PART II
The Fundamental Quadratic Form and the Absolute Differential Calculus

CHAPTER VI
Covariant Differentiation; Invariants and Differential Parameters; Locally Geodesic Co-ordinates

1. Covariant differentiation.

Returning to the remarks made at the end of Chapter IV, we now propose to generalize the operation of differentiation by substituting for the ordinary derivatives of the elements of a tensor certain linear combinations of these derivatives and of the elements of the given system, which will in their turn constitute a mixed (or in particular, covariant) system with one index of covariance more than the given system. Explicitly, if $A_{i_1 \cdots i_m}^{h_1 \cdots h_\mu}$ is the given generic system whose elements are functions of the x's or, in geometrical terms, functions of position, we shall deduce from it another system $A_{i_1 \cdots i_m l}^{h_1 \cdots h_\mu}$ where l is a new index of covariance, which reduces to the system $\dfrac{\partial A_{i_1 \cdots i_m}^{h_1 \cdots h_\mu}}{\partial x_l}$ in the particular case when the co-ordinates are Cartesian.

To simplify the formulae, we shall consider first a mixed system A_i^h with a sin le index i of covariance and a single index h of contravariance.

Fixing our attention on a specific point of V_n (i.e., ignoring the fact that the A's are defined as functions of position), we know that the law of transformation of the functions A_i^h for a change of variables is defined by the invariance of the form

$$F = \sum_{i,h=1}^{n} A_i^h \xi^i u_h \qquad (1)$$

in which the ξ^i's constitute a generic contra variant system, or, in other words, are the contravariant components of a generic vector ξ; similarly the u_h's can be considered as the covariant components of a generic vector $\boldsymbol{u}$.

Now, since a set of values of the A_i^h's is associated with every point of V_n, we can at every point choose two arbitrary vectors $\boldsymbol{\xi}$, $\boldsymbol{u}$, and construct an invariant form with them and the A's.

Suppose this choice made at an arbitrary but determined point P, and consider also a generic point P_1 infinitely near to P. We shall agree to take for $\boldsymbol{\xi}$ and $\boldsymbol{u}$ at P_1 the vectors ***parallel*** to those chosen at P; as the displacement is infinitesimal, the curve of displacement is immaterial. We shall use the operator δ to denote in general the increment of a quantity in passing from P to P_1, and we propose to calculate δF. Differentiating (1) with the operator δ, we have

$$\delta F = \sum_{i,h=1}^{n} \left\{ \delta A_i^h \xi^i u_h + A_i^h \delta\xi^i u_h + A_i^h \xi^i \delta u_h \right\}.$$

Now, by the convention just adopted as to the vectors ξ and $\boldsymbol{u}$, the differentials $\delta\xi^i$ and δu_h must be calculated by the formulae of parallelism ((52) and (52') of Chapter V), while δA_i^h is given by the usual rule of differentiation

$$\delta A_i^h = \sum_{l=1}^{n} \frac{\partial A_i^h}{\partial x_l} \delta x_l \ ,$$

the A's being by hypothesis functions of position. Using these results, we have

$$\delta F = \sum_{i,h,l=1}^{n} \frac{\partial A_i^h}{\partial x_l} \xi^i u_h \delta x_l - \sum_{i,h,j,l=1}^{n} A_i^h \{jl, i\} u_h \xi^j \delta x_l + \sum_{i,h,j,l=1}^{n} A_i^h \{hl, j\} \xi^i u_j \delta x_l$$

Interchanging i and j in the second sum and h and j in the third, so as to get the factor $\xi^i u_h \delta x_l$ in all three sums, and collecting all the terms under a single summation sign, we have

$$\delta F = \sum_{i,h,l=1}^{n} \left[\frac{\partial A_i^h}{\partial x_l} - \sum_{j=1}^{n} A_j^h \{il, j\} + \sum_{j=1}^{n} A_i^j \{jl, h\} \right] \xi^i u_h \delta x_l \ . \tag{2}$$

Now the left-hand side of this equation is invariant on account of its meaning, while ξ^i, δx_l, u_h are arbitrary contra variant or covariant systems; hence the coefficients of this form (the expression in square brackets) constitute by definition a system which is covariant with respect to i and l and contravariant with respect to h. We can therefore put

$$(A_i^h)_l = \frac{\partial A_i^h}{\partial x_l} - \sum_{j=1}^{n} A_j^h \{il, j\} + \sum_{j=1}^{n} A_i^j \{jl, h\} \tag{3}$$

This system is called the ***covariant derivative*** of the system A_i^h. It is sometimes denoted by the symbol $A_{i|l}^h$, and also, when no ambiguity is possible, simply by A_{il}^h.

It is obvious that in Cartesian co-ordinates (which exist when we are dealing with Euclidean forms; cf. § 21 of the preceding chapter) the system reduces to that of ordinary derivatives.

The method used above can be applied, *mutatis mutandis*, to a generic mixed system. We shall always get for δF (as follows at once on carrying out the necessary operations) a multilinear form whose coefficients we shall define as elements of the covariant derived system. These coefficients consist of a first term which is the ordinary derivative, followed by as many terms preceded by the minus sign as there are indices of covariance of the given system, and as many terms preceded by the plus sign as there are indices of contravariance. If we denote by (i) the aggregate of indices $i_1, \ldots, i_m$ and by (h) the aggregate $h_1, \ldots, h_\mu$, the general formula is

$$A_{(i)|l}^{(h)} = \frac{\partial A_{(i)}^{(h)}}{\partial x_l} - \sum_{r=1}^{m} \sum_{j=1}^{n} A_{i_1 \cdots i_{r-1} j i_{r+1} \cdots i_m}^{(h)} \{i_r l, j\} + \sum_{\rho=1}^{\mu} \sum_{j=1}^{n} A_{(i)}^{h_1 \cdots h_{\rho-1} j h_{\rho+1} \cdots h_\mu} \{jl, h_\rho\} \tag{4}$$

2. Particular cases.

Consider first a covariant simple system A_i, which we can always interpret as consisting of the covariant components (moments) of a vector ***A***. In this case the terms contributed by the indices of contravariance are absent, and (4) (or (3)) gives

$$A_{i|l} = \frac{\partial A_i}{\partial x_l} - \sum_{j=1}^{n} \{il, j\} A_j \,. \tag{5}$$

It is easy to see that this double system is not in general symmetrical; from (5) however we get at once the important relation

$$A_{i|l} - A_{l|i} = \frac{\partial A_i}{\partial x_l} - \frac{\partial A_l}{\partial x_i} \,. \tag{6}$$

The vanishing of the covariant derivative $A_{i|l}$ has a simple geometrical significance. In this case, multiplying (5) by dx_l, we have

$$\frac{\partial A_i}{\partial x_l} dx_l = \sum_{j=1}^{n} \{il, j\} A_j dx_l \;;$$

comparing this ,with equation (52') of the preceding chapter, in which we suppose all the *dx*'s to vanish except the *l*th, we see that it expresses the fact that the vector ***A*** undergoes a parallel displacement along the line *l*.

Analogously, for the derivatives of a contravariant simple system A^i, we have

$$A^i_{|l} = \frac{\partial A^i}{\partial x_l} + \sum_{j=1}^{n} A^j \{jl, i\} \,. \tag{5'}$$

Next, consider a system of order zero, i.e., an invariant *f*. In this case (4) becomes

$$f_l = \frac{\partial f}{\partial x_l} \,, \tag{7}$$

or ***the covariant and the ordinary derivatives are identical***. If construct the system of covariant second derivatives, applying formulae (5) to (7). we shall have

$$f_{lk} = \frac{\partial^2 f}{\partial x_l \partial x_k} - \sum_{j=1}^{n} \{lk, j\} \frac{\partial f}{\partial x_j} \;; \tag{8}$$

these are not the same as the ordinary second derivatives but, like them, are symmetrical.

For a covariant double tensor (4) becomes

$$A_{ik|l} = \frac{\partial A_{ik}}{\partial x_l} - \sum_{j=1}^{n} \{il, j\} A_{jk} - \sum_{j=1}^{n} \{kl, j\} A_{ij} \;; \tag{9}$$

and for a contravariant double tensor it becomes

$$A^{ik}_{|l} = \frac{\partial A^{ik}}{\partial x_l} + \sum_{j=1}^{n} \{jl, i\} A^{jk} + \sum_{j=1}^{n} \{jl, k\} A^{ij} \,. \tag{9'}$$

3. Ricci's lemma.

If formula (9) is applied to the system of the coefficients of ds^2, we get, remembering the expression for the derivatives of

these coefficients in terms of Christoffel's symbols (Chap. V, § 16),

$$a_{ik|l} = 0 \qquad (i,k,l = 1,2,\ldots,n) \tag{10}$$

This important theorem, that ***the covariant derivatives of the coefficients*** a_{ik} ***are zero***, can be proved directly from the definition of covariant differentiation. To do so, we must choose two arbitrary vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and construct the expression

$$F = \sum_{i,k=1}^{n} a_{ik} \xi^i \eta^k \ ;$$

we then calculate δF corresponding to a parallel displacement of the vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and we shall get a trilinear form in ξ^i, η^k, δx_l, whose coefficients, by definition, will give the required derived system.

Now F is merely the scalar product of the vectors and $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, which, as we know, is not changed by a parallel displacement; hence we shall have $\delta F = 0$ for any values whatever of $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and $\delta x's$, which means that all the coefficients of this form vanish identically.

Similarly we can show that the covariant derivatives of the reciprocals a^{ik} vanish; in this case we have to use the expression

$$F = \sum_{i,k=1}^{n} a^{ik} u_i v_k$$

which is again the scalar product of the (arbitrary) vectors ***u*** and ***v***.

4. Contravariant differentiation.

There is in the absolute differential calculus a kind of law of reciprocity or duality in accordance with which we can deduce from every theorem or formula a reciprocal theorem or formula, by interchanging the words ***covariant*** and ***contravariant***, and lowering or raising the indices. We have already had several examples of this; we shall now make some brief remarks on the operation of contravariant differentiation, which corresponds to that of covariant differentiation just described.

The shortest way to deduce from a system $A_{(i)}^{(h)}$ the system $A_{(i)}^{(h)k}$ which has the properties reciprocal to those of the covariant derivatives, is to find the covariant derivative of the given system and then compound it with the system of the a^{kl}'s; i.e., to make

$$A_{(i)}^{(h)k} = \sum_{l=1}^{n} a^{kl} A_{(i)l}^{(h)} \ .$$

We could find for this system an expression analogous to (4) and properties corresponding exactly to those of the covariant derivatives; or we could find these properties directly from those of the covariant derivatives, by using the foregoing formula of definition. We shall therefore not pursue the argument in detail, and shall instead resume our discussion of the fundamental properties of covariant differentiation.

5. Conservation of the rules of the ordinary differential calculus.

First, consider a tensor, in general mixed, which is the sum of two others of the same rank and species, i.e.,

$$A_{(i)}^{(h)} = B_{(i)}^{(h)} + C_{(i)}^{(h)} .$$

It will at once be seen that the covariant derivative of the system *A* is obtained, like an ordinary derivative, by adding together that of *B* and that of *C*, or

$$A_{(i)l}^{(h)} = B_{(i)l}^{(h)} + C_{(i)l}^{(h)} . \tag{11}$$

This formula follows either from the linearity of (4), or from the consideration that the form *F* relative to *A* is the sum of a form relative to *B* and a form relative to *C*, so that a similar result holds for δF; the coefficients of the latter expression (which are by definition the derivatives $A_{(i)l}^{(h)}$) will therefore be the sums of the corresponding coefficients of the other two (which are by definition the derivatives $B_{(i)l}^{(h)}$ and $C_{(i)l}^{(h)}$). The reasoning can be extended without difficulty to a sum of any number of terms.

Next, consider the derivative of a product. If $B_{(i')}^{(h')}$, $C_{(i')}^{(h')}$ are two generic tensors, we shall denote their product by

$$A_{(i)}^{(h)} = B_{(i')}^{(h')} \cdot C_{(i")}^{(h")} ,$$

where the symbol (*i*) stands for the aggregate of the indices (*i*') and (*i*") together, and similarly for (*h*). We shall show that

$$A_{(i)l}^{(h)} = B_{(i')l}^{(h')} \cdot C_{(i")}^{(h")} + B_{(i')}^{(h')} \cdot C_{(i")l}^{(h")} . \tag{12}$$

To simplify the formulae we shall suppose that the systems *A* and *B* have each only one index of covariance and one of contravariance. We know (Chapter IV, § 8) that if

$$\phi = \sum B_{i'}^{h'} \xi^{i'} u_{h'} ,$$
$$\psi = \sum C_{i"}^{h"} \eta^{i"} v_{h"}$$

are the invariant forms for the systems *B* and *C*, that for the system *A* is

$$F = \phi\psi .$$

We shall therefore have

$$\delta F = \psi\delta\phi + \phi\delta\psi ,$$

and equating the coefficients of $\xi^{i'}\eta^{i"}u_{h'}v_{h"}\delta x_l$ on both sides of this equation we get equation (12) (for the particular case considered).

Now consider the derivative of a compounded mixed system (Chapter IV, §§9, 10)

$$A_{(i)}^{(h)} = \sum_{(r)(s)=1}^{n} B_{(i')(r)}^{(h')(s)} C_{(i")(s)}^{(h")(r)} , \tag{13}$$

where (*i*) and (*h*) have the meanings already explained, and (*r*) and (*s*) denote the aggregate of all the indices affected by the process of contraction. We shall show that

$$A_{(i)l}^{(h)} = \sum_{(r)(s)=1}^{n} \left[B_{(i')(r)l}^{(h')(s)} C_{(i")(s)}^{(h")(r)} + B_{(i')(r)}^{(h')(s)} C_{(i")(s)l}^{(h")(r)} \right]. \tag{14}$$

In particular, if each aggregate reduces to a single index and if the process of contraction is applied only to one index, (13) becomes

$$A_{ij}^{hk} = \sum_{r=1}^{n} B_{ir}^{h} C_{j}^{kr} , \tag{13'}$$

and (14) becomes

$$A_{ijl}^{hk} = \sum_{r=1}^{n} \left[B_{irl}^{h} C_{j}^{kr} + B_{ir}^{h} C_{jl}^{kr} \right]. \tag{14'}$$

We shall give the **proof** for this simpler case, merely pointing out that it can be immediately extended to the general case.

We start from the invariant forms relative to the systems B and C

$$\phi_\alpha = \sum_{i,h,r=1}^{n} B_{ir}^{h} \xi^i u_h \lambda_\alpha^r ,$$

$$\psi_\alpha = \sum_{j,k,s=1}^{n} C_{j}^{ks} \eta^j v_k \lambda_{\alpha|s} ,$$

where we have followed the same procedure as in Chapter IV, § 9, and introduced a set of n contravariant systems λ_α^r ($\alpha = 1,2,\ldots,n$) and the associated reciprocal set. The invariant form

$$F = \sum_{\alpha=1}^{n} \phi_\alpha \psi_\alpha$$

has the A's as coefficients, as we saw in Chapter IV.

Applying the symbol of operation δ to this we get

$$\delta F = \sum_{\alpha=1}^{n} \left[\psi_\alpha \delta\phi_\alpha + \phi_\alpha \delta\psi_\alpha \right],$$

and equating the coefficients of $\xi^i \eta^i u_h v_k \delta x_l$ on both sides of this equation, we get (14').

To sum up, we have shown that the fundamental rules of ordinary differentiation hold good for covariant differentiation.

6. Applications.

We note first of all that if we start from a generic simple system (a function of position), say a covariant system V_i, and consider its reciprocal V^i, we have by definition

$$V^i = \sum_{k=1}^{n} a^{ik} V_k ;$$

hence, taking the covariant derivative and using Ricci's lemma,

$$V_{|l}^{i} = \sum_{k=1}^{n} a^{ik} V_{k|l} . \tag{15}$$

We shall next calculate the covariant derivative of the scalar product X of two vectors, which, as we know already, is identical with the ordinary derivative.

Let $\boldsymbol{U}$, $\boldsymbol{V}$, be two generic vectors, and put

$$X = U \times V = \sum_{i=1}^{n} U_i V^i .$$

Taking the covariant derivative, we have

$$X_l = \sum_{i=1}^{n} \left[U_{i|l} V^i + U_i V_{|l}^{i} \right].$$

In the second term on the right we can replace $V^i_{|l}$ by the expression for it in (15), so that

$$\sum_{i=1}^n U_i V^i_{|l} = \sum_{i,k=1}^n a^{ik} U_i V_{k|l} = \sum_{k=1}^n U^k V_{k|l} \ .$$

Changing k into i, and substituting in X_l, we get the formula

$$X_l = \sum_{i=1}^n \left[U_{i|l} V^i + U^i V_{i|l} \right], \tag{16}$$

which is often used.

In particular, if $\boldsymbol{V} = \boldsymbol{U}$, we have $X = \boldsymbol{U}^2$, and therefore

$$X_l = 2\boldsymbol{U}\frac{\partial \boldsymbol{U}}{\partial x_l} = 2\sum_{i=1}^n U^i U_{i|l} \tag{16'}$$

7. Divergence of a vector and of a double tensor. Δ_2 of an invariant.

Take a covariant simple system X_i, which we can always think of as the aggregate of the components of a vector $\boldsymbol{X}$, and construct the invariant

$$\Theta = \sum_{i,l=1}^n a^{il} X_{i|l} \ , \tag{17}$$

where the terms $X_{i|l}$ denote covariant derivatives. In the particular case of the fundamental form being Euclidean, we have $a^{il} = \delta^l_i$, and also the covariant and ordinary derivatives are identical; hence in this case (17) becomes

$$\Theta = \sum_{i=1}^n \frac{\partial X_i}{\partial x_i} \ .$$

In three dimensions this expression is called the ***divergence*** of the vector $\boldsymbol{X}$. We shall extend the use of this term to the general case (17).

We can transform (17) by means of (15). Writing X instead of V, (15) becomes, for $l = i$,

$$X^i_{|i} = \sum_{k=1}^n a^{ik} X_{k|i} \ .$$

Summing with respect to i, the right-hand side gives Θ, as can be seen at once from (17) by putting l instead of k and then interchanging l and i. Hence we have

$$\Theta = \sum_{i=1}^n X^i_{|i} \ . \tag{17'}$$

From the general rule for covariant differentiation, or more specifically from (5'), we have

$$X^i_{|i} = \frac{\partial X^i}{\partial x_i} + \sum_{j=1}^n \{ji,i\} X^j \ .$$

Now sum with respect to i. Substituting from (17') on the left, and from the identity

$$\sum_{i=1}^n \{ji,i\} = \frac{1}{\sqrt{a}}\frac{\partial \sqrt{a}}{\partial x_j}$$

(cf. formula (26) in the preceding chapter) on the right, and writing

l as the index of summation on the right instead of i and j, we get

$$\Theta = \sum_{l=1}^{n} \left(\frac{\partial X^l}{\partial x_l} + \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}}{\partial x_l} X^l \right)$$

or, taking the factor $\frac{1}{\sqrt{a}}$ outside the summation sign,

$$\Theta = \frac{1}{\sqrt{a}} \sum_{l=1}^{n} \frac{\partial}{\partial x_l} \left(\sqrt{a} X^l \right). \qquad (17")$$

This expression for the divergence is completely equivalent to the formulae (17) and (17'); it is more useful for purposes of calculation, while (17) and (17') on the contrary are more suited to theoretical discussions.

In particular, consider the case where the vector in question is the ***gradient*** of an invariant u, i.e., where

$$X_i = \frac{\partial u}{\partial x_i} \qquad (i=1,2,\ldots,n).$$

In this case the divergence is denoted by the symbol $\Delta_2 u$ and is called the ***second differential parameter*** of the function u; the expression for it can be deduced at once from (17) or from (17"), using in the calculations the fact that

$$u^l = \sum_{i=1}^{n} a^{il} \frac{\partial u}{\partial x_i}.$$

We thus get

$$\Delta_2 u = \sum_{i,k=1}^{n} a^{ik} u_{ik} = \frac{1}{\sqrt{a}} \sum_{l=1}^{n} \frac{\partial}{\partial x_l} (\sqrt{a} u^l), \qquad (18)$$

both these expressions being generalizations of the ordinary expression for Δ_2 in Cartesian co-ordinates.

Next, take a contra variant double tensor X^{ik}. We note first of all that if instead the given tensor were covariant (X_{ik}) or mixed (X_i^k), we could always compound it with the a^{ik}'s and so obtain an associated tensor in which both indices are indices of contravariance; so that the choice of a contravariant tensor does not really constitute a restriction. From this tensor, taking the covariant derivative and applying the process of contraction, we get the contravariant simple system

$$Y^i = \sum_{k=1}^{n} X^{ik}_{|k}, \qquad (19)$$

which, by an obvious analogy with the former case, is called the ***divergence of the given double tensor***. If the process of contraction were applied to the first instead of to the second index, we should plainly get a contravariant system

$$\sum_{k=1}^{n} X^{ki}_{|k};$$

in general this is distinct from the divergence Y^i, coinciding with it only in the particular case when the given tensor X^{ik} is symmetrical. Vice versa, if X_{ik} is the system reciprocal to X^{ik} (the indices corresponding in the order written), we see at once from the rules in § 5 that the system

$$Y_i = \sum_{k,l=1}^{n} a^{kl} X_{ik|l}$$

is merely the covariant system reciprocal to (19). Returning to (19), it should be added that the expression on the right cannot in general be transformed (as was done for the ordinary divergence (17)) into an expression which is convenient for actual calculations. In the case of an antisymmetrical tensor ($X^{ik} + X^{ki} = 0$), however, the analogy in this respect is perfect. In fact, if we substitute in (19) the values of $X^{ik}_{|k}$ given by (9'), the second term on the right vanishes from the antisymmetry of the X's, while the other two give

$$\sum_{k=1}^{n} \frac{\partial X^{ik}}{\partial x_k} + \sum_{j,k=1}^{n} \{jk,k\} X^{ij} \ .$$

From this expression, by the same method as that just used to pass from (17') to (17"), we get the equation

$$Y^i = \frac{1}{\sqrt{a}} \sum_{k=1}^{n} \frac{\partial(\sqrt{a} X^{ik})}{\partial x_k} . \qquad (19')$$

8. Some laws of transformation, ε-systems. Vector product. Extension of a field.

Consider a set of n covariant simple systems $\lambda_{\alpha|i}$ (where α is the ordinal number of the system and i the index of covariance) and the determinant of the set

$$\nabla = \left\| \lambda_{\alpha|i} \right\| .$$

Changing the co-ordinates from the x's to another set of variables $\overline{x}$, the systems $\lambda_{\alpha|i}$ are transformed (in accordance with the law of covariance) into another set of systems $\overline{\lambda}_{\alpha|i}$. Construct the determinant of these new quantities

$$\overline{\nabla} = \left\| \overline{\lambda}_{\alpha|i} \right\| .$$

We shall show that the relation between $\overline{\nabla}$ and ∇ is

$$\overline{\nabla} = \nabla D , \qquad (20)$$

where D denotes the Jacobian determinant of the transformation, i.e.,

$$D = \begin{pmatrix} x_1 x_2 \cdots x_n \\ \overline{x}_1 \overline{x}_2 \cdots \overline{x}_n \end{pmatrix},$$

which is of course not zero, it being always supposed that we are using a reversible transformation (Chapter I). The relation (20) can be verified at once if we construct the product by rows of the two determinants on the right, viz.,

$$\begin{vmatrix} \lambda_{1|1} & \lambda_{1|2} & \cdots & \lambda_{1|n} \\ \lambda_{2|1} & \lambda_{2|2} & \cdots & \lambda_{2|n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n|1} & \lambda_{n|1} & \cdots & \lambda_{n|n} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x_1}{\partial \overline{x}_1} & \frac{\partial x_2}{\partial \overline{x}_1} & \cdots & \frac{\partial x_n}{\partial \overline{x}_1} \\ \frac{\partial x_1}{\partial \overline{x}_2} & \frac{\partial x_2}{\partial \overline{x}_2} & \cdots & \frac{\partial x_n}{\partial \overline{x}_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial \overline{x}_n} & \frac{\partial x_2}{\partial \overline{x}_n} & \cdots & \frac{\partial x_n}{\partial \overline{x}_n} \end{vmatrix} .$$

Recalling (Chapter IV, § 11) that

$$\overline{\lambda}_{\alpha|i} = \sum_{j=1}^{n} \lambda_{\alpha|j} \frac{\partial x_j}{\partial \overline{x}_i}, \tag{21}$$

we see at once that the elements of the product determinant are precisely the quantities $\overline{\lambda}_{\alpha|i}$.

It is also useful to examine the behaviour of the discriminant

$$a = \|a_{ik}\|$$

of the fundamental form when we change from the variables x to the variables $\overline{x}$. For this purpose, we take the transformation law of the coefficients a_{ik} (Chapter IV, § 129

$$\overline{a}_{ik} = \sum_{j,h=1}^{n} a_{jh} \frac{\partial x_j}{\partial \overline{x}_i} \frac{\partial x_h}{\partial \overline{x}_k};$$

putting

$$b_{jk} = \sum_{h=1}^{n} a_{jh} \frac{\partial x_h}{\partial \overline{x}_k}, \tag{22}$$

we can write this law in the form

$$\overline{a}_{ik} = \sum_{j=1}^{n} b_{jk} \frac{\partial x_j}{\partial \overline{x}_i}.$$

This law, which is completely analogous to (21), enables us to conclude at once, from the example of the preceding case, that the relation between $\overline{a}$ and the determinant b of the quantities b_{jk} is analogous to (20), i.e., that

$$\overline{a} = bD. \tag{23}$$

Further, as (22) is of the same type as (21), the determinant b will be connected with a by the relation

$$b = aD,$$

which, combined with (23), gives us the required relation between a and $\overline{a}$, namely,

$$\overline{a} = aD^2. \tag{24}$$

It follows from (20) and (24) that the ratio $\dfrac{\nabla}{\sqrt{a}}$ is an absolute invariant, i.e., that

$$\frac{\overline{\nabla}}{\sqrt{\overline{a}}} = \frac{\nabla}{\sqrt{a}}.$$

Strictly speaking, this equality holds except for sign; but if we agree to change the sign of the radical when a transformation is made for which D is negative, it holds in sign as well as in numerical value.

The remark just made leads us to define a particularly useful tensor whose elements can be expressed in a very simple form.

In fact, we note that the quantity $\dfrac{\nabla}{\sqrt{a}}$, which we have just seen to be invariant, is merely a multilinear form in the n sets of variables $\lambda_{a|i}$; this is seen at once by expanding the determinant ∇ in the usual way, as the sum of products of its elements n at a time, where no product contains two elements from the same row or column, and with the usual rule as to sign. We may write the result in the form

$$\frac{\nabla}{\sqrt{a}} = \frac{1}{\sqrt{a}} S_{i_1,\ldots,i_n} \pm \lambda_{1|i_1} \lambda_{2|i_2} \cdots \lambda_{n|i_n} , \tag{25}$$

where the symbol S denotes the sum of all the possible products, subject to the conventions stated as to their structure and sign. Since this form is invariant, its coefficients constitute a contravariant system. If we put $\varepsilon^{i_1 i_2 \cdots i_n}$ for the coefficient of the product $\lambda_{1|i_1} \lambda_{2|i_2} \cdots \lambda_{n|i_n}$, we see at once that we have:

$\varepsilon^{i_1 i_2 \cdots i_n} = 0$ if at least two of the indices $i_1 i_2 \cdots i_n$ are equal;

$\varepsilon^{i_1 i_2 \cdots i_n} = \frac{1}{\sqrt{a}}$ if these indices are all different and constitute a permutation of ***even*** order with respect to the fundamental permutation 1, 2,…, n;

$\varepsilon^{i_1 i_2 \cdots i_n} = -\frac{1}{\sqrt{a}}$ if the indices are all different and constitute a permutation of ***odd*** order.

Hence it follows that the system of order n whose elements are 0, $\frac{1}{\sqrt{a}}$, $\frac{1}{\sqrt{a}}$, respectively according to the rules just stated, is ***contravariant***; we shall call it the ***contravariant ε-system***.

We can give an analogous definition of the ***covariant ε-system*** by considering the determinant (reciprocal to ∇)

$$\Delta = \left\| \lambda_\alpha^i \right\|$$

constructed from the reciprocal elements of the systems $\lambda_{\alpha|i}$ in the determinant ∇; these elements, as we know from Chapter IV. § 6, constitute a set of n contra variant simple systems. By a well-known theorem, which can at once be verified, we have

$$\nabla \Delta = 1 ;$$

hence the quantity $\sqrt{a}\Delta$ (the reciprocal of $\frac{\nabla}{\sqrt{a}}$) will be invariant. Expanding the determinant Δ, this can be written in the form

$$\sqrt{a} S_{i_1 \cdots i_n} \pm \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_n^{i_n} ,$$

where the symbol S as before denotes the sum for all permutations of the indices i.

It follows that the system whose elements $\varepsilon_{i_1 i_2 \cdots i_n}$ are zerp if the indices $i_1 i_2 \cdots i_n$ are not all different, and are equal to $\sqrt{a}$ or $-\sqrt{a}$ if the indices are all different according as the permutation $i_1 i_2 \cdots i_n$ is of even or odd order, is ***covariant***.

The use of the same letter ε for both is justified by the fact that this covariant system is the reciprocal of the former system. This statement can easily be verified by the reader.

By means of the ε-systems, when n - 1 vectors $\boldsymbol{v}_a$ (a = 1, 2, ..., n - 1) are given, we can deduce from them (by invariant processes) an nth vector $\boldsymbol{w}$, which is called their ***vector product***, as in three-dimensional Euclidean space it is identical with the ordinary vector product. If v_α^i and $v_{\alpha|i}$ (i = 1, 2,…, n) denote the contravariant and covariant components of the n - 1 given vectors,

the formulae

$$w_i = \sum_{i_1, i_2, \ldots, i_{n-1}=1}^{n} \varepsilon_{i i_1 i_2 \cdots i_{n-1}} v_1^{i_1} v_2^{i_2} \cdots v_{n-1}^{i_{n-1}} ,$$

$$w^i = \sum_{i_1, i_2, \ldots, i_{n-1}=1}^{n} \varepsilon^{i i_1 i_2 \cdots i_{n-1}} v_{1|i_1} v_{2 i_2} \cdots v_{n-1|i_{n-1}}$$

define two reciprocal systems, as can easily be verified; hence either separately defines the same vector, which we call $\boldsymbol{w}$. When $n = 3$ and the space is Euclidean the components of $\boldsymbol{w}$ do in fact reduce to those of the ordinary vector product.

In any case it follows from the preceding definition of the components w_i (or w^i) that $\boldsymbol{w} = 0$ if the vectors $\boldsymbol{v}_\alpha$ are not all linearly independent, i.e., if the characteristic of the matrix composed of their components < n - 1; when they are independent, $\boldsymbol{w} \neq 0$ and is perpendicular to every $\boldsymbol{v}_\alpha$. The flatter property follows from the consideration of a generic *scalar* product $\boldsymbol{w} \times \boldsymbol{v}_\alpha$. Taking, say, the first group of formulae, we have

$$\boldsymbol{w} \times \boldsymbol{v}_\alpha = \sum_{i=1}^{n} w_i v_\alpha^i = \sum_{i, i_1, i_2, \ldots, i_{n-1}=1}^{n} \varepsilon_{i i_1 i_2 \cdots i_{n-1}} v_1^{i_1} v_2^{i_2} \cdots v_{n-1}^{i_{n-1}} v_\alpha^i ,$$

which is zero from the definition of the ε-system, or, in other words, because the sum is the expansion of a determinant with two rows the same.

Lastly, we wish to introduce into the metric of a V_n the notion of the ***extension*** of a field, i.e., to define, for a given field of V_n, a quantity V analogous to the area of a portion of a surface or to the volume of a three-dimensional field. Evidently we have *a priori* a free choice as to the definition of dV, provided that when $n = 2$ it reduces to the expression already given for the element of area (Chapter V, formula (17)), and that when $n = 3$, in Cartesian co-ordinates, we have $dV = dxdydz$; further, from the geometrical meaning of the term, the extension V of a field must be an invariant. All these conditions are satisfied if we assume

$$dV = \sqrt{a}\, dx_1 \cdots dx_n , \tag{26}$$

where $\sqrt{a}$ denotes the arithmetical value of the radical, and therefore

$$V = \int_C \sqrt{a}\, dx_1 \cdots dx_n .$$

We know in fact that on a change of co-ordinates the product $dx_1 dx_2 \cdots dx_n$ must be replaced by $|D|\, d\bar{x}_1 d\bar{x}_2 \cdots d\bar{x}_n$. From (24), extracting the square root, and taking the absolute values of both sides of the equation, we get

$$\left|\sqrt{a}\right| \cdot |D| d\bar{x}_1 d\bar{x}_2 \cdots d\bar{x}_n = \sqrt{\bar{a}}\, d\bar{x}_1 d\bar{x}_2 \cdots d\bar{x}_n .$$

But the left-hand side is $\sqrt{a}\, dx_1 dx_2 \cdots dx_n$, which is therefore invariant.

9. Rotor of a simple tensor in three dimensions.

We can now give a definition of the ***rotor*** (or ***rotation***, or ***curl***)

of a vector $\boldsymbol{X}$ given as a function of position, which shall hold good both when the space considered is not Euclidean, and also when it is Euclidean but the co-ordinates are not Cartesian. For any value of n, the generalization consists in defining as the rotor the covariant double system

$$p_{il} = X_{i|l} - X_{l|i} \,,$$

which is obviously antisymmetrical, since $p_{il} + p_{li} = 0$ identically. As we saw in § 2, the p's can also be written as the differences of the ordinary derivatives $\frac{\partial X_i}{\partial x_l} - \frac{\partial X_l}{\partial x_i}$; if then we consider the $\boldsymbol{X}$'s as coefficients of a Pfaffian

$$\psi = \sum_{i=1}^{n} X_i dx_i \,,$$

it will be seen that the p's are merely the coefficients of the bilinear covariant of this Pfaffian (cf. Chapter II, § 4).

To get the full analogy to the ordinary rotor, however, we should consider a space of only three dimensions. For $n = 3$, there are three different elements $p_{il} = -p_{li}$, corresponding to the pairs of different suffixes 23, 31, 12, pairs of equal suffixes giving zero values of the p's. Bach of the pairs 23, 31, 12, can be associated with the absent suffix (1, 2, or 3, respectively), or, in a general formula, the index h can be associated with the pair of the type h + 1, h + 2, with the convention that suffixes which differ by 3 are to be considered equivalent; for instance, if h = 2, h + 2 represents the suffix 1. It is therefore easy to understand how when n = 3 the rotor can be represented by a simple instead of a double system. If, however, we were to put

$$p_h = p_{h+1,h+2}$$

the simple system so defined would be neither covariant nor contravariant. Instead, it will be convenient to apply the term rotor to a vector $\boldsymbol{R}$ whose contravariant components R^h are defined as follows (with the help of the ε -systems introduced in the preceding section):

$$R^h = \sum_{i,l=1}^{3} \varepsilon^{hil} X_{l|i} \quad (h = 1,2,3).$$

The contravariance of R^h follows immediately from the principle of contraction. In order to see the analogy between this expression and the ordinary rotor, note that in the double sum i and l can take only the values h + 1, h + 2 (since the ε corresponding to the value h would be zero); since i and l must also be unequal, there are two possible cases:

$$i = h + 1,\ l = h + 2, \text{ when } \varepsilon^{hil} = \frac{1}{\sqrt{a}},$$

$$i = h + 2,\ l = h + 1, \text{ when } \varepsilon^{hil} = -\frac{1}{\sqrt{a}}.$$

Hence this sum will have only two terms, and R^h can be written in the following form:

$$R^h = \frac{1}{\sqrt{a}}(X_{h+2|h+1} - X_{h+1|h+2})$$

or

$$R^h = \frac{1}{\sqrt{a}}\left(\frac{\partial X_{h+2}}{\partial x_{h+1}} - \frac{\partial X_{h+1}}{\partial x_{h+2}}\right);$$

the latter being convenient for actual calculations. In Cartesian co-ordinates a = 1, and we get the ordinary expression for the components of a rotor (it being supposed that x_1, x_2, x_3 correspond in order with x, y, z).

10. Sections of a manifold. Geodesic manifolds.

We know that in ordinary space S_3 if we are given two directions $\boldsymbol{\lambda}, \boldsymbol{\mu}$ starting from the same point P and defined by their cosines λ^i, μ^i (i = 1, 2, 3), every other direction ξ through P whose cosines ξ^i are linear combinations of those of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, i.e., $\xi^i = \rho\lambda^i + \sigma\mu^i$ lies in the plane determined by $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$.

The coefficients ρ and σ are of course not independent, as the $\xi's$ must satisfy a quadratic identity; we have in fact

$$\rho^2 + \sigma^2 + 2\rho\sigma\cos\widehat{\boldsymbol{\lambda\mu}} = 1 .$$

The directions ξ so defined are therefore simply infinite in number, and their aggregate is called a ***section***.

All this can easily be extended to a generic V_n, in which m directions $\boldsymbol{\lambda}_\alpha$ ($\alpha = 1,2,\ldots,m$) are given.

Take m multipliers ρ_α, for the moment arbitrary, and consider the directions ξ whose parameters are

$$\xi^i = \sum_{\alpha=1}^{m} \rho_\alpha \lambda_\alpha^i , \tag{27}$$

and consequently whose moments are

$$\xi_i = \sum_{\alpha=1}^{m} \rho_\alpha \lambda_{\alpha|i} . \tag{27'}$$

In order that these expressions may effectively represent parameters and moments, respectively, it is necessary and sufficient that they should satisfy the relation

$$\sum_{i=1}^{n} \xi^i \xi_i = 1 ,$$

that is to say

$$\sum_{\alpha,\beta=1}^{m} \rho_\alpha \rho_\beta \sum_{i=1}^{n} \lambda_\alpha^i \lambda_{\beta|i} = 1 ,$$

or, denoting the angle between the direction $\boldsymbol{\lambda}_\alpha$ and $\boldsymbol{\lambda}_\beta$ by $\widehat{\alpha\beta}$,

$$\sum_{\alpha,\beta=1}^{m} \rho_\alpha \rho_\beta \cos\widehat{\alpha\beta} = 1 . \tag{28}$$

Now suppose that the $\rho's$ are connected by this relation but are otherwise arbitrary. We then see that (27) (or (27')) defines an aggregate of ∞^{m-1} directions (this being the number of arbitrary parameters), including in particular the m given directions; this aggregate is called a ***section***.

A section G being defined in this way by means of m of its

directions $\boldsymbol{\lambda}_\alpha$, take in it any m directions $\boldsymbol{\lambda}'_\alpha$ whatever ($\alpha = 1, 2, \ldots, m$). It is almost obvious that the section G' determined by these directions is again G itself.

This can of course be verified algebraically. In fact, if a direction ξ belongs to G', this is equivalent to saying that its parameters are linear combinations of the parameters λ'^i_α, and therefore also of the parameters λ^i_α; i.e., the direction ξ also belongs to G; and vice versa.

We saw in Chapter V, § 23, that a geodesic is uniquely determined if its starting-point and direction are given. Now let us fix a point P in a V_n, and draw from it two directions $\boldsymbol{\lambda}, \boldsymbol{\mu}$; these will determine a section of ∞^1 directions drawn from P. Consider the ∞^1 geodesies drawn from P in all these directions: they constitute a surface (∞^2 points) which is called a ***geodesic surface with -pole P***.

A geodesic surface is therefore determined by a point and two directions.

A similar definition can be given of an m-dimensional geodesic manifold. Take a point in V_n, and m directions drawn from it, which will define a section of ∞^{m-1} directions, and construct the geodesic corresponding to each of these directions. Since each geodesic contains ∞^1 points, the aggregate of all of them will contain ∞^{m-1+1} points; i.e., it will constitute a manifold V_m, which we shall call a ***geodesic manifold***.

Particularly important cases are the ***geodesic surfaces*** ($m = 2$), and the ***geodesic hypersurfaces*** ($m = n - 1$) determined by $n - 1$ directions drawn from a point; we shall use these in the following section.

11. Locally geodesic (or locally Cartesian) co-ordinates.

In general, a system of co-ordinates in which ds^2 is represented by a form with constant coefficients is called ***Cartesian***. It is not always possible to choose co-ordinates of this kind in a given V_n; it is however always possible to find a system of co-ordinates which behave like Cartesians ***in the immediate vicinity of a point P assigned beforehand***, or, more precisely, which are such that the derivatives of the coefficients of ds^2 (which would vanish identically if the co-ordinates were Cartesian) all vanish at the point P. Such co-ordinates are called ***locally geodesic***, or ***locally Cartesian***, co-ordinates.

Their interest from the point of view of parallelism, or more generally of elementary equipollent displacement, appears plainly from equations (52) and (52') of the preceding chapter, which define the increments of the contravariant and covariant components, respectively. It follows from these equations that when the system of reference is geodesic at P, these increments, in passing to any very near point, are zero, precisely as are those of the ordinary Cartesian components in Euclidean spaces.

Now take a V_n, and in it any system of co-ordinates x; we propose to introduce - if this is possible - a new set of variables

$$\bar{x}_i = f_i(x_1, x_2, \ldots, x_n) \quad (i = 1,2,\ldots,n) \tag{29}$$

such that the $\bar{x}'s$ are geodesic co-ordinates at P, or in other words, putting $\bar{a}_{ik}$ for the coefficients of ds^2 in the new variables, such

that

$$\left(\frac{\partial \bar{a}_{ik}}{\partial \bar{x}_l}\right)_P = 0 \quad (i, k, l = 1,2,\ldots,n), \tag{30}$$

where the use of the suffix P denotes that after differentiation the $\bar{x}$'s are to be replaced by the co-ordinates $\bar{x}_P$ of P. Remembering the definition of Christoffel's symbols (Chapter V, §§ 15, 16), we see that (30) is equivalent to the condition that these symbols themselves are all zero at P, i.e., that

$$\{\overline{jl,i}\}_P = 0 \quad (j, l,i = 1,2,\ldots,n). \tag{30'}$$

The following analysis shows the possibility of finding a set of functions f_i to define a transformation of this kind.

The condition (30) consists of $n \cdot \frac{n}{2}(n+1)$ equations containing the first and second derivatives of the f's (since $\bar{a}_{ik}$, by the law of covariance, can be expressed in terms of the a_{ik}'s and the first derivatives of the f's). Now the number of first derivatives is n^2, and that of second derivatives is $n \cdot \frac{n}{2}(n+1)$, so that the number of both together is greater than the number of equations. Since, as we shall see, the equations are not algebraically inconsistent, it follows that we can solve the equations (30) for the values at P of the first and second derivatives of the f's, or rather for some of them, the others remaining arbitrary; further, the behaviour of the functions at points other than P is a matter of indifference. Thus the choice of the f's can be made with a wide degree of arbitrariness.

To avoid, however, the direct discussion of the equations (30), we shall start from the ideas contained in § 26 of the preceding chapter. We saw there that the expressions

$$\tau^i = d\xi^i + \sum_{j,l=1}^{n} \{jl,i\}\xi^j dx_l$$

constitute a contravariant simple tensor, the vector ξ and therefore its contravariant components ξ^i, and also the differentials dx_l, being all arbitrary. This holds in particular for the hypothesis $\xi^i = dx_i$, i.e., when we suppose

$$\tau^i = d^2 x_i + \sum_{j,l=1}^{n} \{jl,i\} dx_j dx_l \,. \tag{31}$$

If on changing the variables we have at a special point P

$$\frac{\partial \bar{x}_i}{\partial x_k} = \delta_k^i \,, \tag{32}$$

then at that point, from the law of contravariance

$$\bar{\tau}^i = \sum_{k=1}^{n} \tau^k \frac{\partial \bar{x}_i}{\partial x_k} \,,$$

it follows that

$$\bar{\tau}^i = \tau^i \,. \tag{33}$$

If we suppose (as we are always free to do, by making a preliminary change of variables from x_i to $x_i + a$ constant) that the x_i's vanish at P, the equations (32) are satisfied provided the formulae of transformation (29) are of the type

$$\bar{x}_i = x_i + \phi_i(x_1, x_2, \ldots, x_n), \qquad (29')$$

where ϕ_i denotes a function of the x's which is regular at P, and whose expansion in a series of powers of the x's begins with terms of at least the second degree, e.g., a polynomial of the second degree in the x's. In fact, if these conditions are fulfilled, all the first derivatives of the ϕ's vanish at P. The second derivatives $\frac{\partial^2 \phi_i}{\partial x_j \partial x_l}$ are identical with the second derivatives $\frac{\partial^2 \bar{x}_i}{\partial x_j \partial x_l}$, and give the terms of the second degree (by Maclaurin's theorem) on the right-hand side of the equations (29'). By a suitable choice of the numerical values of these second derivatives at P, we can make all the Christoffel's symbols for the variables $\bar{x}$ vanish, so satisfying the equations (30'), as we shall now show.

In fact, writing out both sides of equation (33) in full by means of (31), and considering the x's, in virtue of (29'), as independent variables (with their second differentials zero) and the $\bar{x}$'s as functions of them, we can write (33) in the form

$$d^2\bar{x}_i + \sum_{h,k=1}^{n} \overline{\{hk,i\}} d\bar{x}_h d\bar{x}_k = \sum_{j,l=1}^{n} \{jl,i\} dx_j dx_l .$$

Equating the coefficients of $dx_j dx_l$ on both sides and remembering (32) we get

$$\left(\frac{\partial^2 \bar{x}_i}{\partial x_j \partial x_l}\right)_P + \overline{\{jl,i\}}_P = \{jl,i\}_P ; \qquad (34)$$

from which it appears that we need only take

$$\frac{\partial^2 \bar{x}_i}{\partial x_j \partial x_l} = \{jl,i\}_P \quad (j, l, i = 1, 2, \ldots, n)$$

at P in order to have

$$\overline{\{jl,i\}} = 0$$

for every possible set of values of j, l, i. (QED)

It is not inapposite to give a geometrical interpretation of the conditions imposed on the co-ordinates $\bar{x}$ in order that (30') may be satisfied, or, in other words, in order that they may be geodesic at P. These conditions may be put in the following form:

(***a***) The n co-ordinate hypersurfaces passing through P must behave as geodesic hypersurfaces with respect to points infinitely near P (or, in particular, must be geodesic everywhere).

(***b***) If through a point P', infinitely near to P and on one of the n co-ordinate lines through P - say that along which x_i alone varies - we construct the direction parallel to another of the co-ordinate lines, this parallel must belong to the co-ordinate surface $x_i =$ constant which passes through P'.

(***c***) When the co-ordinate hypersurfaces are fixed in accordance with the foregoing conditions (which, as is geometrically obvious, is always possible), the numbering of these surfaces (i.e., the way in which they are associated with the values of the parameters $\bar{x}_1$, $\bar{x}_2$, …, $\bar{x}_n$) must be carried out so as to satisfy certain numerical conditions which we shall subsequently specify, and which, as we shall see, can always be satisfied.

That (***a***) and (***b***) are consequences of (30') follows immediately from the equations of parallelism and of geodesies. Reciprocally, we shall show that a system of co-ordinates which satisfies the conditions (***a***), (***b***), (***c***) is geodesic at *P*.

We shall begin by expressing the condition (*a*) analytically. Take a direction with parameters $d\overline{x}_k$, drawn from *P* and lying in the hypersurface $\overline{x}_i = \text{constant}$ (so that $d\overline{x}_i = 0$). We have to express the fact that the geodesic in this direction behaves at *P* as if it lay on this hypersurface, i.e., that $d^2 x_i$ vanishes along this geodesic. It follows that $d\overline{x}_i = d^2\overline{x}_i = 0$, and therefore, from the equation of geodesies,

$$\sum_{j,l=1}^{n} \{\overline{jl,i}\}_P \, d\overline{x}_j d\overline{x}_l = 0 .$$

Of the terms in this sum, those in which either *j*, or *l*, or both, are equal to *i* vanish, since $d\overline{x}_i = 0$; the other *dx*'s being arbitrary, the necessary condition for the vanishing of the other terms is that

$$\{\overline{jl,i}\}_P = 0 \quad (j,l \neq i).$$

We thus see the analytical meaning of condition (***a***).

Next consider (***b***). We shall take *P*' on the line *i*, so that, if *dx* represents the increments of the co-ordinates from *P* to *P*', we shall have $dx_l = 0$ for every value of *l* other than *i*. Let $\boldsymbol{\lambda}$ denote the direction of the co-ordinate line *j* at *P*, so that $\lambda^k = 0$ for every *k* other than *j*, and let $\boldsymbol{\lambda}$ undergo a parallel displacement from *P* to *P*'. Applying the usual formula and remembering that dx_i and λ^j are the only components which are not zero, we get

$$d\lambda^i = -\{\overline{ji,i}\}_P \lambda^j dx_i .$$

In order that the direction $\lambda' = \lambda + d\lambda$ may lie on the hypersurface $x_i = \text{constant}$, we must have $\lambda'^i = 0$, or (since, as we noted, $\lambda' = 0$ if $i \neq j$) $d\lambda^i = 0$ if $i \neq j$, so that we must have

$$\{\overline{ji,i}\}_P = 0 \quad (i \neq j).$$

This is the analytical expression of condition (***b***). We must now use the third condition in order to show that the symbols with three equal indices vanish; we shall thus have exhausted all the types of Christoffel's symbols.

Suppose that the co-ordinates *x* satisfy the foregoing conditions. Apply a transformation which leaves the co-ordinate surfaces unchanged; this can be done by putting $x_i = f_i(\overline{x}_i)$ (i.e., every *x* is a function of a single $\overline{x}$), or, which is the same thing,

$$dx_i = X_i(\overline{x}_i) d\overline{x}_i ,$$

where X_i denotes the derivative of *f* with respect to its argument.

We shall now calculate the explicit expression of the symbols which we intend shall vanish. We have

$$[\overline{ii,i}] = \sum_{j=1}^{n} \overline{a}_{ij} \{\overline{ii,j}\} ,$$

or, remembering that all the symbols are already zero except those with three equal indices,

$$[\overline{ii,i}] = \overline{a}_{ii} \{\overline{ii,i}\} .$$

Substituting on the left-hand side the expression which defines

the symbol of the first kind, we get

$$\frac{1}{2}\frac{\partial \bar{a}_{ii}}{\partial \bar{x}_i} = \bar{a}_{ii}\{\overline{ii,i}\} .$$

Hence the condition

$$\{\overline{ii,i}\} = 0 \quad (i = 1, 2, \ldots, n)$$

is equivalent to

$$\left(\frac{\partial \bar{a}_{ii}}{\partial \bar{x}_i}\right)_P = 0 \quad (i = 1, 2, \ldots, n)$$

Now from the law of covariance we have

$$\bar{a}_{ik} = \sum_{j,h=1}^{n} \frac{\partial x_j}{\partial \bar{x}_i}\frac{\partial x_h}{\partial \bar{x}_k} a_{jh} = a_{ik} X_i X_k ,$$

and therefore

$$\frac{\partial \bar{a}_{ii}}{\partial \bar{x}_i} = \frac{\partial a_{ii}}{\partial x_i} X_i^3 + 2a_{ii} X_i X'_i .$$

In order that the required condition may be satisfied, the functions X therefore need only satisfy, at P, the n numerical conditions

$$\frac{\partial \bar{a}_{ii}}{\partial x_i} X_i^3 + 2a_{ii} X_i X'_i = 0 ;$$

otherwise they may be completely arbitrary.

We thus see how to determine a system of co-ordinates $\bar{x}$ which shall be locally geodesic at P.

12. Severi's theorem.

The possibility of choosing co-ordinates which are locally Cartesian at a given point enables us to simplify the proof of some geometrical properties which hold in the neighbourhood of a point. As an example we shall prove, without any calculation, a remarkable theorem due to Professor Severi.

In a given V_n consider two infinitely near points, P and P_1, and a direction $\boldsymbol{u}$ drawn from P. This direction, and the direction PP_1 determine a section of V_n, and therefore a geodesic surface V_2 which passes through P and P_1 and contains $\boldsymbol{u}$.

We can now give $\boldsymbol{u}$ a parallel displacement, from P to P_1, in two ways:

(1) by considering $\boldsymbol{u}$ as a direction in V_n, and therefore using the metric of this variety; this will give a direction $\boldsymbol{u}_1$, which we shall call the ***ambiental parallel***;

(2) by considering $\boldsymbol{u}$ as a surface direction, belonging to the geodesic surface V_2 just defined, and using the metric of V_2; this will give a direction $\boldsymbol{u}_1^*$.

Severi's theorem is that $\boldsymbol{u}_1$ and $\boldsymbol{u}_1^*$ are identical.

We shall examine first the case in which V_n is Euclidean. In this case the geodesies are straight lines (since, with a system of Cartesian co-ordinates y, Christoffel's symbols are zero and the equations of the geodesies become $d^2 y_i = 0 \quad (i = 1, 2, \ldots, n)$) and the geodesic surfaces are planes; Severi's theorem becomes an immediate consequence of the ordinary theory of parallelism in Euclidean spaces.

Next, if V_n is not Euclidean, we note that in the definitions of the ambiental parallel $\boldsymbol{u}_1$; the geodesic surface V_2, and the parallel $\boldsymbol{u}_1^*$ relative to V_2, the only metrical elements used are Christoffel's symbols for the V_n; since all these can be made to vanish by a suitable choice of co-ordinates, the two methods of displacement are applied exactly as if V_n were Euclidean, and therefore lead to the same result.

(End of Chapter VI)