velocity u of the points of the profile in the first approximation is obtained by putting in (1) $v = 0$, i.e., $u = c_0$ (the letters with the index zero denote the equilibrium values of the corresponding quantities), which corresponds to a. displacement of 'the wave profile without any change of its shape. In the next approximation we have

$$u = c_0 + \frac{du}{dp_0} p',$$

or

$$u = c_0\left(1 + \alpha \frac{p'}{p_0}\right), \qquad (2)$$

$$\alpha = \frac{p_0}{2} \frac{c_0^{\ 2}}{V_0^{\ 2}} \left(\frac{\partial^2 V}{\partial p^2}\right)_S$$

(p' - denotes the variable part of the pressure in the wave). For an ideal gas $\alpha = \frac{\gamma + 1}{2\gamma}$ (for air $\alpha = 0.86$), where $\gamma = \frac{c_p}{c_v}$ is the ratio of the specific heats at constant pressures and volume.

When the profile of the wave is distorted to such an extent that the **uniqueness** of the solution disappears, a shock wave arises. The Riemann solution becomes, generally speaking, inapplicable after the formation of discontinuities. It is, however, natural that for waves of small amplitude, in the second approximation under consideration, this solution remains valid in the

presence of such discontinuities. This can be seen in the following way. The jumps in the velocity, pressure, and specific volume in the discontinuity are connected with each other by the relation

$$v_2 - v_1 = \sqrt{(p_2 - p_1)(V_1 - V_2)} .$$

The change of the velocity v over a certain segment of the length of the x-axis in the Riemann solution is equal to the integral

$$v_2 - v_1 = \int_{p_1}^{p_2} \left(-\frac{\partial V}{\partial p} \right)^{1/2} dp .$$

A simple calculation with the help of an expansion into a. series shows that. the preceding expression differ in the terms of the third order only (in carrying out the calculation it must be kept in mind that the change of entropy in the discontinuity is a quantity of the third order, while in the Riemann solution the entropy is constant). Hence it follows that (in an accuracy up to terms of the second order) the motion in the travelling wave in the presence of the discontinuity can be described on each side of it by the Riemann solution with suitable boundary condition on the discontinuity. In the following approximations **this will, however, no longer be the case**, which is connected with the appearance of a reflection from the surface of the discontinuity.

The position of the discontinuity in the wave is determined by a simple geometrical condition, which can easily be derived with the help of formula (2) and the condition of the continuity of the flow of the substance through the discontinuity. Namely, the discontinuity is characterized by the condition that the area of the curve, representing the profile of the wave should remain the same as that of the multivalued curve, determined by Riemann's solution.

Let us now consider a body, performing a steady motion with a velocity U exceeding that of sound. Let the x-axis be parallel to the direction of the motion of the body and let r be the distance from this axis. At large distances from the body the velocity potential $\phi(r, z)$ of the gas is determined, in the first approximation, by the wave equation

$$\frac{1}{{c_0}^2} \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right).$$

The steadiness of the motion of the body is expressed by the equation.

$$\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} = 0 .$$

Combining these two equations we get

$$\left(\frac{U^2}{{c_0}^2}-1\right)\frac{\partial^2\phi}{\partial x^2}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right).$$

Replacing *x* by the variable

$$\tau=\frac{x}{\sqrt{U^2-{c_0}^2}}$$

we obtain the equation

$$\frac{1}{{c_0}^2}\frac{\partial^2\phi}{\partial\tau^2}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right), \tag{3}$$

i.e., the equation of a cylindrical wave in which the role of the time is played by the variable τ.

At sufficiently large distances the **cylindrical** wave within each small region can be treated as a plane one. The displacement velocity of each point of the profile of the wave will then be determined by formula (2). If we wish, however, to follow with the help of this formula the displacement of. a point of the wave profile during large time intervals, it is necessary to take into account the fact that the amplitude of the cylindrical wave decreases with the distance already in the first approximation as $1/\sqrt{r}$. Writing

$$\frac{p'}{p_0}=\frac{\chi}{\sqrt{r}}, \tag{4}$$

and substituting in equation (2), we get

$$u=c_0\left(1+\frac{\alpha\chi}{\sqrt{r}}\right). \tag{5}$$

The first term corresponds to a. displacement of the wave without change of profile {aside from the general decrease of the amplitude as $1/\sqrt{r}$), while the second leads to a. distortion of the profile. The magnitude δr of the additional displacement of the point of the profile at a distance from a. certain given (large) r_0 to *r* is obtained by multiplying by dr/c_0 and integrating from *r* to r_0 for a. constant χ:

$$\delta r=2\alpha\chi(\sqrt{r}-\sqrt{r_0}).$$

If the profile of the wave is defined by the change of *p*' with τ, for a given *r* then the distortion $\delta\tau$ of the profile will be $\delta\tau=\delta r/c_0$, i.e.,

$$\delta\tau=\frac{2\alpha\chi}{c_0}(\sqrt{r}-\sqrt{r_0}). \tag{6}$$

A diverging **cylindrical** wave can be written, in the linear approximation, in the following way:

$$\phi = \int_r^\infty \frac{f(\tau + \xi / c) d\xi}{\sqrt{\xi^2 - r^2}} . \tag{7}$$

The positive sign in $\tau + \xi / c$ corresponds to the fact that in the present case the wave is propagated from positive values to the negative ones (here and below the index zero, corresponding to the equilibrium values of the different quantities, is dropped for the sake of brevity). In our case the "time" τ means in reality the co-ordinate x. We shall take the origin of the co-ordinates inside the body (at a given instant), in this case the regions in front of the body correspond to positive x. Since in the case of: a motion with a velocity exceeding that of sound the perturbations are not propagated in the region of space, lying in front of the body, it can, in any case, be asserted that $\phi \to 0$ for $\tau \to \infty$. Further, at sufficiently large distances behind the body, where the perturbations caused by it are small, even on the axis itself, the potential of the divergent wave determined by formula (7) must remain finite for $\tau = 0$. For the convergence of the integral

$$\phi(0, \tau) = \int_0^\infty f(\tau + \xi / c) \frac{d\xi}{\xi}$$

at the lower limit (for large negative τ) it is necessary, that $f(\tau) \to 0$ for large negative τ. Hence it is easy to conclude that for $\tau \to -\infty$ we have also $\phi \to 0$.

On the other hand, the variable part of the pressure in the linear approximation is connected with ϕ by the equation $p' = -\rho c \frac{\partial \phi}{\partial \tau}$. Integrating with respect to ϕ we get, consequently,

$$\int_{-\infty}^{+\infty} p' d\tau = 0 . \tag{8}$$

This means that if there is a compression in the gas (the region with $p' > 0$), then there must also exist a region of rarefaction, where $p' < 0$ (in this respect a cylindrical wave - the same refers to a spherical one - differs substantially from a plane wave, which can consists of compressions or of rarefactions only).

As is well known, if the velocity of the body is larger than the velocity of sound, there arises in the gas a **shock wave**: in the space lying in front of this wave the gas remains at rest and directly behind the wave there is a region of compression. It follows from (8) that the compression must necessary be followed by a **rarefaction** and that, consequently, there must exist a point in which the **rarefaction reaches its maximum**; owing to the gradual distortion of the profile this point will lag with respect to those situated behind it; this

will, finally, lead to a loss of the uniqueness of the solution and **one more shock wave will arise**.

We arrive at the result that, at least at large distances from the moving body, there exists not a single shock wave, as is usually assumed, but **two shock waves** following each other. In the first wave the pressure suffers a positive jump; thereafter the pressure is gradually decreased, the compression being replaced by a. rarefaction, the pressure, finally, jumping again in the second shock wave.

Figure 1 represents schematically (by a continuous line) the resulting picture of the dependence of pressure p' on τ (i.e., on the co-ordinate x) for a given (large) value of r; ab is the first shock wave, de - the second wave. In the latter the pressure increases up to a certain negative value only, p' tending to zero asymptotically when $\tau \to -\infty$.

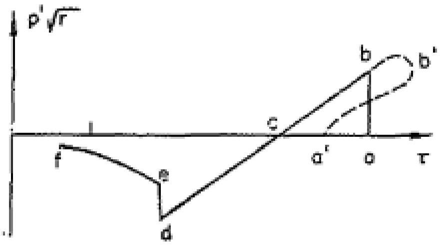

FIG. 1.

Passing to the quantitative calculation of the profile represented in Fig. 1, let us consider the region between the two shock waves. Let the function $\tau = f(\chi)$ (where χ denotes $p'\sqrt{r}/p$) determine the profile at a certain distance r_0.

Taking into account the effect of the distortion of the profile, we obtain the profile at a distance $r >> r_0$ by adding to τ the displacement $\delta\tau$ from (6):

$$\tau = f(\chi) + \frac{2\alpha}{c}(\sqrt{r} - \sqrt{r_0})\chi . \qquad (9)$$

For large r χ is small, and one can with a sufficient accuracy write in (9) the value of the function $f(\chi)$ for $\chi = 0$, while $\sqrt{r_0}$ can be neglected when compared with $\sqrt{r}$:

$$\tau = \frac{2\alpha}{c}\sqrt{r}\chi + const . \qquad (10)$$

Denoting by x_0 the value of the co-ordinate x at the point c (Fig. 1), where $\chi = 0$ (it depends, of course, on :r, according to the law

$x_0 = const \cdot \frac{r}{c}$) and passing to *p*' and *x* instead of χ and τ, we get hence

$$\frac{p'}{p} = \frac{1}{2\alpha\sqrt{\frac{U^2}{c^2} - 1}} \cdot \frac{x - x_0}{r}. \tag{11}$$

The segment *db* of the profile thus proves to be rectilinear. The profile, which is obtained directly by the application the Riemann solution throughout the whole interval, is represented in Fig.1 by a **dotted line**. In reality there exists a discontinuity at a certain point *a*. The position of this point is determined by the above mentioned geometrical condition of the equality of the areas *a'b'c'* and *abc*. Noting that in the points *a*' and *c*' $\chi = 0$, we find with the help of (9) for the are *a'b'c'*:

$$\int\limits_{(c'b'a')} \chi d\tau = \int\limits_{(c'b'a')} \chi f'(\chi) d\chi ,$$

i.e., a value, which is independent of τ. The same must, consequently, hold for the area *abc*. Taking into account the dependence (10) χ on τ we find without difficulty that the length l_1 of the segment from the point *c* (where *p*' = 0) up to the front shock wave (*p*' = p_1) is proportional to

$$l_1 \sim r^{1/4}. \tag{12}$$

Hence the dependence of the jump p'_1 of the pressure in the front shock wave on the distance is expressed by

$$p'_1 = \frac{const}{r^{3/4}}. \tag{13}$$

As regards the second discontinuity (*ed* in Fig. 1), it can easily be shown that the ratio of the pressure behind it (pressure at the point *e*) to the jump p'_2 of the pressure in the discontinuity {length of the segment *ed*) tends slowly to unity for $r \to \infty$. The pressure behind this discontinuity can be regarded as equal to zero at very large distances *r* only; the jump p'_2 is equal here to p'_2 (the total area of the profile must be equal to zero in virtue of (8).

Let us consider now the **spherically symmetric** propagation of a shock wave arising in the case of an **explosion** (and investigated at large distances from the place of the explosion). All the arguments here are exactly similar to those considered above.

In the case of a spherical propagation the amplitude of the wave decreases in the first approximation, as $\frac{const}{r}$, where *r* is now the distance from the

centre. We, therefore, get for the velocity u of the displacement of the points of the profile instead of (5)

$$u = c\left(1 + \frac{\alpha\chi}{r}\right), \tag{14}$$

where χ denotes now the product

$$\chi = \frac{p'}{p} r . \tag{15}$$

We find accordingly for the displacement δr of the points of the profile on the path from a certain r_0 to r

$$\delta r = \alpha\chi \ln\frac{r}{r_0} .$$

If, however, the profile is considered as the course of the change of p' with the time τ, the distortion δr is

$$\delta\tau = -\frac{\alpha\chi}{c}\ln\frac{r}{r_0} . \tag{16}$$

The distortion of the profile of a spherical wave is thus increased with the distance, according to the logarithmical law, i.e., much more slowly than in the case of a plane or a cylindrical wave (where it is proportional respectively to the first power, or to the square root of the distance). Inasmuch as the propagation of an actual wave in gas is accompanied by usual sound absorption, which is connected with the viscosity and the thermal conductivity, then, because of the slowness of the increase of distortion, the spherical sound wave can be absorbed, before the distortion of the profile will lead to the formation of discontinuities. In the particular case of the propagation of an explosive shock wave it may happen that the second shock wave by which (just as in the cylindrical case) it should be followed, may not appear because of the lack of time necessary for its formation.

Instead of equation (9) we now have the equation

$$t = f(\chi) - \frac{\alpha}{c}\ln\frac{r}{r_0}\cdot\chi . \tag{17}$$

Expanding $f(\chi)$ into series of powers of χ and limiting ourselves to the terms of the first order we get

$$t = -\frac{\alpha}{c}\chi\ln\frac{r}{a} \div const , \tag{18}$$

where a is a. certain constant. Hence we again obtain for p' a linear dependence of t in the form

$$\frac{p'}{p} = \frac{1}{\alpha}\frac{c(t_0 - t)}{r\ln(r/a)} . \tag{19}$$

Taking into account the law of conservation of the area, we get in the **spherical** case

$$l_1 \sim \sqrt{\ln \frac{r}{a}} \,, \tag{20}$$

$$p'_1 \sim \frac{1}{r\sqrt{\ln(r/a)}} \,.$$