Ramani Mani, Lectures on Aeroacoustics,
California Institute of Technology, 28 March- 25 May, 1980.

April 4.

The fundamental equations for inviscid, non-heat conducting flow (no viscosity)

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = \dot{M} \\ \dfrac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\dfrac{1}{\rho_0}\nabla p + \dfrac{\vec{F}}{\rho_0} \\ T\left(\dfrac{\partial s}{\partial t} + \vec{u} \cdot \nabla s\right) = \dfrac{q}{\rho_0} \end{cases}$$

where $\dot{M}$ is [mass/volume/time], $\vec{F}$ is [external force/volume], q is [heat/volume/time] and s is [entropy/mass].

If we assume

$$\rho = \rho_0 + \rho'$$
$$p = p_0 + p'$$
$$\vec{u} = \vec{u}'$$

and also assume $\dot{M}, \vec{F}, q$ are in first order smallness.

Then we have

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{\rho'}{\rho_0}\right) + (\nabla \cdot \vec{u}') = \dfrac{\dot{M}}{\rho_0} \\ \dfrac{\partial \vec{u}'}{\partial t} = -\dfrac{1}{\rho_0}\nabla p' + \dfrac{\vec{F}}{\rho_0} \\ \dfrac{\partial s'}{\partial t} = \dfrac{q}{\rho_0 T_0} \end{cases}$$

Since

$$Tds = de + pd\left(\frac{1}{\rho}\right)$$

$$= c_v dT - \frac{p}{\rho^2} d\rho$$

Then,

$$
\begin{aligned}
ds &= c_v \frac{dT}{T} - \frac{p}{\rho T}\frac{d\rho}{\rho} \\
&= c_v\left(\frac{dp}{p} - \frac{d\rho}{\rho}\right) - R\frac{d\rho}{\rho} \\
&= c_v \frac{dp}{p} - c_p \frac{d\rho}{\rho}
\end{aligned}
$$

where we used $R = c_p - c_v$.

Therefore the energy equation becomes

$$
\frac{\partial}{\partial t}\left[\frac{p'}{\gamma p_0} - \frac{\rho'}{\rho_0}\right] = \frac{q}{\rho_0 c_p T_0}.
$$

So if there is no heat transfer $q = 0$, we have

$$
p' = \frac{\gamma p_0}{\rho_0}\rho'
$$

With the definition of speed of sound $c_0^2 = \dfrac{\gamma p_0}{\rho_0}$, we have

$$
p' = c_0^2 \rho'
$$

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The fundamental equations are again written as

$$
\begin{cases}
\dfrac{1}{\rho_0}\dfrac{\partial \rho'}{\partial t} + (\nabla \cdot \vec{u}') = \dfrac{\dot{M}}{\rho_0} \\
\dfrac{\partial \vec{u}'}{\partial t} = -\dfrac{1}{\rho_0}\nabla p' + \dfrac{\vec{F}}{\rho_0} \\
\dfrac{\partial}{\partial t}\left[\dfrac{p'}{\gamma p_0} - \dfrac{\rho'}{\rho_0}\right] = \dfrac{q}{\rho_0 c_p T_0}
\end{cases}.
$$

We calculate as follows.

$\dfrac{\partial}{\partial t}(1) + \dfrac{\partial}{\partial t}(3) - \nabla(2)$:

$$
\frac{\partial^2}{\partial t^2}\left(\frac{p'}{\gamma p_0}\right) = \frac{1}{\rho_0}\frac{\partial \dot{M}}{\partial t} + \frac{1}{\rho_0}\nabla^2 p' - \frac{1}{\rho_0}\nabla \cdot \vec{F} + \frac{1}{\rho_0 c_p T_0}\frac{\partial q}{\partial t}
$$

Putting

$$c_0^2 = \frac{\gamma p_0}{\rho_0}$$

We have

$$\frac{\partial^2}{\partial t^2}\left(\frac{p'}{\gamma p_0}\right) = \frac{1}{\rho_0}\frac{\partial \dot{M}}{\partial t} + \frac{c_0^2}{\gamma p_0}\nabla^2 p' - \frac{1}{\rho_0}\nabla\cdot\vec{F} + \frac{1}{\rho_0 c_p T_0}\frac{\partial q}{\partial t}$$

Rearrangement gives

$$\frac{\partial^2}{\partial t^2}\left(\frac{p'}{\gamma p_0}\right) - c_0^2\nabla^2\left(\frac{p'}{\gamma p_0}\right) = \frac{1}{\rho_0}\frac{\partial \dot{M}}{\partial t} - \frac{1}{\rho_0}\nabla\cdot\vec{F} + \frac{1}{\rho_0 c_p T_0}\frac{\partial q}{\partial t}$$

If we assume time dependence as $e^{-j\omega t}$, then $\partial^2/\partial t^2$ will be replaced by $-\omega^2$.
If we put $q = 0$ and $F = 0$, we have

$$-\omega^2\left(\frac{p'}{\gamma p_0}\right) - c_0^2\nabla^2\left(\frac{p'}{\gamma p_0}\right) = \frac{1}{\rho_0}\frac{\partial \dot{M}}{\partial t}.$$

Rearrangement gives

$$\frac{\omega^2}{c_0^2}\left(\frac{p'}{\gamma p_0}\right) + \nabla^2\left(\frac{p'}{\gamma p_0}\right) = -\frac{1}{c_0^2\rho_0}\frac{\partial \dot{M}}{\partial t}.$$

or

$$\frac{\omega^2}{c_0^2}p' + \nabla^2 p' = -\frac{\gamma p_0}{c_0^2\rho_0}\frac{\partial \dot{M}}{\partial t}$$

Finally we get

$$\nabla^2 p' + \frac{\omega^2}{c_0^2}p' = -\frac{\partial \dot{M}}{\partial t}.$$

If we put $\dot{M} = M_0\delta(\vec{x})$, we have

$$\nabla^2 p' + \frac{\omega^2}{c_0^2}p' = j\omega\dot{M}_0\delta(\vec{x}).$$

Putting $k = \dfrac{\omega}{c_0}$ and $r = \sqrt{x^2 + y^2 + z^2}$, we have

$$p' = \frac{-j\omega M_0 e^{jkr}}{4\pi r}$$

This is the **monopole solution**.

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We consider **unbounded free space**.

We assume the case of $\dot{M} = q = 0$ and $\vec{F} = \vec{F}_0 \delta(\vec{x})$.

Introducing a vector $\vec{A}$ which has the property of $p = \nabla \cdot \vec{A}$ and substituting into the equation, we have

$$\nabla^2 p + k^2 p = \nabla \cdot \vec{F}$$
$$\nabla^2 (\nabla \cdot A) + k^2 \nabla \cdot A = \nabla \cdot \vec{F}$$
$$\nabla^2 A + k^2 A = \vec{F}$$
$$A = \frac{-\vec{F} e^{jkr}}{4\pi r}$$

So

$$p = -\nabla \cdot \left[\frac{\vec{F}_0 e^{jkr}}{4\pi r} \right]$$
$$= \frac{(\vec{F}_0 \cdot \vec{r})}{4\pi r^3} e^{jkr} - \frac{jk(\vec{F}_0 \cdot \vec{r})}{4\pi r^2} e^{jkr}$$

The **first term is near field dominant** and the **second term is far field dominant**. The ratio (second term)/(first term) is $kr = \frac{2\pi r}{\lambda}$. So the **far field approximation** can be given by putting $kr >> 1$ and high frequency dominates in this region.

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April 13

Basic equations for **moving fluid**:

$$\tilde{\rho} = \rho + \rho_0$$
$$\tilde{p} = p + p_0$$
$$\tilde{\vec{u}} = \vec{u} + \vec{u}_0$$
$$\tilde{s} = s + s_0$$

Continuity, momentum and energy equations are written as follows:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \vec{u}_0 \cdot \nabla \rho + \rho_0 \left(\nabla \cdot \vec{u}\right) = 0 \\ \dfrac{\partial \vec{u}}{\partial t} + \left(\vec{u}_0 \cdot \nabla\right)\vec{u} = -\dfrac{1}{\rho_0} \nabla p \\ \dfrac{\partial s}{\partial t} + \vec{u}_0 \cdot \nabla s = 0 \end{cases}$$

Making use of the relations

$$ds = c_v \frac{dp}{p} - c_p \frac{d\rho}{\rho}$$

$$\frac{1}{c_p} ds = \frac{1}{\gamma} \frac{dp}{p} - \frac{d\rho}{\rho}$$

$$\frac{s}{c_p} = \frac{p}{\gamma p_0} - \frac{\rho}{\rho_0}$$

$$p = \rho R T$$

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

$$\frac{T}{T_0} = \frac{p}{p_0} - \frac{\rho}{\rho_0}$$

And define new variables as $P = \dfrac{p}{\gamma p_0}$, $D = \dfrac{\rho}{\rho_0}$, $S = \dfrac{s}{c_p}$, $\vec{U} = \dfrac{\vec{u}}{c_0}$, $\Theta = \dfrac{T}{T_0}$ we have

$$S = P - D$$

$$\Theta = \gamma P - D$$

Then the equations are

$$\begin{cases} \dfrac{\partial \rho_0 D}{\partial t} + \vec{u}_0 \cdot \nabla(\rho_0 D) + \rho_0 \left(\nabla \cdot c_0 \vec{U}\right) = 0 \\ \dfrac{\partial c_0 \vec{U}}{\partial t} + \left(\vec{u}_0 \cdot \nabla\right) c_0 \vec{U} = -\dfrac{1}{\rho_0} \nabla(\gamma p_0 P) \\ \dfrac{\partial c_p S}{\partial t} + \vec{u}_0 \cdot \nabla c_p S = 0 \end{cases}$$

Now we look for solution of the type $\exp\{j[\alpha \vec{n} \cdot \vec{r} - \omega t]\}$. So you can replace $\dfrac{\partial}{\partial t} \rightarrow -j\omega$ and $\nabla \rightarrow -j\alpha$.

The **continuity** equation will be

$$-j\omega D + \alpha \vec{u}_0 \cdot \vec{n} D + \alpha c_0 U_n = 0$$

or

$$[\alpha\vec{u}_0\cdot\vec{n}-\omega]D+\alpha c_0 U_n=0$$

And **momentum** equation will be

$$-j\omega c_0\vec{U}+j\alpha c_0\vec{u}_0\cdot\vec{n}\vec{U}=-c_0^2 j\alpha\vec{n}P$$

or

$$[-j\omega+j\alpha\vec{u}_0\cdot\vec{n}]\vec{U}=-c_0 j\alpha\vec{n}P$$

Taking the scalar product with $\cdot\vec{n}$ and making use of relations $\vec{n}\cdot\vec{n}=1$ and $\vec{U}\cdot\vec{n}=U_n$ we have

$$\alpha c_0 P+[\alpha(\vec{u}_0\cdot\vec{n})-\omega]U_n=0$$

Similarly the **energy** equation becomes the following:

$$-j\omega c_p S+\vec{u}_0\cdot(j\alpha c_p)S=0$$

or

$$(\alpha\vec{u}_0-\omega)S=0$$

And finally we have

$$[\alpha\vec{u}_0\cdot\vec{n}-\omega](P-D)=0$$

There are three roots for this equation. They are

$$\alpha(\vec{u}_0\cdot\vec{n})-\omega=0$$

$$\alpha=\frac{\omega}{c_0+\vec{u}_0\cdot\vec{n}}$$

$$\alpha=\frac{-\omega}{c_0-\vec{u}_0\cdot\vec{n}}$$

April 16

Sound Wave

Shear Wave

Entropy Wave

Landau-Lifshitz problem

Chu and Kovaznay, JFM 1958

Sommerfield diffraction problem

Leading edge problem

Blokhintsev, D. I., Acoustics of a Moving Nonhomogeneous Medium. NACA TM 1399

(1946)

Curle N. Proc. Roy. Soc. 231A, 505 (1955).

April 18

Curle Theory

Continuity equation and the Reynolds form of the momentum equation:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x_i}(\rho v_i) = 0 \\ \dfrac{\partial}{\partial t}(\rho v_i) + \dfrac{\partial}{\partial x_j}(\rho v_i v_j + p_{ij}) = 0 \end{cases} \tag{2.1}$$

where

$$p_{ij} = p\delta_{ij} - \tau_{ij}$$

is compressible stress tensor.

By eliminating ρv_i by $\dfrac{\partial}{\partial t}(1) - \dfrac{\partial}{\partial x_i}(2)$, we have

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x_i \partial x_j}\left[\rho v_i v_j + p_{ij}\right]$$

and hence

$$\frac{\partial^2 \rho}{\partial t^2} - a_0^2 \nabla^2 \rho = \frac{\partial^2}{\partial x_i \partial x_j}\left(T_{ij}\right) \tag{2.2}$$

The term

$$T_{ij} = \rho v_i v_j + p_{ij} - a_0^2 \rho \delta_{ij}$$

is called **Lighthill quadruple tensor**.

Lighthill assumed

$$T_{ij} \approx \rho_0 v_i v_j \tag{2.3}$$

which is valid in low Mach number.

The general solution is given by Stratton as follows:

$$\rho-\rho_0=\frac{1}{4\pi a_0^2}\int_V\frac{\partial^2 T_{ij}}{\partial y_i\partial y_j}\frac{d\vec{y}}{|\vec{x}-\vec{y}|}+\frac{1}{4\pi}\int_S\left\{\frac{1}{r}\frac{\partial\rho}{\partial n}+\frac{1}{r^2}\frac{\partial r}{\partial n}\rho+\frac{1}{a_0 r}\frac{\partial r}{\partial n}\frac{\partial\rho}{\partial t}\right\}dS(\vec{y})$$

(2.4)

In this equation all the quantities $\frac{\partial^2 T_{ij}}{\partial y_i\partial y_j}$, $\frac{\partial\rho}{\partial n}$, ρ, $\frac{\partial\rho}{\partial t}$ are taken at ***retarded times***

$t-\vec{r}/a_0$ where $r=|\vec{x}-\vec{y}|$, and n is the outward normal from the fluid.

We write the volume integral and the surface integral as follows:

$$VI=\int_V\frac{\partial^2 T_{ij}}{\partial y_i\partial y_j}\frac{d\vec{y}}{|\vec{x}-\vec{y}|}$$

$$SI=\int_S\left\{\frac{1}{r}\frac{\partial\rho}{\partial n}+\frac{1}{r^2}\frac{\partial r}{\partial n}\rho+\frac{1}{a_0 r}\frac{\partial r}{\partial n}\frac{\partial\rho}{\partial t}\right\}dS(\vec{y})$$

No solid boundary case.

In the absence of solid boundaries, only the volume integral remains and we have

$$\rho-\rho_0=\frac{1}{4\pi a_0^2}\int_V\frac{\partial^2 T_{ij}}{\partial y_i\partial y_j}\frac{d\vec{y}}{|\vec{x}-\vec{y}|}\qquad(2.5)$$

It is seen that the sound is radiated as if by a distribution of quadrupoles of strength per unit volume in a medium at rest. The alternative form of (2.5) is

$$\rho-\rho_0=\frac{1}{4\pi a_0^2}\frac{\partial^2}{\partial x_i\partial x_j}\int_V\frac{T_{ij}\left(\vec{y},t-\frac{|\vec{x}-\vec{y}|}{a_0}\right)}{|\vec{x}-\vec{y}|}d\vec{y}\qquad(2.6)$$

If $|\vec{x}|$ is sufficiently large, (2.6) may be simplified to

$$\rho-\rho_0=\frac{1}{4\pi a_0^2}\int_V\frac{(x_i-y_i)(x_j-y_j)}{|\vec{x}-\vec{y}|^3}\frac{1}{a_0^2}\frac{\partial^2}{\partial t^2}T_{ij}\left(\vec{y},t-\frac{|\vec{x}-\vec{y}|}{a_0}\right)d\vec{y}\qquad(2.7)$$

Furthermore, if $|\vec{x}|$ is large compared with even the dimensions of the flow then

$|\vec{x}| >> |\vec{y}|$ and (2.7) reduces to

$$\rho - \rho_0 = \frac{1}{4\pi a_0^4}\frac{x_i x_j}{x^3}\frac{\partial^2}{\partial t^2}\int_V T_{ij}\left(\vec{y}, t - \frac{|\vec{x}-\vec{y}|}{a_0}\right)d\vec{y} \tag{2.8}$$

With solid boundary case.

When there are solid boundaries, the solution differs from Lighthill's solution.

(i) The surface integral must be estimated. (ii) The volume integral must take account for inpact of sound waves from the quadrupoles. In other words, it must be noted that the presence of fixed solid boundaries invalidates the idea of regarding the quadrupole distribution as the limiting case of four source distributions. However, this physical idea is exactly equivalent to the mathematical process of twice applying the divergence theorem, and carrying out this process is still permissible.

We have formulas:

$$[f(\vec{y},t)] = f(\vec{y}, t - r/a_0)$$
$$r = |\vec{x}-\vec{y}|$$

$$\left[\frac{\partial f}{r\partial t}\right] = \frac{\partial}{\partial t}\left[\frac{f}{r}\right] \tag{A}$$

$$\frac{\partial}{\partial y_i}\left[\frac{f}{r}\right] = \left[\frac{1}{r}\frac{\partial f}{\partial y_i}\right] - \frac{\partial}{\partial x_i}\left[\frac{f}{r}\right] \qquad \text{(B)??}$$

$$\begin{aligned}\frac{\partial}{\partial x_i}\left[\frac{f}{r}\right] &= -\left(\frac{x_i - y_i}{r}\right)\left\{\frac{1}{r^2}[f] + \frac{1}{a_0 r}\left[\frac{\partial f}{\partial t}\right]\right\} \\ &= \left(\frac{\partial r}{\partial y_i}\right)\left\{\frac{1}{r^2}[f] + \frac{1}{a_0 r}\left[\frac{\partial f}{\partial t}\right]\right\}\end{aligned} \tag{C}$$

$$\frac{\partial r}{\partial y_i} = -\frac{\partial r}{\partial x_i} = \frac{y_i - x_i}{r} \tag{D}$$

So if we put $f = \dfrac{\partial T_{ij}}{\partial y_j}$ in the formula (B) we have

$$\frac{\partial}{\partial y_i}\left[\frac{1}{r}\frac{\partial T_{ij}}{\partial y_j}\right]=\left[\frac{1}{r}\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}\right]-\frac{\partial}{\partial x_i}\left[\frac{1}{r}\frac{\partial T_{ij}}{\partial y_j}\right]$$

or

$$\left[\frac{1}{r}\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}\right]=\frac{\partial}{\partial y_i}\left[\frac{1}{r}\frac{\partial T_{ij}}{\partial y_j}\right]+\frac{\partial}{\partial x_i}\left[\frac{1}{r}\frac{\partial T_{ij}}{\partial y_j}\right]$$

And applying the divergence theorem, we have

$$\int_V \left[\frac{1}{r}\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}\right] d\vec{y}=\int_S \frac{n_i}{r}\left[\frac{\partial T_{ij}}{\partial y_j}\right] dS(\vec{y})+\frac{\partial}{\partial x_i}\int_V \left[\frac{1}{r}\frac{\partial T_{ij}}{\partial y_j}\right] d\vec{y} \tag{2.9}$$

Similarly putting $f=T_{ij}$ in the formula (B), we have

$$\frac{\partial}{\partial y_i}\left[\frac{1}{r}T_{ij}\right]=\left[\frac{1}{r}\frac{\partial T_{ij}}{\partial y_i}\right]-\frac{\partial}{\partial x_i}\left[\frac{1}{r}T_{ij}\right]$$

or

$$\left[\frac{1}{r}\frac{\partial T_{ij}}{\partial y_i}\right]=\frac{\partial}{\partial y_i}\left[\frac{1}{r}T_{ij}\right]+\frac{\partial}{\partial x_i}\left[\frac{1}{r}T_{ij}\right]$$

And applying the divergence theorem again, we have

$$\int_V \left[\frac{1}{r}\frac{\partial T_{ij}}{\partial y_i}\right] d\vec{y}=\int_S \frac{n_i}{r}\left[T_{ij}\right] dS(\vec{y})+\frac{\partial}{\partial x_i}\int_V \left[\frac{1}{r}T_{ij}\right] d\vec{y}$$

Interchanging the subscripts i with j, we have

$$\int_V \left[\frac{1}{r}\frac{\partial T_{ij}}{\partial y_j}\right] d\vec{y}=\int_S \frac{n_j}{r}\left[T_{ij}\right] dS(\vec{y})+\frac{\partial}{\partial x_j}\int_V \left[\frac{1}{r}T_{ij}\right] d\vec{y}$$

And applying the derivative $\dfrac{\partial}{\partial x_i}$, we have

$$\frac{\partial}{\partial x_i}\int_V \left[\frac{1}{r}\frac{\partial T_{ij}}{\partial y_j}\right] d\vec{y}=\frac{\partial}{\partial x_i}\int_S \frac{n_j}{r}\left[T_{ij}\right] dS(\vec{y})+\frac{\partial^2}{\partial x_i \partial x_j}\int_V \left[\frac{1}{r}T_{ij}\right] d\vec{y}$$

Therefore the volume integral becomes as follows:

$$VI = \int_V \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \frac{d\vec{y}}{|\vec{x} - \vec{y}|}$$

$$= \int_S \frac{n_i}{r} \left[\frac{\partial T_{ij}}{\partial y_j} \right] dS(\vec{y}) + \frac{\partial}{\partial x_i} \int_S \frac{n_j}{r} \left[T_{ij} \right] dS(\vec{y}) + \frac{\partial^2}{\partial x_i \partial x_j} \int_V \left[\frac{1}{r} T_{ij} \right] d\vec{y}$$

$$= \int_S \frac{n_i}{r} \frac{\partial T_{ij}}{\partial y_j} dS(\vec{y}) + \frac{\partial}{\partial x_i} \int_S \frac{n_j}{r} T_{ij} \left(\vec{y}, t - \frac{r}{a_0} \right) dS(\vec{y}) + \frac{\partial^2}{\partial x_i \partial x_j} \int_V \frac{T_{ij}\left(\vec{y}, t - \frac{r}{a_0} \right)}{r} d\vec{y}$$

On the other hand the surface integral may be calculated as follows:

$$SI = \int_S \left\{ \frac{1}{r} \frac{\partial \rho}{\partial n} + \frac{1}{r^2} \frac{\partial r}{\partial n} \rho + \frac{1}{a_0 r} \frac{\partial r}{\partial n} \frac{\partial \rho}{\partial t} \right\} dS(\vec{y})$$

$$= \int_S n_i \left\{ \frac{1}{r} \frac{\partial \rho}{\partial y_i} + \frac{1}{r^2} \frac{\partial r}{\partial y_i} \rho + \frac{1}{a_0 r} \frac{\partial r}{\partial y_i} \frac{\partial \rho}{\partial t} \right\} dS(\vec{y})$$

$$= \int_S n_i \frac{1}{r} \frac{\partial}{\partial y_j} \left(\rho \delta_{ij} \right) dS(\vec{y}) - \int_S n_i \left\{ \frac{1}{r^2} \frac{\partial r}{\partial x_i} \rho + \frac{1}{a_0 r} \frac{\partial r}{\partial x_i} \frac{\partial \rho}{\partial t} \right\} dS(\vec{y})$$

And applying the formula

$$\frac{\partial}{\partial x_i} \left\{ \frac{1}{r} f\left(t - \frac{r}{a_0} \right) \right\} = -\left\{ \frac{1}{r^2} f + \frac{1}{a_0 r} f' \right\} \tag{2.14}$$

in the last integral, we have

$$SI = \int_S n_i \frac{1}{r} \frac{\partial}{\partial y_j} \left(\rho \delta_{ij} \right) dS(\vec{y}) + \int_S n_i \frac{\partial}{\partial x_i} \left(\frac{1}{r} \rho \delta_{ij} \right) dS(\vec{y}) \tag{2.13}$$

Therefore finally we have

$$\rho - \rho_0 = \frac{1}{4\pi a_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \int_V \frac{T_{ij}\left(\vec{y}, t - \frac{r}{a_0} \right)}{r} d\vec{y}$$

$$+ \frac{1}{4\pi a_0^2} \int_S n_i \frac{1}{r} \frac{\partial}{\partial y_j} \left(T_{ij} + a_0^2 \rho \delta_{ij} \right) dS(\vec{y})$$

$$+ \frac{1}{4\pi a_0^2} \frac{\partial}{\partial x_i} \int_S n_i \frac{1}{r} \left(T_{ij} + a_0^2 \rho \delta_{ij} \right) dS(\vec{y})$$

And substituting for

$$T_{ij} = \rho v_i v_j + p_{ij} - a_0^2 \rho \delta_{ij}$$

this becomes

$$\rho - \rho_0 = \frac{1}{4\pi a_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \int_V \frac{T_{ij}\left(\vec{y}, t - \frac{r}{a_0}\right)}{r} d\vec{y}$$
$$+ \frac{1}{4\pi a_0^2} \int_S n_i \frac{1}{r} \frac{\partial}{\partial y_j} \left(\rho v_i v_j + p_{ij}\right) dS(\vec{y}) \tag{2.15}$$
$$+ \frac{1}{4\pi a_0^2} \frac{\partial}{\partial x_i} \int_S n_i \frac{1}{r} \left(\rho v_i v_j + p_{ij}\right) dS(\vec{y})$$

and using the momentum equation

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}\left(\rho v_i v_j + p_{ij}\right) = 0 \tag{2.16a}$$

and if there is zero normal velocity at the solid boundaries, that is, if each surface is fixed or vibrating in its own plane, then

$$n_i v_i = 0 \tag{2.17}$$

Hence (2.15) reduces simply to

$$\rho - \rho_0 = \frac{1}{4\pi a_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \int_V \frac{T_{ij}\left(\vec{y}, t - \frac{r}{a_0}\right)}{r} d\vec{y} + \frac{1}{4\pi a_0^2} \frac{\partial}{\partial x_i} \int_S \frac{1}{r} n_j p_{ij} dS(\vec{y})$$
$$= \frac{1}{4\pi a_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \int_V \frac{T_{ij}\left(\vec{y}, t - \frac{r}{a_0}\right)}{r} d\vec{y} - \frac{1}{4\pi a_0^2} \frac{\partial}{\partial x_i} \int_S \frac{P_i\left(\vec{y}, t - \frac{r}{a_0}\right)}{r} dS(\vec{y})$$

(2.18)

where

$$P_i = -n_j p_{ij} \tag{2.19}$$

Therefore one can look upon the sound field as the sum of that generated by a volume distribution of quadrupoles and by a surface distribution of dipoles.

Dimensional analysis

(i) $|x| >> \lambda$

$$\frac{1}{4\pi a_0^2}\frac{\partial}{\partial x_i}\int_S \frac{P_i\left(\vec{y},t-\dfrac{r}{a_0}\right)}{r}dS(\vec{y}) \rightarrow \frac{1}{4\pi a_0^3}\frac{\partial}{\partial t}\int_S \frac{x_i-y_i}{|\vec{x}-\vec{y}|^2}P_i\left(\vec{y},t-\frac{r}{a_0}\right)dS(\vec{y}) \tag{3.1}$$

(ii) $|x| >> L$

$$\frac{1}{4\pi a_0^3}\frac{\partial}{\partial t}\int_S \frac{x_i-y_i}{|\vec{x}-\vec{y}|^2}P_i\left(\vec{y},t-\frac{r}{a_0}\right)dS(\vec{y}) \rightarrow \frac{1}{4\pi a_0^3}\frac{x_i}{x^2}\frac{\partial}{\partial t}\int_S P_i\left(\vec{y},t-\frac{r}{a_0}\right)dS(\vec{y}) \tag{3.2}$$

(iii) $L << \dfrac{a_0}{\omega} \approx \dfrac{a_0 L}{U_0}$

$$\frac{1}{4\pi a_0^3}\frac{x_i}{x^2}\frac{\partial}{\partial t}\int_S P_i\left(\vec{y},t-\frac{r}{a_0}\right)dS(\vec{y}) \rightarrow \frac{1}{4\pi a_0^3}\frac{x_i}{x^2}\frac{\partial}{\partial t}\int_S P_i(\vec{y},t)dS(\vec{y}) \tag{3.3}$$

Hence for flows at low Mach number the 'surface noise' generated is that generated by a single dipole representing the resultant fluctuating force exerted on the fluid by the solid boundaries.

Intensity

For quadrupoles we have

$$I_Q \approx \rho_0 U_0^8 a_0^{-5} L^2 x^{-2} f(R) \tag{3.4}$$

And for dipoles since

$$\begin{aligned}\rho-\rho_0 &\approx a_0^{-3}\frac{1}{x}\frac{U_0}{L}\rho_0 U_0^2 L^2 g(R)\\ &\approx \rho_0 U_0^3 a_0^{-3} L\, x^{-1} g(R)\end{aligned} \tag{3.5}$$

we have

$$I_D \approx \rho_0 U_0^6 a_0^{-3} L^2 x^{-2} g(R) \tag{3.6}$$

Therefore

$$\frac{I_Q}{I_D} \approx \left(\frac{U_0}{a_0}\right)^2 F(R) \tag{3.7}$$

It follows that at sufficiently low Mach numbers the contribution to the sound field from the dipoles should be greater than that for the quadrupoles.
From (3.6) it follows that the total acoustic power output is roughly proportional to

$$\rho_0 U_0^6 a_0^{-3} L^2 \tag{3.8}$$

Acoustic efficiency

The total rate of supply of energy will be proportional to $(\rho_0 U_0^2)(U_0 L^2)$, and the acoustic efficiency, described as the ration of acoustic power output to energy supplied, will then vary roughly as

$$\eta \approx \frac{\rho_0 U_0^6 a_0^{-3} L^2}{(\rho_0 U_0^2)(U_0 L^2)} \approx \left(\frac{U_0}{a_0}\right)^3 \tag{3.9}$$

This is to be compared with the acoustic efficiency for the quadrupoles

$$\eta_Q \approx M^5$$

which was found by Lighthill.