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LECTURES ON PARTIAL DIFFERENTIAL EQUATIONS

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CHAPTER I

INTRODUCTION. CLASSIFICATION OF EQUATIONS

§ 1. Definitions. Examples

1. An equation containing partial derivatives of unknown functions $u_1, u_2, \ldots, u_N$ is said to be of the *nth order* if it contains at least one partial derivative of the nth order and no partial derivatives of order higher than n. By the order of a system of equations containing partial derivatives we mean the order of the highest order equation of the system.

A partial differential equation is called *linear* if it is linear in the unknown functions and in their derivatives; it is called *quasi-linear* if it is linear in the highest order derivatives of the unknown functions. Thus, for instance, the equation

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+u^2=0$$

is quasi-linear of the second order with respect to the unknown function u. The equation

$$\frac{\partial^2 u}{\partial x^2}+a(x,y)\frac{\partial^2 u}{\partial x^2}=2u$$

is linear of the second order with respect to u. The equation

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=u$$

is neither linear nor quasi-linear with respect to u.

By a solution of an equation containing partial derivatives we mean a

system of functions which, when put in the equation in place of the unknown functions, turns the equation into an identity in the independent variables. A solution of a system of equations is defined in an analogous manner.

We shall be primarily interested in linear equations of the second order in one unknown function. The following are examples of equations of this type:

(1) $$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}$$ the 'heat equation',

(2) $$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}$$ the 'wave equation',

(3) $$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}=0$$ the 'Laplace equation'.

Many problems in physics reduce themselves to partial differential equations, in particular, to the partial differential equations listed above.

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§2. The Cauchy problem. The Cauchy-Kowalewski theorem

1. **Formulation of the Cauchy problem.**

Let there be given the following system of equations in the partial derivatives of unknown functions $u_1, u_2, \dots, u_N$ with respect to the independent variables $t, x_1, \dots, x_n$:

$$\frac{\partial^{ni} u_i}{\partial t^{ni}}=F_i(t, x_1, \dots, x_n, u_1, \dots, u_N, \frac{\partial^k u_j}{\partial t^{k0}\partial x_1^{k1}\cdots\partial x_n^{kn}}, \dots)$$
$$(i, j=1,2,\dots, N; k_0+k_1+\dots+k_n=k\le n_j; k_0<n_j)$$

(2.1)

It is clear from our equations that for each of the unknown functions u_j there exists a number n_i which gives the order of the highest order derivatives of n_i appearing in the system. The independent variable *t* is seen to be distinguished in two respects. First, the derivative $\partial^{ni} u_i / \partial t^{ni}$ must appear among the derivatives of the highest order n_i of the function u_i (*i*= 1,2, ...,*N*) and, secondly, the system is solved for these derivatives. In physical problems time occupies this distinguished position and the other variables $x_1, \ldots, x_n$ denote space coordinates. The number of equations is equal to the number of unknown functions.

For some value $t = t_0$ of *t* we prescribe the values ('*initial values*') of the unknown functions and of their derivatives with respect to *t* of orders up to $n_i - 1$. In other words, for $t = t_0$

$$\frac{\partial^k u_i}{\partial t^k} = \phi_i^{(k)}(x_1, x_2, \ldots, x_n) \qquad (k = 0,1,2,\ldots,n_i - 1) \qquad (2.2)$$

All functions $\phi_i^{(k)}(x_1, x_2, \ldots, x_n)$ are prescribed in the same region *G* of the space $(x_1, x_2, \ldots, x_n)$. By the derivative of order zero of the function u_i we mean the function u_i itself.

The Cauchy problem consists in finding a solution of the system (2.1) satisfying the initial conditions (2.2).

CAUCHY-KOWALEWSKI THEOREM. *If all the functions* F_i *are analytic in some neighborhood of the point*

$(t^0, x_1^0, x_2^0, \dots, x_n^0, \phi_{j,k0,k1,\dots,kn,\dots}^0)$ *and if all the functions* $\phi_j^{(k)}$ *are analytic in some neighborhood of the point* $(x_1^0, x_2^0, \dots, x_n^0)$*, then the Cauchy problem has a unique analytic solution in some neighborhood of the point* $(t^0, x_1^0, x_2^0, \dots, x_n^0)$.

§3. The generalized Cauchy problem. Characteristics.

1. **The generalized Cauchy problem**.

Let there be given a system of *N* equations with *N* unknown functions $u_1, u_2, \dots, u_N$

$$\Phi_i\left(x_0, x_1, \dots, x_n; u_1, \dots, u_N; \dots, \frac{\partial^k u_j}{\partial x_0^{k0} \partial x_1^{k1} \cdots \partial x_n^{kn}}, \dots \right) = 0$$

$$(i, j = 1, 2, \dots, N) \qquad (3.1)$$

For each function u_i there exists a highest order n_i of the partial derivatives of u_i with respect to the independent variables $x_0, x_1, \dots, x_n$ which appear in the system (3.1). In the region of points $(x_0, x_1, \dots, x_n)$ under consideration there is given a sufficiently smooth *n*-dimensional surface *S*, and through each point of *S* there passes a curve *l* not tangent to *S* which is assumed to vary sufficiently smoothly as we go from point to point on *S*. The values of the functions u_i and of their derivatives of orders up to $n_i - 1$ in the direction of the curves *l* are prescribed on the surface *S*. These conditions, as prescribed on *S*, are a generalization of the Cauchy conditions (initial data) considered in the

previous section. *We are required to find in some neighborhood of the surface S a solution* $u_1, u_2, \ldots, u_N$ *of the system (3.1) satisfying the conditions prescribed on S.*

Example 1. For the Laplace equation

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0 ,$$

the relation (3.7) assumes the form

$$\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = 0 .$$

This relation together with (3.10) imply that the Laplace equation has no real characteristics.

Example 2. For the wave equation

$$\frac{\partial^2 u}{\partial x_0^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

the relation (3.7) takes the form

$$\alpha_0^2 - \alpha_1^2 - \alpha_2^2 = 0 .$$

Since, according to (3.10), we must also have

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 = 1 ,$$

it follows that

$$2\alpha_0^2 = 1 \quad \text{or} \quad \alpha_0 = \pm 1/\sqrt{2}$$

This means that the tangent planes to all characteristic surfaces form an angle of 45 deg with the *x0*-axis. Using this property of the characteristic surfaces, one can easily picture the form of the characteristic surfaces

going through definite curves in the plane *x0* = const. Thus, for example, if l is a straight line in that plane, the characteristic surface going through l is the plane through l inclined at 45 deg with respect to the plane *x0* = const. Again, if K is the circumference of a circle in the plane *x0* = const., the characteristic surfaces through K will be circular cones whose axes are parallel to the *x0*-axis and whose generators form a 45 deg angle with the plane *x0* = const. (or, which is the same thing, with the *x0*-axis).

Example 3. For the heat equation

$$\frac{\partial u}{\partial x_0} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2},$$

the relation (7,3) takes the form

$$\alpha_1^2 + \cdots \alpha_n^2 = 0.$$

Using (3.10) we conclude that $\alpha_0^2 = 1$. Therefore, the characteristic surfaces are the hyperplanes *x0* = const.

Example 4. For the equation

$$a_1(x_1,\ldots,x_n)\frac{\partial u}{\partial x_1} + a_2(x_1,\ldots,x_n)\frac{\partial u}{\partial x_2} + \ldots + a_n(x_1,\ldots,x_n)\frac{\partial u}{\partial x_n} = 0,$$

(3.7) takes the form

$$a_1(x_1,\ldots,x_n)\alpha_1 + a_{12}(x_1,\ldots,x_n)\alpha_2 + \cdots + a_n(x_1,\ldots,x_n)\alpha_n = 0.$$

Hence all hyperplanes going through the point $(x_1,\ldots,x_n)$ and the vector $(a_1,\ldots,a_n)$ emanating from $(x_1,\ldots,x_n)$ are characteristic at $(x_1,\ldots,x_n)$.

Example 5. For the system of equations in two unknowns

$$\sum_{j=1}^{n} a_{ij}(x_1,x_2)\frac{\partial u_j}{\partial x_1} + \sum_{j=1}^{n} b_{ij}(x_1,x_2)\frac{\partial u_j}{\partial x_2} + \sum_{j=1}^{n} c_{ij}(x_1,x_2)u_j = 0 ,$$

$$(i=1,2,\ldots n),$$

the relation (3.7) assumes the form

$$\left|\alpha_1 a_{ij}(x_1,x_2) + \alpha_2 b_{ij}(x_1,x_2)\right| = 0 .$$

In this case, the characteristic curves are the curves along which $dx_2 / dx_1 = -(\partial\phi / \partial x_1) : (\partial\phi / \partial x_2)$ is equal to a root k of the equation

$$\left|-ka_{ij}(x_1,x_2) + b_{ij}(x_1,x_2)\right| = 0 .$$

Here we are assuming that $\phi(x_1,x_2) = 0$ is the equation of the characteristic curve.

§5. Reduction to canonical form at a point and classification of equations of the second order in one unknown function

1. We consider the linear, second-order equation

$$\sum_{i,j} A_{ij}(x_1,\ldots,x_n)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i} B_i(x_1,\ldots,x_n)\frac{\partial u}{\partial x_i} + C(x_1,\ldots,x_n)u$$
$$+ F((x_1,\ldots,x_n) = 0$$

$$(5.1)$$

in one unknown function u. We assume that $A_{ij} = A_{ji}$, B_i, C and F are all real-valued and defined in some region of the space $(x_1,\ldots,x_n)$. We now define a coordinate transformation by putting

$$\xi_k = \sum_{i=1}^{n} a_{ki} x_i \quad (k=1,\ldots,n), \tag{5.2}$$

where a_{ki} are certain constants. We assume that the transformation (5.2) is non-singular, i.e., that the determinant $|a_{ki}|$ does not vanish. Then the transformation (5.2) is one-to-one. In the new coordinates $\xi_1, \xi_2, \ldots, \xi_n$, equation (5.1) will take the form

$$\sum_{k,l=1}^{n} \left(\sum_{i,j=1}^{n} A_{ij} a_{ki} a_{lj} \right) \frac{\partial^2 u}{\partial \xi_k \partial \xi_l} + \cdots = 0 . \tag{5.3}$$

Here we have written out only terms with second-order derivatives of the unknown function *u*. It is clear from the equality (5.3) that under the transformation (5.2) of the independent variables the coefficients of the second-order derivatives of *u* change in exactly the same manner in which the coefficients of the quadratic form

$$\sum_{i,j=1}^{n} A_{ij} x_i x_j \tag{5.4}$$

change when the x_k are replaced by the ξ_k, according to the formulas

$$x_k = \sum_{i=1}^{n} a_{ik} \xi_i \quad (k=1,..,n) \tag{5.5}$$

We regard the coefficients A_{ij} of the form (5.4) as constants whose values are the same as those of the values of the coefficients $A_{ij}(x_1, \ldots, x_n)$ of equation (5.1) at some point $(x_1^0, \ldots, x_n^0)$ in *G*.

It is proved in algebra that there exists a real non-singular

transformation (5.5) which reduces any form (4.5) with real coefficients A_{ij} to the form

$$\sum_{i=1}^{m} \pm \xi_i^2 \text{, where } m \le n . \qquad (5.6)$$

There are many non-singular real transformations (5.5) which reduce the form (5.4) to the form (5.6), but the number of terms with positive signs and the number of terms with negative signs in the form (5.6) is determined entirely by the form (5.4) and is independent of the choice of the non-singular transformation (5.5). (Law of inertia of quadratic forms.)

If a certain transformation (5.5) reduces the form (5.4) to the form (5.6), then the transformation (5.2) with matrix equal to the transposed inverse of (a_{ik}) reduces (5.1) to the form

$$\sum_{i,j=1}^{n} A^*_{ij}(x_1,\dots,x_n)\frac{\partial^2 u}{\partial\xi_i \partial\xi_j} + \cdots = 0 , \qquad (5.7)$$

where

$$A^*_{ij}(x_1^0,\dots,x_n^0) = \pm 1 \text{, if } i = j \le m ,$$

$$A^*_{ij}(x_1^0,\dots,x_n^0) = 0 , \quad \text{if } i \ne j \text{ or } i = j > m .$$

Here we have written out only terms containing derivatives of the highest order of the function u. The form (5.7) of equation (5.1) is called its *canonical form at the point* $(x_1^0,\dots,x_n^0)$.

It is thus possible to exhibit for each point $(x_1^0,\dots,x_n^0)$ of the region

G a non-singular transformation of the independent variables which reduces equation (5.1) at that point to canonical form.

In general, the transformation (5.2) which reduces equation (5.1) to canonical form at a given point $(x_1^0,\ldots,x_n^0)$ varies with this point, i.e., it may not reduce equation (5.1) to canonical form at a point different from $(x_1^0,\ldots,x_n^0)$. It can be shown by means of examples that as soon as the number of independent variables exceeds two, it is, in general, impossible to exhibit a linear transformation with constant coefficients, or, for that matter, any other transformation which would reduce a given linear equation of the second order to canonical form in an arbitrarily small region. In the case of two variables such a transformation exists under very general assumptions on the coefficients of equation (5.1). This fact will be demonstrated in the following section.

The classification of equations of the second order is based on the possibility of reducing equation (5.1) to canonical form at a point.

2.

(i) Equation (5.1) is said to be *elliptic* at a point $(x_1^0,\ldots,x_n^0)$ if all $A^*_{ii}\,(x_1^0,\ldots,x_n^0)$ are different from zero and have the same sign.

(ii) Equation (5.1) is said to be *hyperbolic* at a point $(x_1^0,\ldots,x_n^0)$ if all but one $A^*_{ii}\,(x_1^0,\ldots,x_n^0)$ in (5.7) have the same sign, the exceptional A^*_{ii} is of opposite sign and $m=n$.

(iii) Equation (5.1) is said to be *ultrahyperbolic* at a point $(x_1^0,\ldots,x_n^0)$ if in (5.7) the number of positive $A^*_{ii}\,(x_1^0,\ldots,x_n^0)$ exceeds one, the

number of negative $A^*_{ii}(x_1^0,\dots,x_n^0)$ exceeds one, and $m = n$.

(iv) Equation (5.1) is said to be *parabolic in the broad sense* at a point $(x_1^0,\dots,x_n^0)$ if some $A^*_{ii}(x_1^0,\dots,x_n^0)$ are zero, i.e., if $m < n$.

(v) Equation (5.1) is said to be parabolic in the restricted sense or, simply, parabolic if only one of the coefficients $A^*_{ii}(x_1^0,\dots,x_n^0)$ (A^*_{11}, say) is zero and all other A^*_{ii} have the same sign, and the coefficient of $\partial u/\partial \xi_1$ is different from zero.

Equation (5.1) is said to be elliptic, hyperbolic, ultrahyperbolic, etc., in the whole region G if it is elliptic, hyperbolic, ultrahyperbolic, etc., at each point of the region G.

3. The non-linear equation of the second order

$$\Phi\left(x_1,\dots,x_n,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n},\dots,\frac{\partial^2 u}{\partial x_i \partial x_j},\dots\right)=0$$

in one unknown function u is said to be elliptic, hyperbolic or parabolic in the broad sense for a given solution $u^*(x_1,\dots,x_n)$ at a point $(x_1^0,\dots,x_n^0)$ (or in the region G) if the equation

$$\sum_{i,j=1}^{N} A_{ij}(x_1,\dots,x_n)\frac{\partial^2 u}{\partial x_i \partial x_j}=0,$$

where

$$A_{ij}(x_1,\dots,x_n)=\frac{\partial \Phi}{\partial\left(\dfrac{\partial^2 u}{\partial x_i \partial x_j}\right)} \tag{5.8}$$

is elliptic, hyperbolic or parabolic in the broad sense at the point

$(x_1^0,\ldots,x_n^0)$ (or in G). On the right side of (5.8) the function u and its derivatives are replaced by the function $u^*(x_1,\ldots,x_n)$ and its appropriate derivatives.

Later we shall study only linear equations of the second order in one unknown function which are either elliptic, or hyperbolic or parabolic in the whole region under consideration. We shall not be interested in ultrahyperbolic equations, for they do not appear in physics or in engineering. We shall, likewise, take no interest in equations which are parabolic in the broad sense, for they turn up seldom in applied work. When speaking of parabolic equations in Chapter IV we shall, accordingly, have in mind equations parabolic in the narrow sense.

§6. Reduction to canonical form in a region of a partial differential equation of the second order in two independent variables

1. Consider the equation

$$A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+F(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})=0\,. \quad (6.1)$$

Here the coefficients A, B and C are functions of x and y which are twice continuously differentiable. We shall assume that the function $u(x, y)$ is also twice continuously differentiable. We now go from the independent variables x, y to the independent variables ξ,η. Let the functions

$$\xi=\xi(x,y)\,,\quad \eta=\eta(x,y) \qquad (6.2)$$

be twice continuously differentiable, and let the Jacobian

$$\begin{vmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{vmatrix}$$

be different from zero in the region under consideration. It is then possible to solve the system (6.2) uniquely for x and y in some region of the (ξ,η) plane. The resulting functions $x(\xi,\eta)$ and $y(\xi,\eta)$ will also be twice continuously differentiable functions of ξ and η. In the new independent variables, equation (6.1) takes the form

$$\begin{aligned}
&\frac{\partial^2 u}{\partial \xi^2}\left[A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2\right] \\
&\quad + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\left[A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + B\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}\right] \\
&\quad + \frac{\partial^2 u}{\partial \eta^2}\left[A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2\right] \\
&\quad + F_1\left(\xi,\eta,u,\frac{\partial u}{\partial \xi},\frac{\partial u}{\partial \eta}\right) = 0
\end{aligned}$$

(6.3)

We now consider the differential equation

$$A\left(\frac{\partial \phi}{\partial x}\right)^2 + 2B\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y} + C\left(\frac{\partial \phi}{\partial y}\right)^2 = 0 \qquad (6.4)$$

in the unknown function $\phi(x,y)$. We shall have to deal separately with the cases $B^2 > AC$, $B^2 < AC$ and $B^2 = AC$ in all of the region under consideration. We shall not consider the case when the expression

$B^2 - AC$ changes sign in the region considered or when it vanishes at some point of the region without vanishing identically in all of it.

2. **Hyperbolic**

We first investigate the case $B^2 > AC$ (in the region under consideration), i.e., the case when equation (6.1) is hyperbolic (cf. the definition of hyperbolicity given in the preceding paragraph).

In this case equation (6.4) is equivalent to the two equations

$$\left.\begin{aligned} A\frac{\partial \phi_1}{\partial x} &= \left[-B-(B^2-AC)^{1/2}\right]\frac{\partial \phi_1}{\partial y} \\ A\frac{\partial \phi_2}{\partial x} &= \left[-B+(B^2-AC)^{1/2}\right]\frac{\partial \phi_2}{\partial y} \end{aligned}\right\} \tag{6.5}$$

If the coefficient A vanishes, then one of the equations (6.5) becomes indeterminate (if, for instance, $B > 0$, then the second equation in (6.5) becomes indeterminate). Its indeterminacy, however, is easily removed by multiplying both sides by the identically equal expressions

$$[-B-(B^2-AC)^{1/2}]/A \equiv C/[-B+(B^2-AC)^{1/2}]. \tag{6.6}$$

We now determine the functions $\phi_i(x, y)$ (i= 1, 2) as solutions of the equations (6.5) by prescribing their values on certain curves l_i which are nowhere tangent to the characteristics of the corresponding equation. If the curves l_i and the values of the functions ϕ_i prescribed on them are sufficiently smooth, we obtain solution functions $\phi_i(x, y)$ (i= 1, 2) having continuous derivatives with respect to x and y of order up to and including the second. If, in addition, we assume that

the initial values $\phi_i(x, y)$ on l_i have been chosen so that the derivative of ϕ_i in the direction of l_i does not vanish anywhere, then at no point will the derivatives of the functions $\phi_i(x, y)$ with respect to x and y vanish simultaneously. Since the vanishing at some point of these two derivatives would imply the vanishing at that point of the derivative of ϕ_i in an arbitrary direction, our statement will have been proved if we can show that at no point A does the derivative of $\phi_i(x, y)$ vanish in all directions. Let us move in a non-characteristic direction from the point A to some point A'. Let us denote by B the point of intersection of l_i and the characteristic through A, and by B' the point of intersection of l_i and the characteristic through A' (Fig. 4). Then, in view of the fact that $\phi_i(x, y)$ is constant along a characteristic, we have

$$\phi_i(A) = \phi_i(B),$$

$$\phi_i(A') = \phi_i(B')$$

$$\phi_i(A') - \phi_i(A) = \phi_i(B') - \phi_i(B)$$

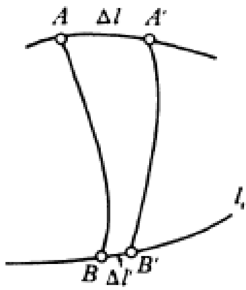

Figure 4

The two distances, Δl between the point A and A' and $\Delta l'$ between the points B and B', approach zero simultaneously. Also, in view of the assumed smoothness of the coefficients of equation (6.1), these two distances are of the same order of smallness. Since, by assumption,

$$\lim_{\Delta l \to 0} \frac{\phi_i(B') - \phi_i(B)}{\Delta l'} \neq 0,$$

we must also have

$$\lim_{\Delta l \to 0} \frac{\phi_i(A') - \phi_i(A)}{\Delta l} \neq 0 .$$

It follows that at no point do the derivatives with respect to *x* and *y* of either one of the functions $\phi_1(x, y)$ and $\phi_2(x, y)$ vanish simultaneously. This means that the Jacobian

$$\begin{vmatrix} \dfrac{\partial \phi_1}{\partial x} & \dfrac{\partial \phi_1}{\partial y} \\ \dfrac{\partial \phi_2}{\partial x} & \dfrac{\partial \phi_2}{\partial y} \end{vmatrix}$$

of these functions cannot vanish at any point of the region under consideration; for, since neither row of the Jacobian vanishes, a necessary condition for its vanishing is the proportionality of its columns, but

$$\frac{\partial \phi_1}{\partial x} : \frac{\partial \phi_1}{\partial y} = \frac{-B - (B^2 - AC)^{1/2}}{A} \neq \frac{-B + (B^2 - AC)^{1/2}}{A} = \frac{\partial \phi_2}{\partial x} : \frac{\partial \phi_2}{\partial y} ,$$

because $B^2 - AC \neq 0$. An indeterminacy in any one of our relations can be removed by using the identity (6.6).

Since the Jacobian of ϕ_1 and ϕ_2 with respect to *x* and *y* does not vanish, we may put in (6.2)

$$\left. \begin{aligned} \xi = \xi(x, y) = \phi_1(x, y) \\ \eta = \eta(x, y) = \phi_2(x, y) \end{aligned} \right\} \qquad (6.7)$$

If we do this, the terms in (6.3) which contain $\partial^2 u / \partial \xi^2$ and $\partial^2 u / \partial \eta^2$ must vanish. At the same time, the coefficient of

$\partial^2 u / \partial\xi\partial\eta$ *will be different from zero* in the whole region considered. Otherwise, a change from the coordinates (x, y) to the coordinates (ξ, η) would decrease the order of the equation, and, consequently, the opposite change, from the coordinates (ξ, η) to the coordinates (x, y) would increase the order of the equation, which is an obvious impossibility.

Dividing equation (6.3) by the coefficient of $\partial^2 u / \partial\xi\partial\eta$ we reduce it to the canonical form

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = F_2\left(\xi, \eta, u, \frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right) \qquad (6.8)$$

in all of the region of definition of the functions $\phi_1(x, y)$ and $\phi_2(x, y)$. If we put $\xi = \alpha + \beta$ and $\eta = \alpha - \beta$, equation (6.8) becomes

$$\frac{\partial^2 u}{\partial\alpha^2} - \frac{\partial^2 u}{\partial\beta^2} = \Phi\left(\alpha, \beta, u, \frac{\partial u}{\partial\alpha}, \frac{\partial u}{\partial\beta}\right). \qquad (6.9)$$

The latter form of the equation is also called *canonical*.

If a hyperbolic equation has been reduced to the canonical form (6.8), it is sometimes possible to integrate it in closed form, i.e., to find a formula giving all solutions of this equation.

Example 1. By means of the substitution

$$\frac{\partial u}{\partial\eta} = v,$$

the equation

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = \frac{1}{2\xi}\frac{\partial u}{\partial\eta} \tag{6.10}$$

becomes

$$\frac{\partial v}{\partial\xi} = \frac{v}{2\xi}\,.$$

The latter equation can be easily integrated by the method of separation of variables, for η enters in v only as a parameter. The constant of integration will be a function of this parameter. We have

$$\ln|v| = \frac{1}{2}\ln|\xi| + \ln|C(\eta)|$$

or

$$v = \frac{\partial u}{\partial\eta} = C(\eta)\sqrt{|\xi|}\,.$$

Whence

$$u = C_1(\eta)\sqrt{|\xi|} + C_2(\xi)\,.$$

Here

$$C_1(\eta) = \int C(\eta)d\eta$$

is an arbitrary (in view of the arbitrariness of $C(\eta)$) differentiable function of η and $C_2(\xi)$ is an arbitrary function of ξ.

Example 2. The transformation

$$x = \xi + \eta\,,\quad y = \xi - \eta$$

reduces the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \qquad (6.11)$$

to the form

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \qquad (6.12)$$

Hence

$$u = \phi(\xi) + \psi(\eta) = \phi_1(x+y) + \psi_1(x-y) \quad (6.13)$$

where ϕ and ψ are arbitrary twice continuously differentiable functions.

3. **Parabolic**

If $B^2 = AC$ in all of the region *G,* then equation (6.1) is parabolic in *G* (cf. the definition of parabolicity in the preceding section). In this case both equations (6.5) are the same and can be replaced by the single equation

$$A\frac{\partial \phi}{\partial x} + B\frac{\partial \phi}{\partial y} = 0\,, \qquad (6.14)$$

or, equivalently, by the equation

$$B\frac{\partial \phi}{\partial x} + C\frac{\partial \phi}{\partial y} = 0\;. \qquad (6.15)$$

Let

$$\phi(x, y) = C$$

be the general integral of this equation in some region *G* of the (*x,y*) plane. We assume that the function $\phi(x, y)$ is twice continuously

differentiable and that its first partial derivatives do not vanish simultaneously (cf. Para. 2). Let

$$\psi(x, y) = C$$

be a family of curves in the region G such that the function $\psi(x, y)$ is sufficiently smooth and the Jacobian

$$\begin{vmatrix} \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \psi}{\partial x} & \dfrac{\partial \psi}{\partial y} \end{vmatrix} \tag{6.16}$$

does not vanish anywhere in the region G. If, for example,

$$\frac{\partial \phi}{\partial x} > 0$$

at all points of the region G, we may assume that

$$\psi(x, y) = y\,.$$

Let us put

$$\xi = \phi(x, y) \quad \text{and} \quad \eta = \psi(x, y)$$

in (6.2). Then the coefficient of $\partial^2 u / \partial \xi^2$ in (6.3) vanishes. The coefficient of $\partial^2 u / \partial \xi \partial \eta$ becomes

$$\left(A \frac{\partial \phi}{\partial x} + B \frac{\partial \phi}{\partial y} \right) \frac{\partial \phi}{\partial x} + \left(B \frac{\partial \phi}{\partial x} + C \frac{\partial \phi}{\partial y} \right) \frac{\partial \psi}{\partial y}\,.$$

According to (6.14) and (6.15), this coefficient will also vanish. We assume that at no point of the region G do all three coefficients A, B and C of equation (6.1) vanish simultaneously. Since $B^2 = AC$, this assumption implies that at each point of G one of the coefficients A and

C is different from zero. Let $A \neq 0$ at some definite point. Then the coefficient of $\partial^2 u / \partial\eta^2$ in equation (6.3) takes the form

$$A\left(\frac{\partial\psi}{\partial x}\right)^2 + 2\boldsymbol{B}\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} + C\left(\frac{\partial\psi}{\partial y}\right)^2 = \frac{1}{A}\left(A\frac{\partial\psi}{\partial x} + B\frac{\partial\psi}{\partial y}\right)^2.$$

This expression cannot be zero: for, otherwise, in view of (6.14), the Jacobian (6.16) would vanish at the point under consideration. We can show in the same way that the coefficient of $\partial^2 u / \partial\eta^2$ does not vanish at points at which $C \neq 0$. This means that this coefficient cannot vanish anywhere in the region *G*. Consequently, we are allowed to divide equation (6.3) by this coefficient. We then obtain

$$\frac{\partial^2 u}{\partial\eta^2} + F_2\left(\xi, \eta, u, \frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right) = 0. \qquad (6.17)$$

According to the definition of canonical form given in §5, equation (6.17) is in *canonical form* in the region *G*.

If equation (6.1) is linear, then so is equation (6.17). Suppose that (6.17) is of the form

$$\frac{\partial^2 u}{\partial\eta^2} = A_1\frac{\partial u}{\partial\xi} + B_1\frac{\partial u}{\partial\eta} + C_1 u + D_1. \qquad (6.18)$$

It is possible to simplify this equation somewhat by introducing a new unknown function *z* in place of *u*. We put

$$u = zv,$$

where $v(\xi,\eta)$ is a function of ξ and η still to be defined. Then equation (6.18) becomes

$$v\frac{\partial^2 z}{\partial \eta^2} + 2\frac{\partial v}{\partial \eta}\frac{\partial z}{\partial \eta} = A_1 v\frac{\partial z}{\partial \xi} + B_1 v\frac{\partial z}{\partial \eta} + C_2 z + D_1 . \tag{6.19}$$

We have written out in detail only terms containing derivatives of *z*. All terms containing the function *z* itself have been combined in $C_2 z$. We choose the function $v(\xi,\eta)$ so that the coefficient of the derivative $\partial z / \partial \eta$ in equation (6.19) should vanish. Equating the coefficient of $\partial z / \partial \eta$ to zero, we get

$$\frac{\partial^2 z}{\partial \eta^2} = A_1 \frac{\partial z}{\partial \xi} + C_3 z + D_2 . \tag{6.20}$$

where $C_3 = C_2 / v$, $D_2 = D_1 / v$, and *v* is

$$v(\xi,\eta) = \exp\left[\frac{1}{2}\int B_1(\xi,\eta)d\eta\right].$$

4. **Elliptic**

Finally, we investigate the case when in all of the region considered $AC > B^2$. Then equation (6.1) will be elliptic in that region (cf. the definition of ellipticity given in § 5). In this case we assume that the coefficients *A*, *B* and *C* are analytic functions of *x* and *y*. Then the coefficients of equations (6.5) are also analytic functions of x and y. Let

$$\phi(x,y) = \phi^*(x,y) + i\phi^{**}(x,y)$$

be an analytic solution of one of the equations in (6.5) in a neighborhood of a point $(x_{0,} y_0)$. We put

$$\xi = \phi^*(x,y) \quad \text{and} \quad \eta = \phi^{**}(x,y) \tag{6.21}$$

in (6.2). Since the Jacobian

$$J = \begin{vmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{vmatrix} \tag{6.22}$$

does not vanish anywhere, we can solve equations (6.21) for x and y. In fact, separating real and imaginary parts in the second equation (6.5), we get

$$\left. \begin{aligned} A\frac{\partial \xi}{\partial x} &= -B\frac{\partial \xi}{\partial y} - (AC - B^2)^{1/2}\frac{\partial \eta}{\partial y} \\ A\frac{\partial \eta}{\partial x} &= -B\frac{\partial \eta}{\partial y} + (AC - B^2)^{1/2}\frac{\partial \xi}{\partial y} \end{aligned} \right\} \tag{6.23}$$

Substituting the expressions for $\partial\xi/\partial x$ and $\partial\eta/\partial x$ obtained from (6.23) in the Jacobian (6.22), we obtain

$$J = -\frac{(AC - B^2)^{1/2}}{A}\left[\left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2\right].$$

It follows that our determinant can be equal to zero only at those points at which

$$\frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial y} = 0,$$

or, in view of equations (6.23), only at those points at which

$$\frac{\partial \xi}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \eta}{\partial x} = 0.$$

But our region contains no such points, for at such points

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0\,.$$

Separating real and imaginary parts in the identity

$$A\left(\frac{\partial \phi}{\partial x}\right)^2 + 2\boldsymbol{B}\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y} + C\left(\frac{\partial \phi}{\partial y}\right)^2 = 0\,,$$

we get

$$A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\boldsymbol{B}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2 = A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2\boldsymbol{B}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2 \tag{6.24}$$

$$A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \boldsymbol{B}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} = 0 \tag{6.25}$$

Since the form

$$A\alpha^2 + 2B\alpha\beta + C\beta^2\,,\ \ (B^2 - AC < 0)$$

is positive-definite, both sides of (6.24) can vanish only if

$$\frac{\partial \xi}{\partial x} = \frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial y} = 0\,. \tag{6.26}$$

We chose the function $\phi(x, y)$ in such a manner, however, that the equations (6.26) are not satisfied at the same time. Hence, we may divide equation (6.3) by one side of the equation (6.24). We get

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = F_2\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right). \tag{6.27}$$

This form of the elliptic equation is called its *canonical form*.

We have reduced our equation to canonical form in a neighborhood of (x_0, y_0) in which there exists an analytic solution of equation (6.5) with non-zero derivatives. It can be shown by means of more intricate considerations that such reduction is possible without assuming that *A(x,y)*, *B(x,y)* and *C(x,y)* are analytic. It suffices to assume that these coefficients are twice continuously differentiable.

CHAPTER II

HYPERBOLIC EQUATIONS

PART I

THE CAUCHY PROBLEM FOR NON-ANALYTIC FUNCTIONS

§8. The reasonableness of the Cauchy problem

§12. Formulas giving the solution of the Cauchy problem for the wave equation

1. Three-Dimensional

Let G_0 in the space (x_1, x_2, x_3) be the domain of definition of a three times continuously differentiate function $\phi_0(x_1, x_2, x_3)$ and a twice continuously differentiate function $\phi_1(x_1, x_2, x_3)$. We wish to determine in *G* (the region described in remark (3) in §11) the solution $u(t, x_1, x_2, x_3)$ of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}, \tag{12.1}$$

satisfying for *t*=0 the conditions

$$u(0, x_1, x_2, x_3) = \phi_0(x_1, x_2, x_3), \tag{12.2a}$$

$$u'_t(0, x_1, x_2, x_3) = \phi_1(x_1, x_2, x_3) \tag{12.2b}$$

We first seek a solution $u_\phi(t, x_1, x_2, x_3)$ of equation (12.1) when the initial conditions have the special form

$$u_\phi(0, x_1, x_2, x_3) = 0, \tag{12.3a}$$

$$u'_{\phi t}(0, x_1, x_2, x_3) = \phi(x_1, x_2, x_3). \tag{12.3b}$$

It is then easily checked that the function

$$v(t, x_1, x_2, x_3) = \frac{\partial u_\phi}{\partial t}$$

satisfies for $t = 0$ the conditions

$$v(0, x_1, x_2, x_3) = \phi(x_1, x_2, x_3)$$

$$v'_t(0, x_1, x_2, x_3) = \frac{\partial^2 u_\phi}{\partial t^2} = \frac{\partial^2 u_\phi}{\partial x_1^2} + \frac{\partial^2 u_\phi}{\partial x_2^2} + \frac{\partial^2 u_\phi}{\partial x_3^2} = 0 .$$

Therefore, the solution of equation (12.1) satisfying both conditions (12.2) is given by the formula

$$u = \frac{\partial u_\phi}{\partial t} + u_{\phi_1} . \tag{12.4}$$

Thus, the general Cauchy problem for equation (12.1) is reduced to the problem of finding u_ϕ. We claim that

$$u_\phi(t, x_1, x_2, x_3) = \frac{1}{4\pi} \iint_{S_t(x_1, x_2, x_3)} \frac{\phi(\alpha_1, \alpha_2, \alpha_3)}{t} d\sigma_t$$

(12.5)(*Kirchhof's formula*)

Here $S_t(x_1, x_2, x_3)$ denotes the sphere with radius t and center (x_1, x_2, x_3) in the hyperplane $t = 0$ where the function ϕ is prescribed, and $d\sigma_t$ denotes an element of the surface of this sphere. We assume that the function $\phi(x_1, x_2, x_3)$ is continuous and bounded together with its derivatives of order up to and including k $(k \geq 2)$. Then, as will become apparent from formula (12.6), the function u_ϕ will have continuous derivatives of order up to and including k.

We first show that u_ϕ given by the formula (12.5), satisfies the

initial conditions (12.3). That u_ϕ satisfies the first of these conditions follows from the fact that

$$\left| \iint_{S_t} \frac{\phi(\alpha_1, \alpha_2, \alpha_3)}{t} d\sigma_t \right| \le least\_upper\_bound\_of\_\left| \phi \cdot \frac{4\pi t^2}{t} \right|,$$

which means that

$$u_\phi(t, x_1, x_2, x_3) \to 0 \quad \text{as} \quad t \to 0 .$$

As for the second condition, we note that if we put

$$\alpha_k = x_k + \beta_k t \, , \; (k=1,2,3)$$

the integral (12.5) becomes

$$u_\phi(t, x_1, x_2, x_3) = \frac{t}{4\pi} \iint_{S_1} \phi(x_1 + t\beta_1, x_2 + t\beta_2, x_3 + t\beta_3) d\sigma_1 \, , \tag{12.6}$$

where the integration is carried out over the surface of the sphere S_1:

$$\beta_1^2 + \beta_2^2 + \beta_3^2 = 1 \, , \quad d\sigma_1 = \frac{d\sigma_t}{t^2}$$

fixed for all values of *x1, x2, x3, t*. Hence

$$\begin{aligned} \frac{\partial u_\phi}{\partial t} &= \frac{1}{4\pi} \iint_{S_1} \phi(x_1 + t\beta_1, x_2 + t\beta_2, x_3 + t\beta_3) d\sigma_1 \\ &\quad + \frac{t}{4\pi} \iint_{S_1} \sum_{k=1}^{3} \beta_k \phi_k (x_1 + t\beta_1, x_2 + t\beta_2, x_3 + t\beta_3) d\sigma_1 \end{aligned} . \tag{12.7}$$

Here ϕ_k denotes the derivative of ϕ with respect to α_k. It is easy to see that as $t \to 0$ the first term on the right approaches $\phi(x_1, x_2, x_3)$ and the second term approaches zero; the latter is true because the

integral in that term stays bounded.

It remains to show that u_ϕ as defined by Kirchhof's formula, satisfies equation (12.1). From equation (12.6) we find that

$$\frac{\partial^2 u_\phi}{\partial x_1^2}+\frac{\partial^2 u_\phi}{\partial x_2^2}+\frac{\partial^2 u_\phi}{\partial x_3^2}=\frac{t}{4\pi}\iint_{S_1}\left(\frac{\partial^2\phi}{\partial\alpha_1^2}+\frac{\partial^2\phi}{\partial\alpha_2^2}+\frac{\partial^2\phi}{\partial\alpha_3^2}\right)d\sigma_1$$
$$=\frac{1}{4\pi t}\iint_{S_1}\left(\frac{\partial^2\phi}{\partial\alpha_1^2}+\frac{\partial^2\phi}{\partial\alpha_2^2}+\frac{\partial^2\phi}{\partial\alpha_3^2}\right)d\sigma_t \quad .$$

(12.8)

To compute $\partial^2 u_\phi / \partial t^2$ we rewrite the equality (12.7) as follows:

$$\frac{\partial u_\phi}{\partial t}=\frac{u_\phi}{t}+\frac{1}{4\pi t}\iint_{S_1}\left(\frac{\partial\phi}{\partial\alpha_1}d\alpha_2 d\alpha_3+\frac{\partial\phi}{\partial\alpha_1}d\alpha_2 d\alpha_3+\frac{\partial\phi}{\partial\alpha_1}d\alpha_2 d\alpha_3\right)$$
$$=\frac{u_\phi}{t}+\frac{1}{4\pi t}\iiint_V\left(\frac{\partial^2\phi}{\partial\alpha_1^2}+\frac{\partial^2\phi}{\partial\alpha_2^2}+\frac{\partial^2\phi}{\partial\alpha_3^2}\right)d\alpha_1 d\alpha_2 d\alpha_3$$
$$=\frac{u_\phi}{t}+\frac{I(t)}{4\pi t}$$

(12.9)

where

$$I(t)=\iiint_{V_t}\left(\frac{\partial^2\phi}{\partial\alpha_1^2}+\frac{\partial^2\phi}{\partial\alpha_2^2}+\frac{\partial^2\phi}{\partial\alpha_3^2}\right)d\alpha_1 d\alpha_2 d\alpha_3$$

and V_t is the surface of the sphere of radius t with center at (x_1, x_2, x_2) in the hyperplane $t = 0$.

From (12.9) we get

$$\begin{aligned}\frac{\partial^2 u_\phi}{\partial t^2} &= -\frac{u_\phi}{t^2} + \frac{1}{t}\left[\frac{u_\phi}{t} + \frac{I(t)}{4\pi t}\right] - \frac{I(t)}{4\pi t^2} + \frac{\partial I(t)}{\partial t} \\ &= \frac{1}{4\pi t}\frac{\partial I(t)}{\partial t}\end{aligned}.$$

(12.10)

Now it is easily seen that

$$\frac{\partial I(t)}{\partial t} = \iint_{S_t} \left(\frac{\partial^2 \phi}{\partial \alpha_1^2} + \frac{\partial^2 \phi}{\partial \alpha_2^2} + \frac{\partial^2 \phi}{\partial \alpha_3^2} \right) d\sigma_t$$

(12.11)

Comparing the equalities (12.8), (12.10) and (12.11) we see that the function u_ϕ defined by *Kirchhof's formula* actually satisfies the wave equation (12.1).

Remark. If the function $\phi_1(x_1, x_2, x_3)$ is only known to be continuous together with its first derivatives, then the function u defined by the equalities (12.4) and (12.5) is only a generalized solution of the Cauchy problem.

2. **Two dimensional**

We now consider the special case when ϕ does not depend on *x3*. It is easy to see that in that case the function u_ϕ given by Kirchhof's formula likewise does not depend on *x3* and so satisfies the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \qquad (12.12)$$

In this case, we can replace the integral over the sphere *St* by the double integral taken over the intersection *Kt* of this sphere with the plane $\alpha_3 = x_3$. Projecting a surface element $d\sigma_t$ of this sphere on the plane $\alpha_3 = x_3$, we get

$$d\sigma_t = \frac{t}{\left[t^2 - (\alpha_1 - x_1)^2 - (\alpha_2 - x_2)^2\right]^{1/2}} d\alpha_1 d\alpha_2,$$

and Kirchhof's formula can then be rewritten as follows:

$$u_\phi(t, x_1, x_2) = \frac{1}{4\pi}\iint_{S_t} \frac{\phi(\alpha_1, \alpha_2)}{t} d\sigma_t$$

$$= \frac{1}{2\pi}\iint_{K_t} \frac{\phi(\alpha_1, \alpha_2) d\alpha_1 d\alpha_2}{\left[t^2 - (\alpha_1 - x_1)^2 - (\alpha_2 - x_2)^2\right]^{1/2}} d\sigma_t$$

Therefore, the solution of equation (12.12) satisfying the conditions

$$u(0, x_1, x_2) = \phi_0(x_1, x_2)$$

$$u'_t(0, x_1, x_2) = \phi_1(x_1, x_2)$$

is given by the formula

$$u(t, x_1, x_2) = \frac{1}{2\pi}\iint_{K_t} \frac{\phi_1(\alpha_1, \alpha_2) d\alpha_1 dg a_2}{\left[t^2 - (\alpha_1 - x_1)^2 - (\alpha_2 - x_2)^2\right]^{1/2}}$$

$$+ \frac{1}{2\pi}\frac{\partial}{\partial t}\iint_{K_t} \frac{\phi_0(\alpha_1, \alpha_2) d\alpha_1 dg a_2}{\left[t^2 - (\alpha_1 - x_1)^2 - (\alpha_2 - x_2)^2\right]^{1/2}} \tag{12.13}$$

This is the so-called *Poisson formula*.

3. **One dimensional**

If the function ϕ does not depend on either *x2* or *x3*, then the function

u_ϕ given by Kirchhof's formula is also independent of *x2* and *x3* and so satisfies the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} \tag{12.14}$$

In that case it is possible to rewrite Kirchhof's formula as follows:

$$u_\phi(t, x_1) = \frac{1}{4\pi}\iint_{S_t} \frac{\phi(\alpha_1)}{t} d\sigma_1 = \frac{1}{2}\int_{x_1-t}^{x_1+t} \phi(\alpha) d\alpha$$

Here we have made use of the fact that the area of the portion of the sphere *St* contained between two planes, $\alpha_1 = const.$ and $\alpha + d\alpha_1 = const.$, which intersect *St* is equal to $2\pi t d\alpha_1$ and the function $\phi(\alpha)$ has almost constant value at all points of this portion of the sphere.

Therefore the solution of equation (12.14), satisfying the conditions

$$u(0, x_1) = \phi_0(x_1), \quad u'_t(0, x) = \phi_1(x_1)$$

is given by the formula

$$\begin{aligned} u(t, x_1) &= \frac{1}{2}\int_{x_1-t}^{x_1+t} \phi_1(\alpha_1) d\alpha_1 + \frac{1}{2}\frac{\partial}{\partial t}\int_{x_1-t}^{x_1+t} \phi_0(\alpha_1) d\alpha_1 \\ &= \frac{\phi_0(x_1+t) + \phi_0(x_1-t)}{2} + \frac{1}{2}\int_{x_1-t}^{x_1+t} \phi_1(\alpha_1) d\alpha_1 \end{aligned} \tag{12.15}$$

This formula is known as *d'Alembert's formula*.

We recall that according to the uniqueness theorem proved in §11 there exist no solutions of the Cauchy problem other than those given by (12.4) for equation (12.1), by (12.13) for equation (12.12), and by

(12.15) for equation (12.14). The method used in obtaining solutions of the Cauchy problem for equations (12.12) and (12.14) from the solution of the Cauchy problem for equation (12.1) is called the method of descent.

We have found the solution of the Cauchy problem for $t > 0$. The case $t < 0$ reduces to the case previously considered if we replace t by $-t$; this transformation does not change equations (12.1), (12.12) and (12.14).

§13. Examination of the formulas which give the solution of the Cauchy problem

1. **Continuous dependence of the solution on the initial conditions**.

2. **Diffusion of waves**.

Formulas (12.4) and (12.5) show that the value at the point $(t, x_1, \dots, x_n)$ of the solution of the wave equation (13.1) for $n = 3$ depends only on the initial conditions on the boundary of the base of the characteristic cone with vertex at $(t, x_1, \dots, x_n)$. On the other hand, for $n = 1$ or $n = 2$, $u(t, x_1, \dots, x_n)$ depends on the conditions as given on the whole base of that cone. This is clear from formulas (12.13) and (12.15).

Let us assume that the initial values of u and $u't$ for $t = 0$ differ from zero only inside a small region Ge about some point $(0, x_1, \dots, x_n)$. Beginning with $t = 0$, we consider the values of u at the points $(t, x_1, \dots, x_n)$ for fixed $x_1, \dots, x_n$ and for increasing t. In the case $n = 3$, the magnitude of $u(t, x_1, \dots, x_n)$ can differ from zero only on a small

portion of the considered straight line (in the space $(t, x_1, \ldots, x_n)$); namely, on that portion of the line on which are located the vertices of those characteristic cones the boundaries of whose bases intersect the region. *Ge*. On the other hand, if $n = 1$ (or $n = 2$) and the point $(0, x_1)$ (or $(0, x_1, x_2)$) does not belong to *Ge*, then $u(t, x_1)$ (or $u(t, x_1, x_2)$) is equal to zero for sufficiently small t and is, in general, different from zero beginning with the values of t for which the segment $|x_1 - \alpha_1| \le t$ (or the circle $(\alpha_1 - x_1)^2 + (\alpha_2 - x_2)^2 \le t^2$) intersects the region *Ge*.

Consequently, a disturbance set up at t =0 in some small neighborhood of the point $(x_1^0, \ldots, x_n^0)$ will, for $n = 3$ and $t > 0$, affect the values of the function only in those points of the space $(x_1, \ldots, x_n)$ which lie *close to* the sphere of radius t with center $(x_1, \ldots, x_n)$. Thus a disturbance set up at $t = 0$ at the point (x_1^0, x_2^0, x_3^0) gives rise to a spherical wave with center at that point, and this wave has a well-defined front and back. On the other hand, if $n = 1$ or $n = 2$, then a disturbance set up at $t = 0$ in a neighborhood of the point $(x_1^0, \ldots, x_n^0)$ affects, in general, *all points in the interior of the sphere with radius t and center* $(x_1^0, \ldots, x_n^0)$. The resulting wave has a well-defined front, but the back of the wave is blurred. We say that in this case the wave is *diffused*. For $n = 3$ no diffusion takes place. It can be shown that no diffusion takes place for the solutions of equation (13.1) for an arbitrary odd $n \ge 3$.

Disturbances set up in a small region *Ge* of a three-dimensional rigid elastic body or of a gas give rise to waves which leave no trace after

them, provided the vibrations satisfy equation (12.1). In the case of a gas, $u(t, x_1, x_2, x_3)$ denotes, for instance, the deviation from normal air pressure at the point (x_1, x_2, x_3) at the moment t. On the other hand, disturbances set up in a small region of a two-dimensional continuum, as, for example, a tight membrane or a water surface, give rise to waves which, in theory, leave a permanent trace after them, provided these vibrations satisfy equation (12.12). In practice they die down very quickly because of the existence of friction which is not taken into consideration in deriving equation (12.12). Likewise, a trace is left after the passing of a wave in a one-dimensional continuum (cf. Para. 3 of this section).

3. **Examination of d'Alembert's formula.**

§14. The Lorentz transformation

1. In §1 we mentioned the fact that except for a constant multiplier the expression

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}$$

is the only linear combination of second derivatives which does not change its form under a rotation of the space, i.e. under an arbitrary orthogonal transformation of the coordinates x_1, x_2, x_3. We now consider in some detail a certain class of linear transformations of the variables (t, x_1, x_2, x_3) with constant real coefficients which are closely connected with the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} = 0\,. \tag{14.1}$$

By a *Lorentz transformation* of the variables x_0, x_1, x_2, x_3 we mean any linear homogeneous transformation of these variables

$$y_i = \sum_{j=0}^{3} a_{ij} x_j \quad (i=0,1,2,3) \tag{14.2}$$

with real coefficients a_{ij} which leaves invariant the quadratic form

$$x_0^2 - x_1^2 - x_2^2 - x_3^2\,. \tag{14.3}$$

This means that in the new variables this quadratic form becomes

$$y_0^2 - y_1^2 - y_2^2 - y_3^2$$

It is easy to check that the totality of Lorentz transformations forms a group. In particular, it is easy to see that the product of two Lorentz transformations (substitutions) is again a Lorentz transformation.

We now write down a formula for a special class of Lorentz transformations which have the property of leaving invariant two of the last three (space) coordinates. Such transformation has the form

$$\left.\begin{aligned} y_0 &= \alpha x_0 + \beta x_1 \\ y_1 &= \gamma x_0 + \delta x_1 \\ y_2 &= x_2 \\ y_3 &= x_3 \end{aligned}\right\} \tag{14.4}$$

For such transformations the identity

$$y_0^2 - y_1^2 \equiv x_0^2 - x_1^2$$

must hold. Substituting in this identity the expressions for y_0 and y_1 in (14.4), we get

$$(\alpha x_0 + \beta x_1)^2 - (\gamma x_0 + \delta x_1)^2 \equiv x_0^2 - x_1^2 .$$

Whence

$$\left.\begin{aligned} \alpha^2 - \gamma^2 &= 1 \\ \beta^2 - \delta^2 &= -1 \\ \alpha\beta - \gamma\delta &= 0 \end{aligned}\right\} \tag{14.5}$$

In particular, these equations are satisfied if we put

$$\alpha = \delta = \cosh\psi ,$$

$$\beta = \gamma = \sinh\psi ,$$

where ψ is an arbitrary number. Then

$$\left.\begin{aligned} y_0 &= \cosh\psi x_0 + \sinh\psi x_1 \\ y_1 &= \sinh\psi x_0 + \cosh\psi x_1 \\ y_2 &= x_2 \\ y_3 &= x_3 \end{aligned}\right\}$$

Let us put $\tanh\psi = \beta$. We then obtain the usual formulas for the class of Lorentz transformations under consideration:

$$\left.\begin{aligned} y_0 &= (x_0 + \beta x_1)/(1-\beta^2)^{1/2} \\ y_1 &= (\beta x_0 + x_1)/(1-\beta^2)^{1/2} \\ y_2 &= x_2 \\ y_3 &= x_3 \end{aligned}\right\} \tag{14.6}$$

Here β is an arbitrary number smaller than one in absolute value because $|\tanh\psi| < 1$ for any ψ.

The formulas (14.6) are of fundamental importance, since, as we are going to show, *any Lorentz transformation is a combination of an orthogonal transformation of the variables* x_1, x_2, x_3 *which leaves* x_0

fixed, a transformation of the form (14.6), and a possible change of sign of one of the variables (*a reflexion*).

If we transpose the matrix of each of the intermediate transformations, we will again obtain a matrix of a transformation of the same type. It follows from this that *the transpose of a matrix of a Lorentz transformation is again a matrix of a Lorentz transformation*. From the definition of a Lorentz transformation it follows that *the inverse of a Lorentz transformation is also a Lorentz transformation*.

2.

We now prove a fundamental fact which clarifies the close connection between Lorentz transformations and the wave equation.

THEOREM. *Every non-singular linear transformation of the variables* t, x_1, x_2, x_3 *with real constant coefficients which does not change the form of equation (14.1) is a combination of a Lorentz transformation, a translation of the origin in the space* (t, x_1, x_2, x_3), *and a similarity transformation in that space.*

§15. The mathematical foundations of the special principle of relativity

The special principle of relativity asserts that all laws of nature have the

same form for all observers who move with respect to one another with uniform, straight-line motion. More precisely, for each of these observers there exists a 'local' space-time coordinate system (t, x_1, x_2, x_3) (t is the time coordinate, and x_1, x_2, x_3 are the space coordinates) in which any given natural law is expressible by means of the same equations. In particular, for each of these observers the speed of light in his local coordinate system is the same in all directions. For the sake of simplicity we assume that this speed is equal to 1.

We wish to find a connexion between the local space-time coordinates of two observers A' and A'' of whom A'' is assumed to be moving uniformly and along a straight line with respect to A'. The velocity of A'' is equal to β, $|\beta| < 1$. In view of the assumption that space and time are homogeneous and isotropic, we consider the required connexion to be linear and its coefficients to be functions of β only. We denote the local space-time coordinates of A' by (t', x'_1, x'_2, x'_3) and the local space-time coordinates of A'' by $(t'', x''_1, x''_2, x''_3)$. For simplicity in notation we shall sometimes write x'_0 instead of t' and x''_0 instead of t''.

Thus, let

$$x''_i = \sum_{j=0}^{3} a_{ij}(\beta) x'_i + \alpha_i \quad (i{=}0,1,2,3). \qquad (15.1)$$

We shall find the connexion between the coordinates (t', x'_1, x'_2, x'_3) and the coordinates $(t'', x''_1, x''_2, x''_3)$, using only the assumption that the velocity of light is constant for the observers A' and A''.

We describe the rectilinear propagation of a plane light wave by means of some non-constant function

$$f(a_0 t' + a_1 x'_1 + a_2 x'_2 + a_3 x'_3), \qquad (15.2)$$

whose values move as a result of the changing time t' in a direction perpendicular to the plane

$$a_1 x'_1 + a_2 x'_2 + a_3 x'_3 = const.$$

with the speed

$$-\frac{a_0}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

which is equal to 1 by assumption. Here a_0, a_1, a_2, a_3 are constants. It follows that

$$a_0^2 = a_1^2 + a_2^2 + a_3^2. \qquad (15.3)$$

Since the velocity of light for the observer A'' in his local coordinates t'', $x''1$, $x''2$, $x''3$ is also equal to 1, we find, on going from the coordinates x' to the coordinates x'', that the expression

$$a_0 t' + a_1 x'_1 + a_2 x'_2 + a_3 x'_3$$

goes over into the expression

$$a'_{0'} t'' + a'_1 x''_1 + a_2 x''_2 + a_3 x''_3 + b$$

and

$$a'^2_0 = a'^2_1 + a'^2_2 + a'^2_3. \qquad (15.4)$$

We shall show that the coordinates t'', x''_1, x''_2, x''_3 are obtained from the coordinates t', x'_1, x'_2, x'_3 by means of a Lorentz transformation and a translation of the origin. By means of a translation of the origin we can replace the coordinates t'', x''_1, x''_2, x''_3 by

coordinates t, x_1, x_2, x_3 whose connexion with t', x'_1, x'_2, x'_3 is given by the homogeneous linear equations

$$x_i = \sum_{j=0}^{3} a_{ij}(\beta) x'_j \quad (i=0,1,2,3). \tag{15.5}$$

Now let the function

$$f(a_0 x'_0 + a_1 x'_1 + a_2 x'_2 + a_3 x'_3)$$

go over into the function

$$f(a'_0\, x_0 + a'_1\, x_1 + a'_2\, x_2 + a'_3\, x_3),$$

where the numbers a'_0, a'_1, a'_2, a'_3 satisfy the relation

$$a'^2_0 - a'^2_1 - a'^2_2 - a'^2_3 = 0$$

as soon as the numbers a_0, a_1, a_2, a_3 satisfy the relation

$$a_0^2 - a_1^2 - a_2^2 - a_3^2 = 0\,.$$

Here a_0, a_1, a_2, a_3 is an *arbitrary* system of numbers satisfying equation (15.3) and a'_0, a'_1, a'_2, a'_3 is the corresponding system of numbers resulting from the transformation (15.5). We shall show that it now follows that (15.5) yields the Lorentz transformation for the coefficients a, that is, that

$$a_0^2 - a_1^2 - a_2^2 - a_3^2 = a'^2_0 - a'^2_1 - a'^2_2 - a'^2_3\,.$$

In fact, what we do know from the form of the substitution (15.5) is that the relation between the a_i and their transforms under (15.5) must be of the form

$$a_0^2 - a_1^2 - a_2^2 - a_3^2 \equiv \sum_{i,j=0}^{3} k_{ij}(\beta) a'_i\, a'_j \tag{15.6}$$

We first show that

$$\sum_{i,j=0}^{3} k_{ij}(\beta)a'_i\, a'_j \equiv k(\beta)(a'^2_0 - a'^2_1 - a'^2_2 - a'^2_3)\ .$$

(15.7)

In fact,

$$\sum k_{ij}(\beta)a'_i\, a'_j = 0 \qquad (15.8)$$

implies

$$a'^2_0 - a'^2_1 - a'^2_2 - a'^2_3 = 0\,, \qquad (15.9)$$

and conversely. This means that the surfaces given in the four-dimensional space a'_0, a'_1, a'_2, a'_3 by (15.8) and (15.9) coincide. It is easy to see that this implies the correctness of (15.7). Consequently

$$a_0^2 - a_1^2 - a_2^2 - a_3^2 = k(\beta)(a'^2_0 - a'^2_1 - a'^2_2 - a'^2_3)\ .$$

If we consider the motion of the first system with respect to the second, we have, analogously,

$$a'^2_0 - a'^2_1 - a'^2_2 - a'^2_3 = k(-\beta)(a_0^2 - a_1^2 - a_2^2 - a_3^2)$$

whence

$$k(\beta)k(-\beta) = 1\ .$$

On the other hand, since neither system is in any way distinguished, we must have

$$k(\beta) = k(-\beta)\,,$$

and, consequently, $k(\beta) = \pm 1$.

When the variables x'_i are subjected to the transformation (15.5), the variables a_i are also subjected to a linear transformation. It follows that the number of plus and minus signs in the quadratic form in the a_i

cannot change. Hence $k(\beta)=1$, and the form

$$a_0^2 - a_1^2 - a_2^2 - a_3^2$$

must remain invariant under the transformation (15.5). Thus, the transformation of the variables a_i is a Lorentz transformation. The linear transformation to which the a_i are subjected when the x_i are transformed by means of (15.5) is given by a matrix which is the transposed inverse of the matrix (15.5). Consequently, (15.5) is itself a Lorentz transformation (cf. Para. 1, §14), which is what we set out to prove.

(to be continued)