Mathematical Principles of Classical Fluid Mechanics.

By

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A. Preface and introductory remarks.

1. Classical fluid mechanics

Classical fluid mechanics is a branch of continuum mechanics; that is, it proceeds on the assumption

that a fluid is practically continuous and homogeneous in structure. The fundamental property which distinguishes a fluid from other continuous media is that it cannot be in equilibrium in a state of stress such that the mutual action between two adjacent parts is oblique to the common surface. Though this property is the basis of hydrostatics and hydrodynamics, it is by itself insufficient for the description of fluid motion. In order to characterize the physical behavior of a fluid the property must be extended, given suitable analytical form, and introduced into the equations of motion of a general continuous medium, this leading ultimately to a system of differential equations which are to be satisfied by the velocity, density, pressure, etc. of an arbitrary fluid motion. In this article we shall consider these differential equations, their derivation **from fundamental axioms**, and the various forms which they take when more or less special assumptions concerning the fluid or the fluid motion are made.

Our intent, then, is to present in a **mathematically correct way**, in concise form, and with more than passing attention to the foundations, the principles of classical fluid mechanics. The work includes the body of **exact theoretical knowledge** which accompanies the fundamental equations, and at the same time excludes relativistic and quantum effects, most of the kinetic theory, special fields such as turbulence, and all numerical or approximate work. Other topics which have been omitted, but which properly come within the scope of the article, are hydrostatics, rotating fluid masses, one-diinensiona1 gas flows, and stability theory; these subjects are treated elsewhere in this Encyclopedia. A basic knowledge of vector analysis and partial differential equations is expected of the reader, and some experience in hydrodynamics will prove helpful.

The paper proper begins with Division B, where the equations of motion are derived; we have attempted to give **rigorous** and **complete** discussions of the basic points, establishing the entire work on the concept of motion as a **continuous point transformation**. In the final part of this chapter we have discussed t**ransformation of coordinates** and **variational principles**. The material in Part C is to some extent standard in textbooks, but its omission would affect the unity of the article. Moreover, it is here that we first meet many of the ideas which are of importance in the more complex situations treated later. Part D returns to the foundations of the subject with a concise treatment of the thermodynamics of fluid motion, including a postulational summary of the relevant parts of classical thermodynamics. The presentation here may serve as a model for the discussion of multicomponent hydrodynamical systems.

In Part E we present the general theory of **perfect** (i.e., nonviscous) **gases**. We have attempted as much as possible to include results on non-isentropic motion and to avoid the ideal gas assumption $pV = RT$. Rather surprisingly, this point of view leads in many cases to a considerable economy of thought. Part F deals with the theory of **shock waves** in a perfect fluid. The treatment is based entirely on the postulates of motion (Parts B and D) and requires no new dynamical assumptions. The section on shock layers should be useful as an introduction to the specialized literature on the subject. The concluding chapter

begins with a clearcut derivation of the constitutive equations of a **viscous fluid** and covers other theoretical work of recent years.

Some of the sections contain new material or improved treatment of known work. In particular we refer to the following items: the discussion of **variational principles** (Sects. 14, 15, 24 and 47), the theory of **dynamical similarity** (Sects. 36 and 66), the theory of the **stress tensor** (Sect. 59), the **energy method** (Sect. 73), an extension of the **Helmholtz-Rayleigh theorem** (Sect. 75), and several new formulas or equations, e.g., Eqs. (29.9), (40.6), (42.8), etc. An attempt has been made to cite original authorities whenever possible; on the other hand, complete references to a subject are seldom given, since they can usually be traced through the papers which are quoted. Finally, we must add that in a number of places proofs have been considerably modified and shortened from their original form.

This work owes much to the stimulating lectures and penetrating scholarship of my teachers David Gilbarg and Clifford Truesdell. Although the responsibility for the material presented is solely mine, their influence is apparent in many places. Also to my wife Barbara I owe sincerest thanks and gratitude, specifically for typing the entire manuscript and generally for smoothing the whole project to completion. Every work on fluid dynamics is the better for whatever degree of closeness it attains to the style, clarity, and thoroughness of Sir Horace Lamb's Hydrodynamics. The author hopes he has stayed to the path there laid out.

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2. Vectors and tensors.

The mathematical notation used in this article is that of ordinary Cartesian or Gibbsian vector analysis. This notation leads to the utmost conciseness of expression, and at the same time illuminates the physical meaning of the phenomena represented. Most of the vector operations which we use are standard, but occasionally an expression is needed which may appear unusual or ambiguous. For this reason it is convenient to define all operations in terms of vector components: then the meaning of an equation can always be made clear simply by rewriting it in component form. Another advantage accrues to this method, namely that any equation admits an immediate tensorial interpretation if so desired.

Except in a few special situations we shall use lower case bold face to denote vectors; in a fixed rectangular coordinate system, the components of vectors $\boldsymbol{b}$, $\boldsymbol{c}$, etc., will be denoted by b^i, c^i, etc., or equivalently b_i, c_i, etc., where i =1, 2, 3. In this notation the scalar product $\boldsymbol{b}\cdot\boldsymbol{c}$ is defined by

$$\boldsymbol{b}\cdot\boldsymbol{c} = b^i c_i = b_i c^i ,$$

with the usual convention that a repeated index is summed from 1 to 3.[1] Similarly the vector product

[1] The simultaneous use of upper and lower indices has been adopted in order to conform with the standard notation of tensor

$\boldsymbol{b} \times \boldsymbol{c}$ is defined by its components

$$(\boldsymbol{b} \times \boldsymbol{c})^i = e^{ijk} b_j c_k ,$$

where e^{ijk} is the usual permutation symbol.[2] The magnitude of a vector $\boldsymbol{b}$ is denoted by the corresponding italic lower case letter, thus

$$b = |\boldsymbol{b}| = \sqrt{\boldsymbol{b} \cdot \boldsymbol{b}} .$$

(One important exception to this rule will be made: the magnitude of the velocity vector $\boldsymbol{v}$ will be denoted by q, the letter v being reserved to stand for a velocity component.)

The symbols $\mathrm{grad}\phi$, $\mathrm{div}\boldsymbol{b}$ and $\mathrm{curl}\boldsymbol{b}$ will be employed in their usual senses, thus

$$\mathrm{div}\boldsymbol{b} = b^i{}_{,i}$$

and

$$(\mathrm{curl}\boldsymbol{b})^i = e^{ijk} b_{k,j} , \quad (\mathrm{grad}\phi)_i = \phi_{,i} .$$

The comma in these formulas is a. standard convention denoting differentiation. That is, if F is an arbitrary scalar or vector function of position we define

$$F_{,i} \equiv \frac{\partial F}{\partial x^i} , \quad i = 1, 2, 3.$$

[This definition of $F_{,i}$ must be modified in case one wishes to consider curvilinear coordinate systems, as in Sect. 12. The modification need not concern us here, however, since except for a few instances the article is couched exclusively in the notation of Cartesian vector analysis.]

Second order tensors (dyadics) occur frequently in this work. They wi1l.be represented by **uppercase bold face letters**: $\boldsymbol{\Sigma}, \boldsymbol{T}$, etc. The components of a tensor $\boldsymbol{\Sigma}$ will be denoted by Σ^{ij}, and also, upon occasion, by Σ^i_j and Σ_{ij}. By the equations

$$\boldsymbol{b} = \boldsymbol{c} \cdot \boldsymbol{\Sigma} \quad \text{and} \quad \boldsymbol{b} = \boldsymbol{\Sigma} \cdot \boldsymbol{c}$$

we mean, respectively

$$b^i = c_j \Sigma^{ji} \quad \text{and} \quad b^i = \Sigma^{ij} c_j .$$

Finally $\boldsymbol{\Sigma} : \boldsymbol{T}$ stands for the scalar product $\Sigma^{ij} T_{ij}$.

Several special notation are convenient. By $\boldsymbol{\Sigma}_x$ we mean the **vector** with components $e^{ijk} \Sigma_{jk}$. By

analysis.

[2] That is, $e^{123} = e^{231} = e^{312} = 1$, $e^{213} = e^{132} = e^{321} = -1$, and all other components are 0.

grad$\boldsymbol{b}$ we mean the **tensor** with components $b_{j,i}$, that is

$$(\mathrm{grad}\boldsymbol{b})_{ij} = b_{j,i} .$$

Finally, div$\boldsymbol{\Sigma}$ stands for the **vector** with components $\Sigma^{ji}{}_{,j}$. From these definitions it follows that

$$\mathrm{curl}\boldsymbol{b} = (\mathrm{grad}\boldsymbol{b})_x \quad \text{and} \quad (\boldsymbol{c}\cdot\mathrm{grad}\boldsymbol{b})_i = c^j b_{i,j} .$$

The reader familiar with tensor analysis will observe that if $\boldsymbol{b}$ is regarded as a. short name for the set of contravariant components b^i or covariant components b_i of a vector in a general curvilinear coordinate system, and if $\boldsymbol{\Sigma}$ is likewise regarded as a short name for the components of a tensor, then the above definitions are tensorially invariant. Thus the vector symbols we have introduced could equally well serve as a shorthand for writing tensor formulas.

A general transformation of volume integrals into surface integrals is embodied in the symbolic formula[3]

$$\int_v F_{,i}dv = \oint_\sigma F n_i da . \tag{2.1}$$

Here F is any scalar, vector, or tensor, with or without an index i to be summed out; v is a volume in which F is continuously differentiable; σ is the surface of this volume (assumed suitably smooth); and n_i are the components of the *outer* normal $\boldsymbol{n}$ to the surface σ. Replacing F by b^i gives

$$\int_v \mathrm{div}\boldsymbol{b}dv = \oint_\sigma \boldsymbol{b}\cdot\boldsymbol{n}da , \tag{2.2}$$

usually called the divergence heorem; replacing F by $e^{ijk}b_j$ gives

$$\int_v \mathrm{curl}\boldsymbol{b}dv = \oint_\sigma \boldsymbol{n}\times\boldsymbol{b}da . \tag{2.3}$$

These formulas, and others like them, will be used frequently in this work.

List of frequently used symbols.

Within a single section sometimes these same symbols are defined and used in a different sense. Numbers refer to section where symbol is first used.

c: sound speed, Sect. 35.

E: internal energy, Sects. 30, 33.

F: arbitrary function.

H: total enthalpy, Sects. 18, 38.

I: enthalpy, Sect. 38.

[3] H. B. PHILLIPS [48], formula (127).

J: Jacobian, Sect. 3.

M: Mach number, Sect. 36.

n: distance normal to streamline.

p: pressure.

q: speed.

Q: mass flow, Sect. 37.

r: radial distance.

s: distance along streamline.

S: entropy, Sects. 30,33.

t: time.

T: absolute temperature.

u,v,w: velocity components.

$\boldsymbol{a}$: acceleration vector.

$\boldsymbol{D}$: deformation tensor, Sect. 11.

$\boldsymbol{f}$: extraneous force vector. with the fluid.

$\boldsymbol{I}$: unit matrix.

$\boldsymbol{n}$: unit (outer) normal vector to a surface.

$\boldsymbol{t}$: stress vector, Sect. 6.

$\boldsymbol{T}$: stress tensor, Sect. 6.

$\boldsymbol{v}$: velocity vector.

ρ : density.

θ : polar coordinate.

θ : velocity inclination.

Θ : expansion, Sect. 26.

ϕ : velocity potential.

Φ : dissipation function, Scots. 34, 61.

ψ : stream function, Sects. 19, 42.

ω : vorticity magnitude.

Ω : extraneous force potential, Sect. 9.

$\boldsymbol{\omega}$: vorticity vector.

$\boldsymbol{\Omega}$: vorticity tensor, Sect. 11.

$\widetilde{T}$: kinetic energy, Sect. 9.

$\widetilde{W}$: vorticity measure, Sect. 27.

$\widetilde{C},\widetilde{S},\widetilde{V}$: curves, surfaces, volumes moving with the fluid.

v,σ : fixed volume in space, and its bounding surface.

Other standard notations are introduced in Sects. 2 and 3.