PART I1

A RAPID COURSE IN RIEMANNIAN GEOMETRY

§8. Covariant Differentiation

The object of Part II will be to give a rapid outline of some basic concepts of Riemannian geometry which will be needed later. For more information the reader should consult Nomizu, "*Lie groups and differential geometry*," The Mathematical Society of Japan, 1956; Laugwitz, "*Differentialgeometrie*," Teubner 1960; or Helgason, "*Differential geometry and symmetric spaces*," Academic Press, 1962.

Let M be a smooth manifold.

DEFINITION. An ***affine connection*** at a point $p \in M$ is a function which assigns to each tangent vector $X_p \in TM_p$ and to each vector field Y a new tangent vector

$$X_p \succ Y \in TM_p$$

called the ***convariant derivative***[1] of Y in the direction X_p. This is required to be bilinear as a function of X_p and Y. Furthermore, if

$$f : M \to R$$

is a real valued function, and if fY denotes the vector field

$$(fY)_q = f(q)Y_q$$

then $\succ$ is required to satisfy the identity

$$X_p \succ (fY) = (X_p f)Y_p + f(p)X_p \succ Y \, .$$

(As usual, $X_p f$ denotes the directional derivative of f in the direction X_p.)

A ***global affine connection*** (or briefly a ***connection***) on M is a function which assigns to each $p \in M$ an affine connection $\succ_p$ at p, satisfying the following smoothness condition.

(1) If X and Y are smooth vector fields on M then the vector field $X \succ Y$, defined by the identity

[1] Note that our $X \succ Y$ coincides with Nomizu's $\nabla_X Y$. The notation is intended to suggest that the differential operator X acts on the vector field Y.

$$(X \succ Y)_p = X_p \succ_p Y ,$$

must also be smooth.

Note that:

(2) $X \succ Y$ is bilinear as a function of X and Y.

(3) $(fX) \succ Y = f(X \succ Y)$.

(4) $(X \succ (fY) = (Xf)Y + f(X \succ Y)$.

Conditions (1), (2), (3), (4) can be taken as the definition of a connection.

In terms of local coordinates $u^1,\ldots,u^n$ defined on a coordinate neighborhood $U \subset M$, the connection $\succ$ is determined by n^3 smooth real valued functions $\Gamma^k{}_{ij}$ on U, as follows. Let ∂_k denote the vector field $\frac{\partial}{\partial u^k}$ on U. Then any vector field X on U can be expressed uniquely as

$$X = \sum_{k=1}^{n} x^k \partial_k$$

where the x^k are real valued functions on U. In particular the vector field $\partial_i \succ \partial_j$ can be expressed as

(5) $$\partial_i \succ \partial_j = \sum_k \Gamma^k{}_{ij} \partial_k .$$

These functions $\Gamma^k{}_{ij}$ determine the connection completely on U. In fact given vector fields $X = \sum x^i \partial_i$ and $Y = \sum y^j \partial_j$ one can expand $X \succ Y$ by the rules (2), (3), (4); yielding the formula

(6) $$X \succ Y = \sum_k \left(\sum_i x^i y^k{}_{,i} \right) \partial_k ,$$

$$\nabla_X Y = \sum_k \left(\sum_i x^i y^k{}_{,i} \right) \frac{\partial}{\partial x^k}$$

where the symbol $y^k{}_{,i}$ stands for the real valued function

$$y^k{}_{,i} = \partial_i y^k + \sum_j \Gamma^k{}_{ij} y^j .$$

Conversely, given any smooth real valued functions $\Gamma^k{}_{ij}$ on U, one can define $X \succ Y$ by the formula (6). The result clearly satisfies the conditions (1), (2), (3), (4), (5).

Using the connection $\succ$ one can define the covariant derivative of a vector field along a curve in M. First some definitions.

A ***parametrized curve*** in M is a smooth function c from the real numbers to M. A ***vector field*** V along the curve c is a function which assigns to each $t \in R$ a tangent vector

$$V_t \in TM_{c(t)}.$$

This is required to be smooth in the following sense: For any smooth function f on M the correspondence

$$t \to V_t f$$

should define a smooth function on R.

As an example the ***velocity vector field*** $\frac{dc}{dt}$ of the curve is the vector field along c which is defined by the rule

$$\frac{dc}{dt} = c_* \frac{d}{dt}.$$

Here $\frac{d}{dt}$ denotes the standard vector field on the real numbers, and

$$c_* : TR_t \to TM_{c(t)}$$

denotes the ***homomorphism*** of tangent spaces induced by the map c. (Compare Diagram 9.)

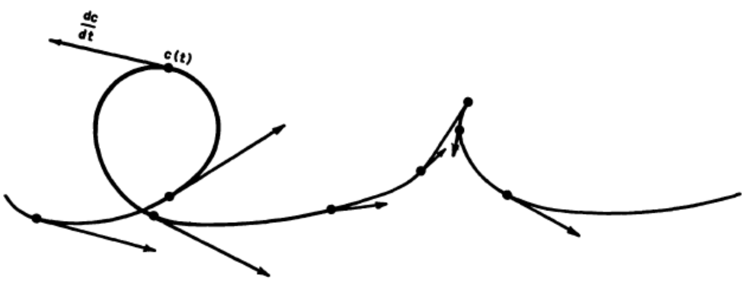


Diagram 9

Now suppose that M is provided with an affine connection. Then any vector field V along c determines a new vector field $\frac{DV}{dt}$ along c called the ***covariant derivative*** of V. The operation

$$V \to \frac{DV}{dt}$$

is characterized by the following three axioms.

a) $\frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt}$.

b) If f is a smooth real valued function on R then

$$\frac{D(fV)}{dt} = \frac{df}{dt}V + f\frac{DV}{dt}.$$

c) If V is induced by a vector field Y on M, that is if $V_t = Y_{c(t)}$ for

each t, then $\frac{DV}{dt}$ is equal to $\frac{dc}{dt} \succ Y$ (= the covariant derivative of Y in the direction of the velocity vector of c).

LEMMA 8.1. *There is one and only one operation* $V \to \frac{DV}{dt}$ *which satisfies these three conditions.*

PROOF: Choose a local coordinate system for M, and let $u^1(t),\ldots,u^n(t)$ denote the coordinates of the point $c(t)$. The vector field V can be expressed uniquely in the form

$$V = \sum v^j \partial_j$$

where $v^1,\ldots,v^n$ are real valued functions on R (or an appropriate open subset of R), and $\partial_1,\ldots,\partial_n$ are the standard vector fields on the coordinate neighborhood. It follows from (a), (b), and (c) that

$$\begin{aligned}\frac{DV}{dt} &= \sum_j \left(\frac{dv^j}{dt}\partial_j + v^j \frac{dc}{dt} \succ \partial_j \right) \\ &= \sum_k \left(\frac{dv^k}{dt} + \sum_{i,j} \frac{du^i}{dt} \Gamma^k{}_{ij} v^j \right) \partial_k\end{aligned}.$$

Conversely, defining $\frac{DV}{dt}$ by this formula, it is not difficult to verify that conditions (a), (b), and (c) are satisfied.

A vector field V along c is said to be a ***parallel vector field*** if the covariant derivative $\frac{DV}{dt}$ is identically zero.

LEMMA 8.2. *Given a curve c and a tangent vector* V_0 *at the point c(0), there is one and only one parallel vector field V along c which extends* V_0.

PROOF. The differential equations

$$\frac{dv^k}{dt} + \sum_{i,j} \frac{du^i}{dt} \Gamma^k{}_{ij} v^j = 0$$

have solutions $v^k(t)$ which are uniquely determined by the initial values $v^k(0)$. Since these equations are linear, the solutions can be defined for all relevant values of t. (Compare Graves, "The Theory of Functions of Real Variables," p. 152.)

The vector V_t is said to be obtained from V_0 by ***parallel translation*** along c.

Now suppose that M is a Riemannian manifold. The inner product

of two vectors X_p, Y_p will be denoted by $\langle X_p, Y_p \rangle$.

DEFINITION. A connection $\succ$ on M is ***compatible*** with the Riemannian metric if parallel translation preserves inner products. In other words, for any parametrized curve c and any pair P, P' of parallel vector fields along c, the inner product $\langle P, P' \rangle$ should be constant.

LEMMA 8.3. *Suppose that the connection is compatible with the metric. Let V, W be any two vector fields along c. Then*

$$\frac{d}{dt}\langle V,W\rangle = \left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{DW}{dt} \right\rangle.$$

PROOF: Choose parallel vector fields $P_1,\ldots,P_n$ along c which are orthonormal at one point of c and hence at every point of c. Then the given fields V and W can be expressed as $\sum v^i P_i$ and $\sum w^j P_j$, respectively (where $v^i = \langle V, P_i \rangle$ is a real valued function on R). It follows that $\langle V, W \rangle = \sum v^i w^i$ and that

$$\frac{DV}{dt} + \sum \frac{dv^i}{dt} P_i\ , \quad \frac{DW}{dt} + \sum \frac{dw^j}{dt} P_j\ .$$

Therefore

$$\left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{DW}{dt} \right\rangle = \sum \left(\frac{dv^i}{dt} w^i + v^i \frac{dw^i}{dt} \right) = \frac{d}{dt}\langle V,W\rangle,$$

which completes the proof.

COROLLARY 8.4. For any vector fields Y, Y' on M and any vector $X_p \in TM_p$:

$$X_p\langle Y,Y'\rangle = \langle X_p \succ Y, Y'_p \rangle + \langle Y_p, X_p \succ X_p Y' \rangle.$$

PROOF. Choose a curve c whose velocity vector at $t = 0$ is X_p; and apply 8.3.

DEFINITION 8.5. A connection $\succ$ is called ***symmetric*** if it satisfies the identity

$$(X \succ Y) - (Y \succ X) = [X, Y].$$

(As usual, $[X, Y]$ denotes the poison bracket $[X, Y]f = X(Yf) - Y(Xf)$ of two vector fields.) Applying this identity to the case $X = \partial_i$, $Y = \partial_j$,

since $[\partial_i, \partial_j] = 0$ one obtains the relation

$$\Gamma^k{}_{ij} - \Gamma^k{}_{ji} = 0 .$$

Conversely if $\Gamma^k{}_{ij} = \Gamma^k{}_{ji}$ then using formula (6) it is not difficult to verify that the connection $\succ$ is symmetric throughout the coordinate neighborhood.

LEMMA 8.6. (***Fundamental lemma of Riemannian geometry***.) *A Riemannian manifold possesses one and only one symmetric connection which is compatible with its metric.*

(Compare Nomizu p. 76, Laugwitz p. 95.)

PROOF of uniqueness. Applying 8.4 to the vector fields $\partial_i, \partial_j, \partial_k$ and setting $\langle \partial_j, \partial_k \rangle = g_{jk}$ one obtains the identity

$$\partial_i g_{jk} = \langle \partial_i \succ \partial_j, \partial_k \rangle + \langle \partial_j, \partial_i \succ \partial_k \rangle .$$

Permuting *i*, *j*, and *k* this gives three linear equations relating the three quantities

$$\langle \partial_i \succ \partial_j, \partial_k \rangle, \ \langle \partial_j \succ \partial_k, \partial_i \rangle \ \text{ and } \ \langle \partial_k \succ \partial_i, \partial_j \rangle .$$

(There are only three such quantities since $\partial_i \succ \partial_j = \partial_j \succ \partial_i$.)

These equations can be solved uniquely; yielding the ***first Christoffel identity***

$$\langle \partial_i \succ \partial_j, \partial_k \rangle = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) .$$

The left hand side of this identity is equal to $\sum_l \Gamma^l{}_{ij} g_{lk}$. Multiplying by the inverse (g^{kl}) of the matrix (g_{lk}) this yields the ***second Christoffel identity***

$$\Gamma^l{}_{ij} = \sum_k \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) g^{kl} .$$

$$\nabla_\lambda g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - g_{\mu\sigma}\Gamma^\sigma{}_{\nu\lambda} - g_{\nu\sigma}\Gamma^\sigma{}_{\mu\lambda} = 0$$

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\frac{\partial g_{\rho\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right)$$

Thus the connection is uniquely determined by the metric.

Conversely, defining $\Gamma^l{}_{ij}$ by this formula, one can verify that the resulting connection is symmetric and compatible with the metric. This completes the proof.

An alternative characterization of symmetry will be very useful

later. Consider a "parametrized surface" in M: that is a smooth function

$$s : R^2 \to M\ .$$

By a ***vector field*** V along s is meant a function which assigns to each $(x{,}y) \in R^2$ a tangent vector

$$V_{(x,y)} \in TM_{s(x,y)}\ .$$

As examples, the two standard vector fields $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ give rise to vector fields $s_* \dfrac{\partial}{\partial x}$ and $s_* \dfrac{\partial}{\partial y}$ along s. These will be denoted briefly by $\dfrac{\partial s}{\partial x}$ and $\dfrac{\partial s}{\partial y}$; and called the "***velocity vector fields***" of s.

For any smooth vector field V along s the ***covariant derivatives*** $\dfrac{DV}{\partial x}$ and $\dfrac{DV}{\partial y}$ are new vector fields, constructed as follows. For each fixed y_0, restricting V to the curve

$$x \to s(x, y_0)$$

one obtains a vector field along this curve. Its covariant derivative with respect to x is defined to be $\left(\dfrac{DV}{\partial x}\right)_{(x,y_0)}$. This defines $\dfrac{DV}{\partial x}$ along the entire parametrized surface s.

As examples, we can form the two covariant derivatives of the two vector fields $\dfrac{\partial s}{\partial x}$ and $\dfrac{\partial s}{\partial y}$. The derivatives $\dfrac{D}{\partial x}\dfrac{\partial s}{\partial x}$ and $\dfrac{D}{\partial y}\dfrac{\partial s}{\partial y}$ are simply the ***acceleration vectors*** of suitable coordinate curves. However, the mixed derivatives $\dfrac{D}{\partial x}\dfrac{\partial s}{\partial y}$ and $\dfrac{D}{\partial y}\dfrac{\partial s}{\partial x}$ cannot be described so simply.

LEMMA 8.7. *If the connection is symmetric then* $\dfrac{D}{\partial x}\dfrac{\partial s}{\partial y} = \dfrac{D}{\partial y}\dfrac{\partial s}{\partial x}$.

PROOF. Express both sides in terms of a local coordinate system, and compute.

§9. The Curvature Tensor

The curvature tensor R of an affine connection $\succ$ measures the extent to which the second covariant derivative $\partial_i \succ (\partial_j \succ Z)$ is symmetric in i and j. Given vector fields X, Y, Z define a new vector field $R(X\ ,Y)Z$ by the identity

$$R(X,Y)Z = -X \succ (Y \succ Z) + Y \succ (X \succ Z) + [X,Y] \succ Z\ .$$

LEMMA 9.1. *The value of $R(X,Y)Z$ at a point $p \in M$ depends only on the vectors X_p, Y_p, Z_p at this point p and not on their values at nearby points. Furthermore the correspondence*

$$X_p, Y_p, Z_p \to R(X_p, Y_p)Z_p$$

from $TM_p \times TM_p \times TM_p$ to TM_p is tri-linear.

Briefly, this lemma can be expressed by saying that R is a "***tensor***."

PROOF: Clearly $R(X,Y)Z$ is a tri-linear function of X, Y, and Z. If X is replaced by a multiple fX then the three terms $-X \succ (Y \succ Z)$, $Y \succ (X \succ Z)$, $[X,Y] \succ Z$ are replaced respectively by

i) $-fX \succ (Y \succ Z)$,

ii) $(Yf) \succ (X \succ Z) + fY \succ (X \succ Z)$,

iii) $-(Yf)(X \succ Z) + f[X,Y] \succ Z$.

Adding these three terms one obtains the identity

$$R(fX,Y)Z = fR(X,Y)Z.$$

Corresponding identities for Y and Z are easily obtained by similar computations.

Now suppose that $X = \sum x^i \partial_i$, $Y = \sum y^j \partial_j$ and $Z = \sum z^k \partial_k$.

Then

$$\begin{aligned} R(X,Y)Z &= \sum R(x^i\partial_i, y^j\partial_j)(z^k\partial_k) \\ &= \sum x^i y^j z^k R(\partial_i, \partial_j)\partial_k \end{aligned}.$$

Evaluating this expression at p one obtains the formula

$$(R(X,Y)Z)_p = \sum x^i(p) y^j(p) z^k(p) \big(R(\partial_i, \partial_j)\partial_k\big)_p$$

which depends only on the values of the functions x^i, y^j, x^k at p, and not on their values at nearby points. This completes the proof.

Now consider a parametrized surface

$$s : R^2 \to M .$$

Given any vector field V along s. one can apply the two covariant differentiation operators $\frac{D}{\partial x}$ and $\frac{D}{\partial y}$ to V. In general these operators will not commute with each other.

LEMMA 9.2. $\frac{D}{\partial y}\frac{D}{\partial x}V - \frac{D}{\partial x}\frac{D}{\partial y}V = R\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right)V$.

PROOF: Express both sides in terms of a local coordinate system,

and compute, making use of the identity

$$\partial_j \succ (\partial_i \succ \partial_k) - \partial_i \succ (\partial_j \succ \partial_k) = R(\partial_i, \partial_j)\partial_k .$$

[It is interesting to ask whether one can construct a vector field P along s which is parallel, in the sense that

$$\frac{D}{\partial x} P = \frac{D}{\partial y} P = 0 ,$$

and which has a given value $P_{(0,0)}$ at the origin. In general no such vector field exists. However, if the curvature tensor happens to be zero then P can be constructed as follows. Let $P_{(x,0)}$ be a parallel vector field along the x-axis, satisfying the given initial condition. For each fixed x_0 let $P_{(x,y)}$ be a parallel vector field along the curve

$$y \to s(x_0, y) ,$$

having the right value for $y = 0$. This defines P everywhere along s. Clearly $\frac{D}{\partial y} P$ is identically zero; and $\frac{D}{\partial x} P$ is zero along the x-axis. Now the identity

$$\frac{D}{\partial y}\frac{D}{\partial x} P - \frac{D}{\partial x}\frac{D}{\partial y} P = R\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right) P = 0$$

implies that $\frac{D}{\partial y}\frac{D}{\partial x} P = 0$. In other words, the vector field $\frac{D}{\partial x} P$ is parallel along the curves

$$y \to s(x_0, y) .$$

Since $\left(\frac{D}{\partial x} P\right)_{(x_0,0)} = 0$, this implies that $\frac{D}{\partial x} P$ is identically zero; and completes the proof that P is parallel along s.]

Henceforth we will assume that M is a Riemannian manifold, provided with the unique symmetric connection which is compatible with its metric. In conclusion we will prove that the tensor R satisfies four symmetry relations.

LEMMA 9.3. *The curvature tensor of a Riemannian manifold satisfies*:

(1) $R(X,Y)Z + R(Y,X)Z = 0$.

(2) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$,

(3) $\langle R(X,Y)Z, W\rangle + \langle R(X,Y)W, Z\rangle = 0$,

(4) $\langle R(X,Y)Z,W\rangle = \langle R(Z,W)X,Y\rangle$.

PROOF: The skew-symmetry relation (1) follows immediately from the definition of *R*.

Since all three terms of (2) are tensors, it is sufficient to prove (2) when the bracket products $[X,Y]$, $[X,Z]$ and $[Y,Z]$ are all zero. Under this hypothesis we must verify the identity

$$\begin{aligned} &-X \succ (Y \succ Z) + Y \succ (X \succ Z) \\ &-Y \succ (Z \succ X) + Z \succ (Y \succ X) \\ &-Z \succ (X \succ Y) + X \succ (Z \succ Y) = 0 \end{aligned} .$$

But the symmetry of the connection implies that

$$Y \succ Z - Z \succ Y = [Y,Z] = 0 .$$

Thus the upper left term cancels the lower right term. Similarly the remaining terms cancel in pairs. This proves (2).

To prove (3) we must show that the expression $< R(X,Y)Z,W >$ is skew-symmetric in Z and *W*. This is clearly equivalent to the assertion that

$$\langle R(X,Y)Z,Z\rangle = 0$$

for all *X*,*Y*,*Z*. Again we may assume that $[X,Y]= 0$, so that $<R(X,Y)Z,Z>$ is equal to

$$\langle -X \succ (Y \succ Z) + Y \succ (X \succ Z), Z\rangle .$$

In other words we must prove that the expression

$$\langle Y \succ (X \succ Z), Z\rangle$$

is symmetric in *X* and *Y*.

Since $[X,Y] = 0$ the expression $YX < Z,Z >$ is symmetric in *X* and *Y*. Since the connection is compatible with the metric, we have

$$X\langle Z,Z\rangle = 2\langle X \succ Z, Z\rangle$$

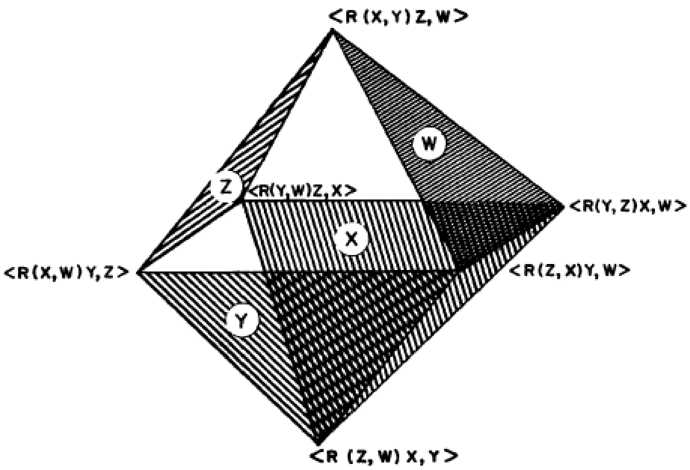
hence

$$YX\langle Z,Z\rangle = 2\langle Y \succ (X \succ Z), Z\rangle + 2\langle X \succ Z, Y \succ Z\rangle .$$

But the right hand term is clearly symmetric in *X* and *Y*. Therefore $\langle Y \succ (X \succ Z), Z\rangle$ is symmetric in *X* and *Y*; which proves property (3).

Property (4) may be proved from (1), (2), and (3) as follows.

Formula (2) asserts that the sum of the quantities at the vertices of shaded triangle *W* is zero. Similarly (making use of (1) and (3)) the sum of the vertices of each of the other shaded triangles is zero.



Adding these identities for the top two shaded triangles, and subtracting the identities for the bottom ones, this means that twice the top vertex minus twice the bottom vertex is zero. This proves (4), and completes the proof.

§10. Geodesics and Completeness

Let **M** be a connected Riemannian manifold.

DEFINITION. A parametrized path

$$\gamma : I \to M \ ,$$

where I denotes any interval of real numbers, is called a ***geodesic*** if the **acceleration vector field** $\frac{D}{dt}\frac{d\gamma}{dt}$ is identically zero.

Thus the velocity vector field $\frac{d\gamma}{dt}$ must be parallel along γ. If γ is a geodesic, then the identity

$$\frac{d}{dt}\left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle = 2\left\langle \frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle = 0$$

shows that the length $\left\| \frac{d\gamma}{dt} \right\| = \left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle^{1/2}$ of the velocity vector is constant along γ. Introducing the are-length function

$$s(t) = \int \left\| \frac{d\gamma}{dt} \right\| dt + \text{constant}$$

this statement can be rephrased as follows: The parameter t along a geodesic is a linear function of the arc-length. The parameter t is actually equal to the arc-length if and only if $\left\| \frac{d\gamma}{dt} \right\| = 1$.

In terms of a local coordinate system with coordinates $u^1, \ldots, u^n$ a curve $t \to \gamma(t) \in M$ determines n smooth functions

$u^1(t),\ldots,u^n(t)$. The equation $\frac{D}{dt}\frac{d\gamma}{dt}$ for a geodesic then takes the form

$$\frac{d^2u^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma^k{}_{ij}(u^1,\ldots,u^n)\frac{du^i}{dt}\frac{du^j}{dt} = 0 .$$

The existence of geodesics depends, therefore, on the solutions of a certain system of second order differential equations.

More generally consider any system of equations of the form

$$\frac{d^2\vec{u}}{dt^2} = \vec{F}(\vec{u}, \frac{d\vec{u}}{dt}) .$$

Here $\vec{u}$ stands for $(u^1,\ldots,u^n)$ and $\vec{F}$ stands for an n-tuple of C^∞ functions, all defined throughout some neighborhood U of a point

$$(\vec{u}_1, \vec{v}_1) \in R^{2n} .$$

EXISTENCE AND UNIQUENESS THEOREM 10.1. *There exists a neighborhood W of the point* $(\vec{u}_1, \vec{v}_1)$ *and a number* $\varepsilon > 0$ *so that, for each* $(\vec{u}_0, \vec{v}_0) \in W$ *the differential equation*

$$\frac{d^2\vec{u}}{dt^2} = \vec{F}(\vec{u}, \frac{d\vec{u}}{dt})$$

has a unique solution $t \to \vec{u}(t)$ *which is defined for* $|t| < \varepsilon$*, and satisfies the initial conditions*

$$\vec{u}(0) = \vec{u}_0 , \quad \frac{d\vec{u}}{dt}(0) = \vec{v}_0 .$$

Furthermore, the solution depends smoothly on the initial conditions. In other words, the correspondence

$$(\vec{u}_0, \vec{v}_0, t) \to \vec{u}(t)$$

from $W \times (-\varepsilon, \varepsilon)$ *to* R^n *is a* C^∞ *function of all* $2n+1$ *variables.*

PROOF: Introducing the new variables $v^i = \frac{du^i}{dt}$ this system of n second order equations becomes a system of $2n$ first order equations:

$$\begin{cases} \dfrac{d\vec{u}}{dt} = \vec{v} \\ \dfrac{d\vec{v}}{dt} = \vec{F}(\vec{u}, \vec{v}) \end{cases} .$$

The assertion then follows from Graves, "Theory of Functions of Real Variables," p. 166. (Compare our §2.4.)

Applying this theorem to the differential equation for geodesics, one obtains the following.

LEMMA 10.2. *For every point* p_0 *on a Riemannian manifold M there exists a neighborhood U of* p_0 *and a number* $\varepsilon > 0$ *so that: for each* $p \in U$ *and each tangent vector* $v \in TM_p$ *with length <* ε *there is a unique geodesic*

$$\gamma_v : (-2,2) \to M$$

satisfying the conditions

$$\gamma_v(0) = p\,,\ \frac{d\gamma_v}{dt}(0) = v\,.$$

PROOF. If we were willing to replace the interval (-2,2) by an arbitrarily small interval, then this statement would follow immediately from 10.1. To be more precise; there exists a neighborhood U of p_0 and numbers $\varepsilon_1, \varepsilon_2 > 0$ so that: for each $p \in U$ and each $v \in TM_p$ with $\|v\| < \varepsilon_1$ there is a unique geodesic

$$\gamma_v : (-2\varepsilon_2, 2\varepsilon_2) \to M$$

satisfying the required initial conditions.

To obtain the sharper statement it is only necessary to observe that the differential equation for geodesics has the following homogeneity property. Let c be any constant. If the parametrized curve

$$t \to \gamma(t)$$

is a geodesic, then the parametrized curve

$$t \to \gamma(ct)$$

will also be a geodesic.

Now suppose that ε is smaller than $\varepsilon_1\varepsilon_2$. Then if $\|v\| < \varepsilon$ and $|t| < 2$ note that

$$\|v/\varepsilon_2\| < \varepsilon_1 \ \text{ and } \ |\varepsilon_2 t| < 2\varepsilon_2\,.$$

Hence we can define $\gamma_v(t)$ to be $\gamma_{v/\varepsilon_2}(\varepsilon_2 t)$. This proves 10.2.

This following notation will be convenient. Let $v \in TM_q$ be a tangent vector, and suppose that there exists a geodesic

$$\gamma : [0,1] \to M$$

satisfying the conditions

$$\gamma(0) = q\,,\ \frac{d\gamma}{dt}(0) = v\,.$$

Then the point $\gamma(1) \in M$ will be denoted by $\exp_q(v)$ and called

the ***exponential*** of the tangent vector v. The geodesic γ can thus be described by the formula

$$\gamma(t) = \exp_q(tv) .$$

Lemma 10.2 says that $\exp_q(v)$ is defined providing that $\|v\|$ is small enough. In general, $\exp_q(v)$ is not defined for large vectors v. However, if defined at all, $\exp_q(v)$ is always uniquely defined.

DEFINITION. The manifold M is ***geodesically complete*** if $\gamma(t) = \exp_q(tv)$ is defined for all $q \in M$ and all vectors $v \in TM_q$.

This is clearly equivalent to the following requirement:

For every geodesic segment $\gamma_0 : [a,b] \to M$ it should be possible to extend γ_0 to an infinite geodesic

$$\gamma : R \to M .$$

We will return to a study of completeness after proving some local results.

Let *TM* be the tangent manifold of *M*, consisting of all pairs (p,v) with $p \in M$, $v \in TM_p$. We give *TM* the following C^∞ structure: if $(u^1,\ldots,u^n)$ is a coordinate system in an open set $U \subset M$ then every tangent vector at $q \in M$ can be expressed uniquely as $t^1\partial_1 + \cdots + t^n\partial_n$, where $\partial_i = \left.\dfrac{\partial}{\partial u^i}\right|_q$. Then the functions $u^1,\ldots,u^n$, $t^1,\ldots,t^n$ constitute a coordinate system on the open set $TU \subset TM$.

Lemma 10.2 says that for each $p \in M$ the map

$$(q,v) \to \exp_q(v)$$

is defined throughout a neighborhood V of the point $(p,0) \in TM$. Furthermore this map is differentiable throughout V.

Now consider the smooth function $F : V \to M \times M$ defined by $F(q,v) = (q, \exp_q(v))$. We claim that the Jacobian of F at the point $(p_0,0)$ is non-singular. In fact, denoting the induced coordinates on $U \times U \subset M \times M$ by $(u_1^1,\ldots,u_1^n,u_2^1,\ldots,u_2^n)$, we have

$$F_*\left(\frac{\partial}{\partial u^i}\right) = \frac{\partial}{\partial u_1^i} + \frac{\partial}{\partial u_2^i}\ ,\quad F_*\left(\frac{\partial}{\partial t^j}\right) = \frac{\partial}{\partial u_2^j}$$

Thus the Jacobian matrix of F at $(p,0)$ has the form $\begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$, and hence is non-singular.

It follows from the **implicit function theorem** that F maps some neighborhood V' of $(p,0) \in TM$ diffeomorphically onto some neighborhood of $(p,p) \in M \times M$. We may assume that the first neighborhood V', consists of all pairs (q,v) such that q belongs to a given neighborhood U' of p and such that $\|v\| < \varepsilon$. Choose a smaller neighborhood W of p so that $F(V') \supset W \times W$. Then we have proven the following.

LEMMA 10.3. *For each* $p \in M$ *there exists a neighborhood W and a number* $\varepsilon > 0$ *so that*:

(1) *Any two points of W are joined by a unique geodesic in M of length* $< \varepsilon$.

(2) *This geodesic depends smoothly upon the two points. (I.e., if* $t \to \exp_{q_1}(tv)$, $0 \le t \le 1$, *is the geodesic joining* q_1 *and* q_2, *then the pair* $(q_1, v) \in TM$ *depends differentiably on* (q_1, q_2).)

(3) *For each* $q \in W$ *the map* $\exp_q$ *maps the open* ε*-ball in* TM_q *diffeomorphically onto an open set* $U_q \supset W$.

REMARK. With more care it would be possible to choose W so that the geodesic joining any two of its points lies completely within W. Compare J. H. C. Whitehead, Convex regions in the geometry of paths, Quarterly Journal of Mathematics (Oxford) Vol. 3, (1932), pp. 33-42.

Now let us study the relationship between geodesics and arc-length.

THEOREM 10.4. *Let W and* ε *be as in Lemma 10.3. Let*

$$\gamma : [0,1] \to M$$

be the geodesic of length $< \varepsilon$ *joining two points of W, and let*

$$\omega : [0,1] \to M$$

be any other piecewise smooth path joining the same two points. Then,

$$\int_0^1 \left\|\frac{d\gamma}{dt}\right\| dt \le \int_0^1 \left\|\frac{d\omega}{dt}\right\| dt\,,$$

where equality can hold only if the point set $\omega([0,1])$ *coincides with*

$\gamma([0,1])$.

Thus γ is the shortest path joining its end points.

The proof will be based on two lemmas. Let $q = \gamma(0)$ and let U_q be as in 10.3.

LEMMA 10.5. *In U_q, the geodesics through q are the orthogonal trajectories of hypersurfaces*

$$\{\exp_q(v) : v \in TM_q, \|v\| = \text{constant}\}.$$

PROOF. Let $t \to v(t)$ denote any curve in TM_q with $\|v(t)\| = 1$. We must show that the corresponding curves

$$t \to \exp_q(r_0 v(t))$$

in U_q, where $0 < r_0 < \varepsilon$, are orthogonal to the radial geodesics

$$r \to \exp_q(r v(t_0)).$$

In terms of the parametrized surface *f* given by

$$f(r,t) = \exp_q(r v(t)), \quad 0 \le r \le \varepsilon,$$

we must prove that

$$\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle = 0$$

for all (*r*, *t*).

Now

$$\frac{\partial}{\partial r}\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle = \left\langle \frac{D}{\partial r}\frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle + \left\langle \frac{\partial f}{\partial r}, \frac{D}{\partial r}\frac{\partial f}{\partial t} \right\rangle.$$

The first expression on the right is zero since the curves

$$r \to f(,t)$$

are geodesics. The second expression is equal to

$$\left\langle \frac{\partial f}{\partial r}, \frac{D}{\partial t}\frac{\partial f}{\partial r} \right\rangle = \frac{1}{2}\frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial r} \right\rangle = 0,$$

since $\left\| \frac{\partial f}{\partial r} \right\| = \|v(t)\| = 1$. Therefore the quantity $\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle$ is independent of *r*. But for *r*=0 we have

$$f(0,t) = \exp_q(0) = q,$$

hence $\frac{\partial f}{\partial t}(0,t) = 0$. Therefore $\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle$ is identically zero, which completes the proof.

Now consider any piecewise smooth curve

$$\omega : [a,b] \to U_q - [q] .$$

Each point $\omega(t)$ can be expressed uniquely in the form

$\exp_q(r(t)v(t))$ with $0 < r(t) < \varepsilon$, and $\|v(t)\| = 1$, $v(t) \in TM_q$.

LEMMA 10.6. *The length* $\int_a^b \left\| \frac{d\omega}{dt} \right\| dt$ *is greater than or equal to* $|r(b) - r(a)|$, *where equality holds only if the function* $r(t)$ *is monotone, and the function* $v(t)$ *is constant.*

Thus the shortest path joining two concentric spherical shells around q is a radial geodesic.

PROOF. Let $f(r,t) = \exp_q(v(t))$, so that $\omega(t) = f(r(t),t)$. Then

$$\frac{d\omega}{dt} = \frac{\partial f}{\partial r} r'(t) + \frac{\partial f}{\partial t} .$$

Since the two vectors on the right are mutually orthogonal, and since $\left\| \frac{\partial f}{\partial r} \right\| = 1$, this gives

$$\left\| \frac{\partial \omega}{\partial t} \right\|^2 = |r'(t)|^2 + \left\| \frac{\partial f}{\partial t} \right\|^2 \geq |r'(t)|^2$$

where equality holds only if $\frac{\partial f}{\partial t} = 0$; hence only if $\frac{dv}{dt} = 0$. Thus

$$\int_a^b \left\| \frac{d\omega}{dt} \right\| dt \geq \int_a^b |r'(t)| dt \geq |r(b) - r(a)|$$

where equality holds only if $r(t)$ is monotone and v(t) is constant. This completes the proof.

The **proof of Theorem 10.4** is now straightforward. Consider any piecewise smooth path m from q to a point

$$q' = \exp_q(rv) \in U_q ;$$

where $0 < r < \varepsilon$, $\|v(t)\| = 1$. Then for any δ > o the path ω must contain a segment joining the spherical shell of radius δ to the spherical shell of radius r, and lying between these two shells. The length of this segment will be $\geq r - \delta$; hence letting δ tend to 0 the length of ω will be $\geq r$. If $\omega([0,1])$ does not coincide with

$\gamma([0,1])$, then we easily obtain a strict inequality. This completes the proof of 10.4.

An important consequence of Theorem 10.4 is the following.

COROLLARY 10.7. Suppose that a path $\omega:[0,l]\to M$, parametrized by arc-length, has length less than or equal to the length of any other path from $\omega(0)$ to $\omega(l)$. Then ω is a geodesic.

PROOF. Consider any segment of ω lying within an open set *W*, as above, and having length < ε. This segment must be a geodesic by Theorem 10.4. Hence the entire path ω is a geodesic.

DEFINITION. A geodesic $\gamma:[a,b]\to M$ will be called ***minimal*** if its length is less than or equal to the length of any other piecewise smooth path joining its endpoints.

Theorem 10.4 asserts that any sufficiently small segment of a geodesic is minimal. On the other hand a long geodesic may not be minimal. For example we will see shortly that a great circle arc on the unit sphere is a geodesic. If such an arc has length greater than π, it is certainly not minimal.

In general, minimal geodesics are not unique. For example two antipodal points on a unit sphere are joined by infinitely many minimal geodesics. However, the following assertion is true.

Define the ***distance*** $\rho(p,q)$ between two points $p,q\in M$ to be the greatest lower bound for the arc-lengths of piecewise smooth paths joining these points. This clearly makes *M* into a **metric space**. It follows easily from 10.4 that this metric is **compatible** with the usual topology of *M*.

COROLLARY 10.8. *Given a compact set* $K\subset M$ *there exists a number* δ *> 0 so that any two points of K with distance less than* δ *are joined by a unique geodesic of length less than* δ. *Furthermore this geodesic is minimal; and depends differentiably on its endpoints.*

PROOF. Cover *K* by open sets W_α, as in 10.3, and let δ be small enough so that any two points in *K* with distance less than δ lie in a common W_α. This completes the proof.

Recall that the manifold *M* is ***geodesically complete*** if every geodesic segment can be extended indifinitely.

THEOREM 10.9 (Hopf and Rinow). *If M is geodesically complete, then any two points can be joined by a minimal geodesic.*

PROOF. Given $p,q\in M$ with distance r > 0, choose a

neighborhood U_p as in Lemma 10.3. Let $S \subset U_p$ denote a spherical shell of radius $\delta < \varepsilon$ about *p*. Since *S* is compact, there exists a point

$$p_0 = \exp_q(\delta v)\ ,\ \ \|v\| = 1\ ,$$

on *S* for which the distance to *q* is minimized. We will prove that

$$\exp_p(rv) = q\ .$$

This implies that the geodesic segment $t \to \gamma(t) = \exp_q(tv)$, $0 \le t \le r$, is actually a minimal geodesic from *p* to *q*.

The proof will amount to showing that a point which moves along the geodesic γ must get closer and closer to *q*. In fact for each $t \in [\delta, r]$ we will prove that

(1t) $$\rho(\gamma(t), q) = r - t\ .$$

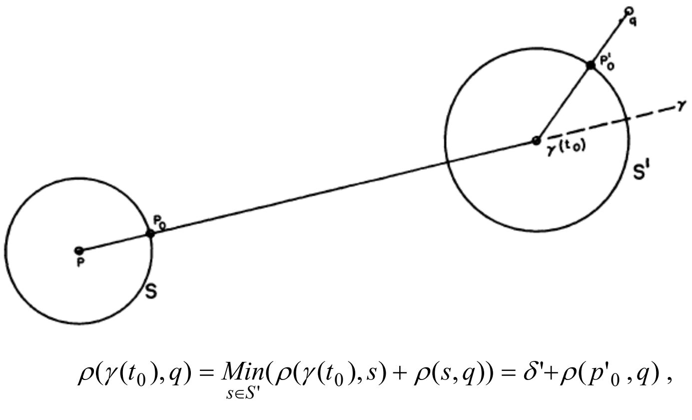
This identity, for *t* = *r*, will complete the proof.

First we will show that the equality (1 δ) is true. Since every path from *p* to *q* must pass through *S*, we have

$$\rho(p,q) = \underset{s\in S}{Min}(\rho(p,s) + \rho(s,q)) = \delta + \rho(p_0, q)\ .$$

Therefore $\rho(p_0, q) = r - t$. Since $p_0 = \gamma(\delta)$, this proves (1 δ).

Let $t_0 \in [\delta, r]$ denote the supremum of those numbers *t* for which (1t) is true. Then by continuity the equality (1 t_0) is true also. If $t_0 < r$ we will obtain a contradiction. Let *S'* denote a small spherical shell of radius δ' about the point $\gamma(t_0)$; and let $p'_0 \in S'$ be a point of *S'* with minimum distance from *q*. (Compare Diagram 10.) Then



$$\rho(\gamma(t_0), q) = \underset{s\in S'}{Min}(\rho(\gamma(t_0), s) + \rho(s,q)) = \delta' + \rho(p'_0, q)\ ,$$

hence

(2) $$\rho(p'_0, q) = (r - t_0) - \delta'\ .$$

We claim that p'_0 is equal to $\gamma(t_0 + \delta')$. In fact the triangle inequality states that

$$\rho(p, p'_0) \ge \rho(p,q) - \rho(p'_0, q) = t_0 + \delta'$$

(making use of (2)). But a path of length precisely $t_0 + \delta'$ from p to p'_0 is obtained by following γ from p to $\gamma(t_0)$, and then following a minimal geodesic from $\gamma(t_0)$ to p'_0. Since this broken geodesic has minimal length, it follows from Corollary 10.7 that it is an (unbroken) geodesic, and hence coincides with γ.

Thus $\gamma(t_0 + \delta') = p'_0$. Now the equality (2) becomes

$(1\ t_{0+\delta'})$ $$\rho(\gamma(t_0 + \delta'), q) = r - (t_0 + \delta').$$

This contradicts the definition of t_0; and completes the proof.

COROLLARY 10.10. *If M is geodesically complete then every bounded subset of M has compact closure. Consequently M is complete as a metric space* (i.e., *every Cauchy sequence converges*).

PROOF. If $X \subset M$ has diameter d then for any $p \in X$ the map $\exp_p : TM_p \to M$ maps the disk of radius d in TM_p onto a compact subset of M which (making use of Theorem 10.9) contains X. Hence the closure of X is compact.

Conversely, if M is complete as a metric space, then it is not difficult, using Lemma 10.3, to prove that M is geodesically complete. For details the reader is referred to Hopf and Rinow. Henceforth we will not distinguish between geodesic completeness and metric completeness, but will refer simply to a ***complete Riemannian manifold***.

FAMILIAR EXAMPLES OF GEODESICS. In Euclidean n-space, R^n, with the usual coordinate system $x_1, \ldots, x_n$ and the usual Riemannian metric $dx_1 \otimes dx_1 + \cdots + dx_n \otimes dx_n$ we have $\Gamma^k{}_{ij} = 0$ and the equations for a geodesic γ, given by $t \to (x_1(t), \ldots, x_n(t))$ become

$$\frac{d^2 x_i}{dt^2} = 0,$$

whose solutions are the straight lines. This could also have been seen as follows: it is easy to show that the formula for arc length

$$\int \left(\sum_{i=1}^{n} \left(\frac{dx_i}{dt} \right)^2 \right)^{1/2} dt$$

coincides with the usual definition of arc length as the least upper bound of the lengths of inscribed polygons; from this definition it is clear that straight lines have minimal length, and are therefore geodesics.

The geodesics on S^n are precisely the great circles, that is, the intersections of S^n with the planes through the center of S^n.

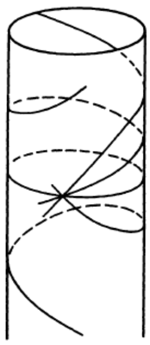
PROOF. Reflection through a plane E^2 is an isometry $I : S^n \to S^n$ whose fixed point set is $C = S^n \cap S^2$. Let x and y be two points of C with a unique geodesic C' of minimal length between them. Then, since I is an isometry, the curve $I(C')$ is a geodesic of the same length as C' between $I(x) = x$ and I(y) = y. Therefore $C' = I(C')$. This implies that $C' \subset C$.

Finally, since there is a great circle through any point of S^n in any given direction, these are all the geodesics.

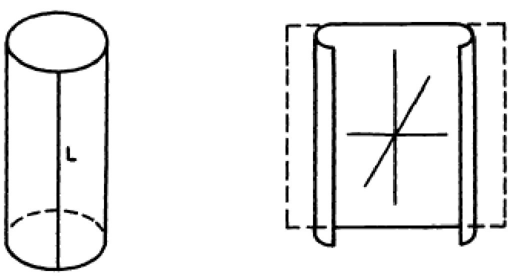
Antipodal points on the sphere have a continium of geodesics of minimal length between them. All other pairs of points have a unique geodesic of minimal length between them, but an infinite family of non-minimal geodesics, depending on how many times the geodesic goes around the sphere and in which direction it starts.

By the same reasoning every meridian line on a surface of revolution is a geodesic.

The geodesics on a right circular cylinder Z are the generating lines, the circles cut by planes perpendicular to the generating lines, and the helices on Z.



PROOF. If L is a generating line of Z then we can set up an isometry $I : Z - L \to R^2$ by rolling Z onto R^2:



The geodesics on Z are just the images under I^{-1} of the straight lines in R^2. Two points on Z have infinitely many geodesics between them.

(End of Part II)