B. The equation of motion.

I. Kinematics and dynamics of fluid motion.

3. Kinematical preliminaries.

Fluid flow is an intuitive physical notion which is represented mathematically by a **continuous transformation** of three-dimensional Euclidean space into itself. The parameter t describing the transformation is identified with the time, and we may suppose its range to be $-\infty < t < \infty$, where t = 0 is an arbitrary initial instant.

In order to describe the transformation analytically let us introduce a **fixed rectangular coordinate system** (x^1, x^2, x^3). We refer to the coordinate triple (x^1, x^2, x^3) as the **position** and denote it by $\boldsymbol{x}$. Now consider a typical point or particle P moving with the fluid. At time t = 0 let it occupy the position $\boldsymbol{X}$ = (X^1, X^2, X^3) and at time t suppose it has moved to the position $\boldsymbol{x}$ = (x^1, x^2, x^3). Then $\boldsymbol{x}$ is determined as a function of $\boldsymbol{X}$ and t, and the flow may be represented by the transformation

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t) \quad (\text{or} \quad x^i = \phi^i(\boldsymbol{X}, t)). \tag{3.1}$$

If $\boldsymbol{X}$ is fixed while t varies, Eq. (3.1) specifies the **path** of the particle P initially at $\boldsymbol{X}$; on the other hand, for fixed t:, Eq. (3.1) determines a transformation of the region initially occupied by the fluid into its position at time t.

We assume that initially distinct points remain distinct throughout the entire motion, or, in other words, that; the transformation (3.1) possesses an inverse,[1]

$$\boldsymbol{X} = \boldsymbol{\Phi}(\boldsymbol{x}, t) \quad (\text{or} \quad X^\alpha = \Phi^\alpha(\boldsymbol{x}, t)). \tag{3.2}$$

It is also assumed that ϕ^i and Φ^α possess continuous derivatives up to the third order in all variables, except possibly at certain singular surfaces, curves, or points. Unless otherwise specified, we shall be concerned only with those portions of a flow which ***do not*** contain singularities. Cases of exception (singular surfaces in particular) require a separate examination, and are dealt with in Sects. 51 and 54. Finally, notice that any closed surface whatever, which moves with the fluid, completely and permanently separates the matter on the two sides of it.

Although a flow is completely determined by the transformation (3.1), it is also important to consider the state of motion at a given point during the course of time. This is described by the functions

$$\rho = \rho(\boldsymbol{x}, t), \quad \boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t), \text{ etc.} \tag{3.3}$$

which give the density and velocity, etc., of the particle which happens to be at the position $\boldsymbol{x}$ at the time t. It was d'Alembert in 1749 and Euler in 1752 who first recognized the importance of the field description (3.3) in the study of fluid motion, and Euler who conceived the magnificent idea of studying the motion

[1] Greek letters will be used as indices for particle coordinates.

directly through partial differential equations relating the quantities (3.3).[2] We must now develop the ideas just outlined.

The variables ($\boldsymbol{x}$, t) used in. the field description (3.3) of the flow will be called **spatial variables**; the variables ($\boldsymbol{X}$, t), which single out individual particles will correspondingly be called **material variables**.[3] By means of Eq. (3.1) any quantity F which is a function of the spatial variables ($\boldsymbol{x}$, t) is also a function of the material variables ($\boldsymbol{X}$, t), and conversely. If we wish to indicate the dependence of F on a particular set of variables we write either

$$F = F(\boldsymbol{x},t) \quad \text{or} \quad F = F(\boldsymbol{X},t),$$

the functions $F(\boldsymbol{x}, t)$ and $F(\boldsymbol{X}, t)$ of course being related by the change of variables (3.1) and (3.2). Geometrically, $F(\boldsymbol{X}, t)$ is the value of F experienced at time t by the particle initially at $\boldsymbol{X}$, and $F(\boldsymbol{x}, t)$ is the value of F felt by the particle instantaneously at the position $\boldsymbol{x}$. We shall use the symbols

$$\frac{\partial F}{\partial t} \equiv \frac{\partial F(\boldsymbol{x},t)}{\partial t} \quad \text{and} \quad \frac{dF}{dt} \equiv \frac{\partial F(\boldsymbol{X},t)}{\partial t}$$

for the two possible time derivatives of F; obviously they are quite different quantities. $\frac{dF}{dt}$ is called the **material derivative** of F. It measures the rate of change of F following a particle, and it can of course be expressed in either material or spatial variables. $\frac{\partial F}{\partial t}$, on the other hand, gives the rate of change of F apparent to a viewer stationed at the position $\boldsymbol{x}$.

The velocity $\boldsymbol{v}$ of a particle is given by the definition

$$\boldsymbol{v} \equiv \frac{d\boldsymbol{x}}{dt}, \quad \left(v^i \equiv \frac{dx^i}{dt} \equiv \frac{\partial \phi^i(\boldsymbol{X},t)}{\partial t} \right).$$

As defined, $\boldsymbol{v}$ is a function of the material variables; in practice, however, one usually deals with the spatial form

$$\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t).$$

In most problems it is sufficient to know $\boldsymbol{v}(\boldsymbol{x}, t)$ rather than the actual motion (3.1).

We have introduced the velocity field in terms of the motion (3.1). It is naturally important to be able to proceed in the opposite direction, that is, to determine Eq. (3.1) from $\boldsymbol{v}(\boldsymbol{x}, t)$. This transition is effected by solving the system

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[2] Euler's work on fluid mechanics will be found, for the most part, in volumes II 12, 13 of his collected works (Opera Omnia, Zurich). Professor Truesdell's introductions to these volumes lucidly describe Euler's contributions to fluid mechanics in relation to those of his predecessors and contemporaries, and firmly establish Euler as the founder of rational fluid mechanics.

[3] The two sets of variables just introduced are usually called Eulerian and Lagrangian, respectively, though both are in fact due to Euler; cf. [26]. § 14.

of ordinary differential equations

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}(\boldsymbol{x}, t) \tag{3.4}$$

with the conditions $\boldsymbol{x}(0) = \boldsymbol{X}$. The integration of Eq. (3.4) should be carried out "in the large" and is therefore not always an easy problem.[4]

Acceleration is the rate of change of velocity experienced by a moving particle. Denoting the acceleration vector by $\boldsymbol{a}$, we have then $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$. We observe that acceleration can be computed directly in terms of the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$, for we have

$$a^i = \frac{dv^i}{dt} = \frac{\partial v^i}{\partial t} + \frac{\partial v^i}{\partial x^j}\frac{\partial x^j}{dt},$$

or

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \mathrm{grad}\boldsymbol{v}\ . \tag{3.5}$$

Eq. (3.5) is a special case of the general formula

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \boldsymbol{v} \cdot \mathrm{grad}F \tag{3.6}$$

relating the material derivative to spatial derivatives. Eq. (3.6) may be interpreted as expressing, for an arbitrary quantity $F = F(\boldsymbol{x}, t)$, the time rate of change of F apparent to a viewer situated on the moving particle instantaneously at the position $\boldsymbol{x}$.

The Jacobian of the transformation (3.1), namely

$$J = \frac{\partial(x^1, x^2, x^3)}{\partial(X^1, X^2, X^3)} = \det\left(\frac{\partial x^i}{\partial X^\alpha}\right)$$

represents the dilatation of an infinitesimal volume as it follows the motion. From the assumption that Eq. (3.1) possesses a differentiable inverse it follows that

$$0 < J < \infty\ . \tag{3.7}$$

In the sequel we shall make use of the elegant formula

$$\frac{dJ}{dt} = J\mathrm{div}\boldsymbol{v}\ , \tag{3.8}$$

due originally to Euler. To prove this, let A_i^α be the cofactor of $\frac{\partial x^i}{\partial X^\alpha}$ in the expansion of the Jacobian determinant, so that

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4 In [10], §9.21 there is a particularly interesting example of the integration of equation (3.4), due originally to Maxwell, Proc. Lond. Math. Soc. **3**, 82 (1870). Other examples are discussed in [10], §9.71 and [8], §§72, 159. The general problem of integration is considered by Lichtenstein [9], pp.

$$\frac{\partial x^i}{\partial X^\alpha} A_j^\alpha = J\delta_j^i .$$

Then clearly

$$\frac{dJ}{dt} = \frac{d}{dt}\left(\frac{\partial x^i}{\partial X^\alpha}\right) A_i^\alpha = \frac{\partial v^i}{\partial X^\alpha} A_i^\alpha = \frac{\partial v^i}{\partial x^j}\frac{\partial x^j}{\partial X^\alpha} A_i^\alpha = \frac{\partial v^i}{\partial x^i} J .$$

Incompressible fluids. If a fluid is assumed to he incompressible, that is, to move without change in volume, then by Eq. (3.8) we have

$$\mathrm{div}\boldsymbol{v} = 0 . \qquad (3.9)$$

Further study of incompressible fluid motion must involve dynamical considerations; in particular, the common assumption $\mathrm{curl}\boldsymbol{v} = 0$ needs dynamical justification whenever it is applied.

4. The transport theorem.

Let $\widetilde{V} = \widetilde{V}(t)$ denote an arbitrary volume which is moving with the fluid,[5] and let $F(\boldsymbol{x}, t)$ be a scalar or vector function of position. The volume integral

$$\int_{\widetilde{V}} F dv$$

is then a well-defined function of time. Its derivative is given by the important formula

$$\frac{d}{dt}\int_{\widetilde{V}} F dv = \int_{\widetilde{V}} \left(\frac{dF}{dt} + F\mathrm{div}\boldsymbol{v}\right) dv . \qquad (4.1)$$

To prove Eq. (4.1), we introduce (X^1, X^2, X^3) as new variables of integration by means of Eq. (3.1). Then the moving region $\widetilde{V}(t)$) in the $\boldsymbol{x}$-variables is replaced by the fixed region $\widetilde{V}_0 = \widetilde{V}(0)$ in the $\boldsymbol{X}$-variables (recall that $\widetilde{V}$ is at all times composed of the same particles), and

$$\int_{\widetilde{V}} F dv = \int_{\widetilde{V}_0} F(\boldsymbol{X}, t) J dv_0 ,$$

where the formula $dv = J dv_0$ relates the element of volume dv in the $\boldsymbol{x}$-variables to the element of volume dv_0 in the $\boldsymbol{X}$-variables. The integral on the right involves t only under the integral sign, hence

$$\frac{d}{dt}\int_{\widetilde{V}} F dv = \int_{\widetilde{V}_0} \left(J\frac{dF}{dt} + F\frac{dJ}{dt}\right) dv ,$$

and Eq. (4.1) follows at once by transformation of the last integral using Euler's formula (3.8).

159 to 170.

[5] We shall generally use script ***capital*** letters to denote volumes, surfaces, and curves which move with the particles of fluid. On the other hand, volumes, surfaces, and curves which are fixed in the physical space will be denoted by script ***lower case*** letters. This notation will prove to be a convenient one for the formulation of a number of the basic principles of hydrodynamics.

Eq. (4.1) can be expressed in an alternate way which brings out clearly its kinematical significance. Indeed, by virtue of Eq. (3.6) the integrand on the right of Eq. (4.1) can be written

$$\frac{\partial F}{\partial t} + \operatorname{div}(\boldsymbol{v}F),$$

and then by application of the divergence theorem (2.2) we find

$$\frac{d}{dt}\int_{\tilde{V}} F dv = \frac{\partial}{\partial t}\int_{\tilde{V}} F dv + \oint_{\tilde{S}} F\boldsymbol{v}\cdot\boldsymbol{n} da .$$

Here $\tilde{S}$ is the surface of $\tilde{V}$, $\boldsymbol{v}\cdot\boldsymbol{n}$ is the component of $\boldsymbol{v}$ along the outward normal to $\tilde{S}$, and $\frac{\partial}{\partial t}$ denotes differentiation with $\tilde{V}$ held fixed. Eq. (4.2) expresses that the rate of change of the total F over a material volume $\tilde{V}$ equals the rate of change of the total F over the fixed volume instantaneously coinciding with $\tilde{V}$ plus the flux of F out of the bounding surface. It should be emphasized that Eqs. (4.1) and (4.2) express a **kinematical theorem**, independent of any meaning attached to F.

5. The equation of continuity.

We suppose that the fluid possesses a density function $\rho = \rho(\boldsymbol{x}, t)$, which serves by means of the formula

$$\tilde{M} = \int_{\tilde{V}} \rho dv \tag{5.1}$$

to determine the mass $\tilde{M}$ of fluid occupying a region $\tilde{V}$. We naturally assume $\rho > 0$, and assign to ρ the physical dimension "**mass per unit volume**".

Turning now to the physical significance of the concept of mass, we postulate the following **principle of conservation of mass**: *the mass of fluid in a material volume* $\tilde{V}$ *does not change as* $\tilde{V}$ *moves with the fluid*. The principle of conservation of mass is otherwise expressed by the statement

$$\frac{d}{dt}\int_{\tilde{V}} \rho dv = 0 . \tag{5.2}$$

Now from Eqs. (4.1) and (5.2) it follows easily that

$$\int_{\tilde{V}} \left(\frac{d\rho}{dt} + \rho \operatorname{div}\boldsymbol{v} \right) dv = 0 ,$$

and since $\tilde{V}$ is arbitrary this implies

$$\frac{d\rho}{dt} + \rho \operatorname{div}\boldsymbol{v} = 0 . \tag{5.3}$$

This is the **spatia**l, or **Eulerian**, **form of the equation of continuity** and is a necessary and sufficient condition for a motion to conserve the mass of each

moving volume. In virtue of Eq. (3.6) we can express the equation of continuity in the alternate form

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0 . \tag{5.4}$$

The derivation just given is substantially due to Euler.[6]

Multiplying Eq. (5.3) by J and using Eq. (3.8), we derive two forms of the material, or Lagrangian, equation of continuity:

$$\frac{d}{dt}(\rho J) = 0 , \quad \rho J = \rho_0 , \tag{5.5}$$

where $\rho_0 = \rho_0(\boldsymbol{X})$ is the initial density distribution.

The principle of conservation of mass is sometimes expressed in an equivalent form involving a fixed volume: the rate of change of mass in a fixed volume v is equal to the mass flux through its surface, i.e.,

$$\frac{\partial}{\partial t} \int_v \rho dv = -\oint_\sigma \rho \boldsymbol{v} \cdot \boldsymbol{n} da . \tag{5.6}$$

Applying the divergence theorem to the right hand side of Eq. (5.6) leads to

$$\int_v \left(\frac{\partial \rho}{\partial t} + div(\rho \boldsymbol{v}) \right) dv = 0 .$$

from which Eq. (5.4) is easily obtained. It is essentially this derivation which is found in most texts, but with application of the divergence theorem disguised in a discussion of the variation of $\rho \boldsymbol{v}$ 0 over a small box. The only objection to this derivation is that the principle of conservation of mass in its first form is more convincing.

We conclude this section with an important formula, valid for an arbitrary function $F = F(\boldsymbol{x}, t)$, namely

$$\frac{d}{dt} \int_{\tilde{V}} \rho F dv = \int_{\tilde{V}} \rho \frac{dF}{dt} dv . \tag{5.7}$$

Eq. (5.7) is an easy consequence of Eqs. (4.1) and (5.3).

6. The equations of motion.

We consider now the **dynamics** of fluid motion; our intention is to derive the equations which govern the action of forces, external and internal, upon the fluid. In this section we shall present what seems to be the most straight-forward and compelling treatment of this topic, stemming from the pioneer work of Euler and Cauchy.

[6] L. Euler: Principes généraux du mouvement des fluids. Hist. Acad. Berlin (1755) (Opera Omnia. II **12**, pp. 54 to 92). As early as 1751 Euler had corresponding ideas for incompressible fluids, but this material did not appear in published form until 1761.

We adopt the **stress principle** of Cauchy,[7] which states that "upon any imagined closed surface $\widetilde{S}$ there exists a distribution of **stress vectors *t*** whose resultant and moment are equivalent to those of the actual forces of material continuity exerted by the material outside $\widetilde{S}$ upon that inside".[8] It is assumed that $\boldsymbol{t}$ depends at any given time only on the position and the orientation of the surface element da; in other words, if $\boldsymbol{n}$ denotes the (outward) normal to $\widetilde{S}$, then $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{x}, t; \boldsymbol{n})$. As Truesdell remarks, the above principle "*has the simplicity of genius. Its profound originality can be grasped only when one realizes that a whole century of brilliant geometers had treated very special elastic problems in very complicated and sometimes incorrect ways without ever hitting upon this basic idea, which immediately became the foundation of the mechanics of distributed matter*".[9]

We now set forth the fundamental principle of the dynamics of fluid motion; the **principle of conservation of linear momentum**: ***the rate of change of linear momentum of a material volume*** $\widetilde{V}$ ***equals the resultant force on the volume***.[10] This principle is otherwise expressed by the statement

$$\frac{d}{dt}\int_{\widetilde{V}} \rho \boldsymbol{v} dv = \int_{\widetilde{V}} \rho \boldsymbol{f} dv + \oint_{\widetilde{S}} \boldsymbol{t} da \,, \tag{6.1}$$

where $\boldsymbol{f}$ is the **extraneous force per unit mass**. In setting down axiom (6.1) it is tacitly assumed that the force $\boldsymbol{f}$ is a **known function** of position and time, and perhaps also of the state of motion of the fluid. This point of view bypasses one of the prime problems in the foundations of mechanics, namely the **recognition**, and even the **existence**, **of a coordinate system** in which $\boldsymbol{f}$ is known. Of course, in the situations to which fluid mechanics is usually applied, an inertial frame is generally evident beforehand, and the **axiom** (6.1) is patently applicable. By means of Eq. (5.7), Eq. (6.1) may be written in the form

$$\int_{\widetilde{V}} \rho \frac{d\boldsymbol{v}}{dt} dv = \int_{\widetilde{V}} \rho \boldsymbol{f} dv + \oint_{\widetilde{S}} \boldsymbol{t} da \,; \tag{6.2}$$

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[7] A.-L. Cauchy: Ex. de Math. **2** (1827). (Oeuvres (2) **7**, pp.l79 to 81). A similar statement, but restricted to the case of perfect fluids, was given by Euler.

[8] This statement of Cauchy's principle is due to Truesdell, J. Rational Mech. Anal. **1**,125 (1952).

[9] C. Truesdell: Amer. Math. Monthly **60**, 445 (1953).

[10] The necessity for a clearcut statement of the postulates on which continuum mechanics rests was pointed out by Felix Klein and David Hilbert. The first axiomatic presentation is due to G. Hamel, Math. Ann. 66. 350 (1908); also [38], pp. 1 to 42. In a recent paper, W. Noll has developed the foundations of continuum mechanics at a level of rigor comparable to that of advanced mathematical analysis. It should be emphasized that the above postulate cannot be derived from classical mass-point mechanics by simple limiting processes; rather it is a plausible analogue of the basic equations of that subject.

here integration over a moving volume can be replaced, without loss of generality, by integration over a fixed volume.

From the form alone of Eq. (6.2) follows a result of great importance. Let l^3 be the volume of *v*; dividing both sides of (6.2) by l^2, letting *v* tend to zero, and noting that the integrands are bounded, we obtain

$$\lim_{v\to 0}\frac{1}{l^2}\oint_{S}\boldsymbol{t}da = 0\,, \tag{6.3}$$

that is, the stress forces are in local equilibrium. Consider the tetrahedron of Fig. 1, with vertex at an arbitrary point $\boldsymbol{x}$, and with three of its faces parallel to the

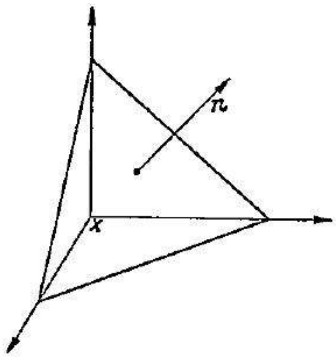

Fig. 1. Stress tetrahedron.

coordinate planes. Let the slanted face have normal $\boldsymbol{n}$ and area Σ. The normals to the other faces are $-\boldsymbol{i}$, $-\boldsymbol{j}$, and $-\boldsymbol{k}$, and their areas are $n_1\Sigma$, $n_2\Sigma$ and $n_3\Sigma$. Now let us apply Eq. (6.3) to the family of tetrahedrons obtained by letting $\Sigma \to 0$. Since $\boldsymbol{t}$ is a continuous function of position, and $l^2 \sim \Sigma$, we obtain easily

$$\boldsymbol{t}(\boldsymbol{n}) + n_1\boldsymbol{t}(-\boldsymbol{i}) + n_2\boldsymbol{t}(-\boldsymbol{j}) + n_3\boldsymbol{t}(-\boldsymbol{k}) = 0\,, \tag{6.4}$$

where $\boldsymbol{t}(\boldsymbol{n})$ is an abbreviation for $\boldsymbol{t}(\boldsymbol{x}, t; \boldsymbol{n})$. This formula has been proved, of course, only for the case when all the components n_i are positive. To extend its validity, we first note that by continuity it holds if all the n_i are ≥ 0. Thus, in particular,

$$\boldsymbol{t}(\boldsymbol{i}) = -\boldsymbol{t}(\boldsymbol{i})\,,\;\; \boldsymbol{t}(\boldsymbol{j}) = -\boldsymbol{t}(\boldsymbol{j})\,,\;\; \boldsymbol{t}(\boldsymbol{k}) = -\boldsymbol{t}(\boldsymbol{k})\,. \tag{6.5}$$

Now applying the "tetrahedron" argument in the other octants, and using Eq. (6.5), we find that, in all cases,

$$\boldsymbol{t}(\boldsymbol{n}) = n_1\boldsymbol{t}(\boldsymbol{i}) + n_2\boldsymbol{t}(\boldsymbol{j}) + n_3\boldsymbol{t}(\boldsymbol{k})\,. \tag{6.6}$$

t** may therefore be expressed as a linear function of components of* ***n, that is

$$t^i = n_j T^{ji} \quad \text{where} \quad T^{ji} = T^{ji}(\boldsymbol{x}, t)\,.$$

The matrix of coefficients T^{ij} obviously forms a tensor, called the stress tensor and here denoted by $\boldsymbol{T}$. Each component of $\boldsymbol{T}$ has a simple physical interpretation, namely, T^{ij} is the *j*-component of the force on the surface element with outer

normal in the i-direction. The foregoing argument is due in principle to Cauchy.[11]

Replacing t by $\boldsymbol{n}\cdot\boldsymbol{T}$ in (6.2) and applying the divergence theorem, we find

$$\int_v \rho\frac{d\boldsymbol{v}}{dt}dv = \int_v (\rho\boldsymbol{f} + \mathrm{div}\boldsymbol{T})dv\,,$$

and since v is arbitrary it follows that

$$\rho\frac{d\boldsymbol{v}}{dt} = \rho\boldsymbol{f} + \mathrm{div}\boldsymbol{T}\,. \tag{6.7}$$

This is the simple and elegant **equation of motion** discovered by Cauchy.[12] It is valid for any fluid, and indeed for any continuous medium, regardless of the form which the stress tensor may take.

Perfect fluids. All real fluids obviously can exert tangential stresses across surface elements, so that $\boldsymbol{t}$ generally will fail to be normal to the surface element on which it acts. The effect of the tangential stresses is small in many practical cases, however, and therefore it is not unreasonable to study the idealized situation in which the tangential stresses are neglected altogether. A **perfect fluid** is then by definition a material for which

$$\boldsymbol{t} = -p\boldsymbol{n}\,. \tag{6.8}$$

p is called the **pressure**: when $p > 0$, the vectors $\boldsymbol{t}$ acting on a closed surface tend to compress the fluid inside. Comparing Eqs. (6.6) and (6.8), we find $p(\boldsymbol{n}) = p(\boldsymbol{i}) = p(\boldsymbol{j}) = p(\boldsymbol{k})$. That is, p is independent: of $\boldsymbol{n}$,

$$p = p(\boldsymbol{x},t).$$

The equations of motion now take the simple form[13]

$$\rho\frac{d\boldsymbol{v}}{dt} = \rho\boldsymbol{f} - \mathrm{grad}p\,. \tag{6.9}$$

It is satisfying to note that we have obtained four equations, namely Eq. (5.3) and the three equations embodied in Eqs. (6.7) or (6.9), relating the four quantities ρ and the components of $\boldsymbol{v}$. To be sure, further variables $\boldsymbol{T}$ or p enter, but one may reasonably expect to express them in terms of ρ and $\boldsymbol{v}$ by direct mechanical or thermodynamical assumptions. The various possibilities for this form the material of the following chapters.

Material forms of the equations of motion. For the case of a perfect fluid it is relatively simple to find equations satisfied by $\boldsymbol{v}$, ρ, and p as functions of the variables X^α, t. Indeed, noting that $\frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{x}}{dt^2}$, and multiplying both sides of Eq. (6.9) by $x_{i,\alpha} \equiv x^i{}_{,\alpha}$, we obtain

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[11] A.-L. Cauchy: Ex. de Math. 2 (1827), (Oeuvres (2) 7, pp. 79 to 81).

[12] A.-L. Cauchy: Ex. de Math. 3 (1823), (Oeuvres (2) 8, pp. 195 to 226).

[13] L. Euler: Cf. footnote 9.

$$\left(\frac{d^2 x^i}{dt^2} - f^i\right) x_{i,\alpha} = -\frac{1}{\rho} p_{,\alpha}$$

which may be written vectorially as

$$\mathrm{Grad}\boldsymbol{x} \cdot \left(\frac{d^2 \boldsymbol{x}}{dt^2} - \boldsymbol{f}\right) = -\frac{1}{\rho} \mathrm{grad} p \,. \tag{6.10}$$

These equations are inconvenient to handle and infrequently used except for one dimensional flows. They are necessary, however, when one wishes to distinguish one article from another, as in the case of a non-homogeneous fluid. The material equations for fluids sucseptible of tangential stresses are extremely cumbersome and never seem to be used.[14]

7. Conservation of angular momentum

The principle of conservation of angular momentum is usually stated as a theorem in the classical dynamics of mass points or rigid bodies. Its proof, however, depends on certain axioms concerning the nature of the "inner forces" between the particles or bodies making up the dynamical system in question. The situation can be treated similarly in continuum mechanics.[15] Here, in order to guarantee the conservation of angular momentum it is necessary to make certain assumptions concerning the forces exerted across surface elements, or, in other words, concerning the stress tensor. Specifically, we postulate that the stress tensor is symmetric, i.e.,

$$T^{ij} = T^{ji} \,. \tag{7.1}$$

(When extraneous **couples** are present this needs modification. However, we specifically exclude extraneous couples from this study, since they arise generally only for polarized media and thus are not important in fluid mechanics.) As a theorem, Eqs. (7.1) are due to Cauchy[16]; that they can equally well serve as **axioms** was first recognized by Boltzmann.[17] As a consequence of Eqs. (7.1) the following result now holds:

Theorem (**conservation of angular momentum**). *For an arbitrary continuous medium satisfying the continuity equation* (5.3), *the dynamical equation* (6.7), *and the Boltzmann postulate* (7.1), *we have*

$$\frac{d}{dt}\int_{\tilde{V}} \rho(\boldsymbol{r} \times \boldsymbol{v}) dv = \int_{\tilde{V}} \rho(\boldsymbol{r} \times \boldsymbol{f}) dv + \oint_{\tilde{S}} \boldsymbol{r} \times \boldsymbol{t} da \,, \tag{7.2}$$

where $\tilde{V}$ *is an arbitrary material volume.*

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14 In non-linear elasticity, on the other hand, great importance is attached to the material form of the equation of motion.

15 The following presentation is similar to that of Hamel, [38], p. 9. A different point of view is adopted by Truesdell and Toupin (this Encyclopedia, Vol. III, Part 1), who postulate a generalized law of conservation of angular momentum in which extraneous torques are admitted.

16 A.-L. Cauchy: Cf. footnote 14.

Proof. From Eqs. (5.7) and (6.7) it is easy to show that

$$\frac{d}{dt}\int_{\tilde{V}}\rho(\boldsymbol{r}\times\boldsymbol{v})dv=\int_{\tilde{V}}\rho(\boldsymbol{r}\times\frac{d\boldsymbol{v}}{dt})dv$$
$$=\int_{\tilde{V}}\rho(\boldsymbol{r}\times\boldsymbol{f})dv+\oint_{\tilde{S}}\boldsymbol{r}\times\boldsymbol{t}da-\int_{\tilde{V}}\boldsymbol{T}_x dv.$$

Here $\boldsymbol{T}_x$ is the axial vector field defined by $(\boldsymbol{T}_x)^i = e^{ijk}T_{jk}$. By virtue of Eq. (7.1) we have $\boldsymbol{T}_x = 0$, and Eq. (7.2) is proved. Conversely, if Eq. (7.2) holds for arbitrary volumes then $\boldsymbol{T}$ must be symmetric.

For certain types of fluids the stress tensor turns out to be symmetric on purely mechanical grounds, irrespective of any other considerations. We mention in particular **perfect fluids**, where $\boldsymbol{T} = -p\boldsymbol{I}$, and isotropic viscous fluids in which **stress is a function of the rate of deformation** (Sect. 59). For these important cases, then, the **Boltzmann postulate is a tautology** and Eq. (7.2) can be obtained directly from the equations of motion.

It is possible to imagine a. mechanical system for which $\boldsymbol{T}$ is not symmetric, and Hamel, in the reference already cited, gives several examples. In cases of this sort, which are not of interest in fluid mechanics, the principle of conservation of momentum as given in Eq. (7.2) no longer holds, but must be generalized to allow for “apparent” extraneous torques.

8. Surface conditions.

If a surface in a moving fluid always consists of the same particles, it is clearly a possible bounding surface of the fluid. The converse proposition, namely that every bounding surface must be a material surface, is less obvious.

Suppose a fluid to be in continuous motion according to the conditions set down in Sect. 3, and let $F(\boldsymbol{x}, t) = 0$ be the equation of its boundary surface. Then F must satisfy the condition

$$\frac{dF}{dt}=\frac{\partial F}{\partial t}+\boldsymbol{v}\cdot\mathrm{grad}F=0 \quad \text{when } F=0, \tag{8.4}$$

(Kelvin[18]), and this condition in turn implies that the surface always consists of the same particles (Lagrange[19]).

Proof. It is well known that the normal velocity of a moving surface $F(\boldsymbol{x}, t) = 0$ is given by the formula

[17] Cf. [38], p. 9.
[18] W. Thomson (Lord Kelvin): Cambridge and Dublin Math. J. (1848). (Papers 1, p. 83).
[19] J. -L. Lagrange: Nouv. Mém. Acad. Sci. Berlin (1781), (Oeuvres 4, p. 706).

$$V = \frac{-\dfrac{\partial F}{\partial t}}{|\,\mathrm{grad} F\,|}.$$

But if $F = 0$ is a bounding surface, then

$$V = \boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{v} \cdot \frac{\mathrm{grad} F}{|\,\mathrm{grad} F\,|},$$

and Eq. (8.1) follows at once. On the other hand, if Eq. (8.1) holds, we wish to show that $F = 0$ always consists of the same particles. Set

$$G(\boldsymbol{X}, t) = F(\boldsymbol{\varphi}(\boldsymbol{X}, t), t),$$

so that $G(\boldsymbol{X}, t) = 0$ describes the initial positions of particles which at time t are on the surface $F = 0$. Clearly

$$\frac{\partial G}{\partial t} = 0 \quad \text{when } G = 0.$$

Therefore the normal velocity of propagation of the surface $G = 0$ through the $\boldsymbol{X}$-space is zero. It follows that $G = 0$ is fixed in the $\boldsymbol{X}$-space, and hence always the same particles make up the moving surface $F = 0$.

At a fixed boundary we have the obvious condition $\boldsymbol{v} \cdot \boldsymbol{n} = 0$, independent of the preceding analysis.

II. Energy and momentum transfer.

9. The energy transfer equation.

Let $\widetilde{T}$ denote the kinetic energy of a volume $\widetilde{V}$,

$$\widetilde{T} = \frac{1}{2} \int_{\widetilde{V}} \rho q^2 dv,$$

and let $\boldsymbol{D}$ be the deformation tensor, $D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$. Then for an arbitrary material volume $\widetilde{V}$ we have

$$\frac{d\widetilde{T}}{dt} = \int_{\widetilde{V}} \rho \boldsymbol{f} \cdot \boldsymbol{v} dv + \oint_{\widetilde{S}} \boldsymbol{t} \cdot \boldsymbol{v} da - \int_{\widetilde{V}} \boldsymbol{T} : \boldsymbol{D} dv. \tag{9.1}$$

The **proof** is a simple exercise in use of Eqs. (5.7), (6.7), and the symmetry of $\boldsymbol{T}$. Eq. (9.1) states that *the rate of change of kinetic energy of a moving volume is equal to the rate at which work is being done on the volume by external forces, diminished by a "dissipation" term involving the interaction of stress and deformation*. This latter term must represent the rate at which work is being done in changing the volume and shape of fluid elements. Part of the power connected with this term may well be recoverable, but the rest must be accounted for as heat.[20] For a perfect fluid the energy equation takes the simpler form

[20] See Sect. 34.

$$\frac{d\widetilde{T}}{dt} = \int_{\widetilde{V}} \rho \boldsymbol{f} \cdot \boldsymbol{v} dv - \oint_{\widetilde{S}} p\boldsymbol{v} \cdot \boldsymbol{n} da + \int_{\widetilde{V}} p \operatorname{div} \boldsymbol{v} dv . \qquad (9.2)$$

The last term is the rate at which work is done by the pressure in changing the volume of fluid elements.

A slight **simplification** of the energy equation may be effected if $\boldsymbol{f}$ is derivable from a time-independent potential; $\boldsymbol{f} = -\operatorname{grad} \Omega$, $\Omega = \Omega(\boldsymbol{x})$. In this case, setting $\widetilde{U} = \int_{\widetilde{V}} \rho \Omega dv$, Eq. (9.1) becomes

$$\frac{d}{dt}(\widetilde{T} + \widetilde{U}) = \oint_{\widetilde{S}} \boldsymbol{t} \cdot \boldsymbol{v} da - \int_{\widetilde{V}} \boldsymbol{T} : \boldsymbol{D} dv$$

10. The momentum transfer equation.

The principle of conservation of linear momentum, stated in Eq. (6.1), may be transformed by Eq. (4.2) into the form

$$\frac{\partial}{\partial t} \int_{v} \rho \boldsymbol{v} dv = \int_{v} \rho \boldsymbol{f} dv + \oint_{\sigma} (\boldsymbol{t} - \rho \boldsymbol{v}\boldsymbol{v} \cdot \boldsymbol{n}) da , \qquad (10.1)$$

expressing the rate of change of momentum of a fixed volume v. Because of the physical interpretation of the final term, Eq. (10.1) is known as the **momentum transfer equation**. Eq. (10.1) is sometimes used instead of Eq. (6.1) as the basic expression of the law of conservation of linear momentum.

The momentum transfer equation is often used to determine the force on an obstacle immersed in a steady flow. To illustrate this with a single example, suppose that the fluid occupies the entire exterior of some obstacle, and that the external force field is zero. Then if σ denotes the surface of the obstacle and Σ denotes a "control surface" enclosing σ, we have the following formula for the force $\boldsymbol{F}$ acting on the obstacle,

$$\boldsymbol{F} = -\int_{\sigma} \boldsymbol{t} da = \int_{\Sigma} (\boldsymbol{t} - \rho \boldsymbol{v}\boldsymbol{v} \cdot \boldsymbol{n}) da , \qquad (10.2)$$

(note that $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on σ). By an analogous argument proceeding from the Eq. (7.2) we find for the moment $\boldsymbol{L}$ on σ the formula

$$\boldsymbol{L} = \int_{\Sigma} \boldsymbol{r} \times (\boldsymbol{t} - \rho \boldsymbol{v}\boldsymbol{v} \cdot \boldsymbol{n}) dv .$$

Another force formula of a different type can be derived from the energy equation (9.1). Consider a rigid body moving with rectilinear velocity $\boldsymbol{U}$ through a fluid, the fluid being bounded externally by fixed walls. Let $\widetilde{V}$ denote the flow region, σ its external boundary, and σ_0 the surface of the moving body. Then

$$\int_{\sigma_0} \boldsymbol{t}\cdot\boldsymbol{v}da = \boldsymbol{U}\cdot\int_{\sigma_0} \boldsymbol{t}da \tag{10.3}$$

(for a perfect fluid this follows from the boundary condition $\boldsymbol{v}\cdot\boldsymbol{n} = \boldsymbol{U}\cdot\boldsymbol{n}$; for a viscous fluid it depends on the assumption $\mathbf{v} = \boldsymbol{U}$ on σ_0). Combining Eq. (10.3) with Eq. (9.1) gives

$$\boldsymbol{F}\cdot\boldsymbol{U} = \frac{d\widetilde{T}}{dt} + \int_{\widetilde{V}} \boldsymbol{T} : \boldsymbol{D}dv, \tag{10.4}$$

thus determining the component of $\boldsymbol{F}$ in the direction of motion. (The case where the flow region is infinite in extent can be handled similarly, given suitable asymptotic behavior of the flow at infinity. Further applications of the momentum principle will be found in [23], pp. 203 to 234, and in [12].)

11. Kinematics of deformation. The vorticity vector.

This subject is based upon a simple decomposition of the tensor grad$\boldsymbol{v}$, namely

$$\text{grad}\boldsymbol{v} = \boldsymbol{D} + \boldsymbol{\Omega}, \tag{11.1}$$

where

$$D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad \Omega_{ij} = \frac{1}{2}(v_{j,i} - v_{i,j}).$$

The tensors $\boldsymbol{D}$ and $\boldsymbol{\Omega}$ are respectively the symmetric and skew-symmetric parts of grad$\boldsymbol{v}$. The discussion is conveniently divided into two parts.

1. ***The deformation tensor.*** Let $d\boldsymbol{x}$ denote a material element of arc. Its rate of change during the fluid motion is given by the formula

$$\frac{d}{dt}(dx^i) = \frac{d}{dt}\left(\frac{\partial x^i}{\partial X^\alpha}dX^\alpha\right) = \frac{\partial v^i}{\partial X^\alpha}dX^\alpha = \frac{\partial v^i}{\partial x^j}dx^j,$$

or simply

$$\frac{d}{dt}(d\boldsymbol{x}) = d\boldsymbol{x}\cdot\text{grad}\boldsymbol{v}. \tag{11.2}$$

From Eq. (11.2) we have easily

$$\frac{d}{dt}(ds^2) = 2d\boldsymbol{x}\cdot\boldsymbol{D}\cdot d\boldsymbol{x},$$

where $ds = |d\boldsymbol{x}|$. The tensor $\boldsymbol{D}$ thus is a measure of the rate of change of the squared element of arc following a fluid motion. In a rigid motion ds = const, whence *a necessary and sufficient condition that a motion be locally and instantaneously rigid is that* $\boldsymbol{D}$ =0. For this reason, $\boldsymbol{D}$ is called the **deformation tensor**. The tensor $\boldsymbol{D}$ – 1/3 (Trace $\boldsymbol{D}$) $\boldsymbol{I}$ is also of interest, for its vanishing is the necessary and sufficient condition that the motion locally and instantaneously

preserves angles.

If $\boldsymbol{D}$ =0 everywhere in the fluid, the motion is rigid and

$$v = \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{r} + const\,, \tag{11.3}$$

where $\boldsymbol{\omega}$ is twice the (constant) **angular velocity** of the motion. Eq. (11.3) can also be derived analytically as the integral of the system of first order partial differential equations $\boldsymbol{D}$ =0.

2. ***General motion of a fluid.*** Let us consider the velocity field in the neighborhood of a fixed point P. Denoting the evaluation of a quantity at the point P by a subscript, we have near P,

$$\boldsymbol{v} = \boldsymbol{v}_P + \boldsymbol{r} \cdot (\mathrm{grad}\boldsymbol{v})_P + O(r^2)\,,$$

where $\boldsymbol{r}$ denotes the radius vector from P. Neglecting terms of order r^2 and using Eq. (11.1), we obtain

$$\boldsymbol{v} = \boldsymbol{v}_P + \boldsymbol{r} \cdot \boldsymbol{D}_P + \boldsymbol{r} \cdot \boldsymbol{\Omega}_P\,. \tag{11.4}$$

We must now interpret the various terms in this formula.

The first term on the right represents a **uniform translation** of velocity v_P. If we set $D = \boldsymbol{r} \cdot \boldsymbol{D}_P \cdot \boldsymbol{r}$, then the second term can be written in the form

$$\mathrm{grad}\frac{1}{2}D\,. \tag{11.5}$$

This term represents a **velocity field** normal at each point to the quadric surface D = const which passes through that point. In this velocity field there are three mutually perpendicular directions which are suffering no instantaneous rotation (the axes of strain). The principal (or eigen-) values of D measure the rates of extension per unit length of fluid elements in these directions.

The final term in Eq. (11.4) may be written

$$\frac{1}{2}\boldsymbol{\omega}_P \times \boldsymbol{r}\,, \tag{11.6}$$

where $\boldsymbol{\omega} = \mathrm{curl}\boldsymbol{v}$ is the **vorticity vector**. [The simplest way to verify Eq. (11.6) is to note that

$$\boldsymbol{\omega} = (\mathrm{grad}\boldsymbol{v})_x = \boldsymbol{\Omega}_x = 2(\Omega_{23}, \Omega_{31}, \Omega_{12})\,,$$

whence the components of Eq. (11.6) are equal to those of $\boldsymbol{r} \cdot \boldsymbol{\Omega}_P$.] The vector form of Eq. (11.6) shows clearly that the final term $\boldsymbol{r} \cdot \boldsymbol{\Omega}_P$ represents a **rigid rotation** of angular velocity $\frac{1}{2}\boldsymbol{\omega}_P$.

By combining the results of the two previous paragraphs, the identity (11.1) can be fully interpreted. For an arbitrary motion, the velocity $\boldsymbol{v}$ in the

neighborhood of a fixed point P is given, up to terms of order r^2, by

$$\boldsymbol{v} = \boldsymbol{v}_P + \mathrm{grad}\frac{1}{2}D + \frac{1}{2}\boldsymbol{\omega}_P \times \boldsymbol{r}\,, \tag{11.7}$$

where D = $\boldsymbol{r}\cdot\boldsymbol{D}\cdot\boldsymbol{r}$ is the rate of strain quadric and $\boldsymbol{\omega} = \mathrm{curl}\boldsymbol{v}$ is the vorticity vector: thus *an arbitrary instantaneous state of continuous motion is at each point the superposition of a uniform velocity of translation, a dilatation along three mutually perpendicular axes, and a rigid rotation of these axes*.[21] The angular velocity of the rotation is $\frac{1}{2}\boldsymbol{\omega}_P$. This result amply establishes that $\boldsymbol{\omega}$ represents the local and instantaneous rate of rotation of the fluid.

If **D** = 0 at a point it is apparent from Eq. (11.7) that the motion is locally and instantaneously a **rotation**, -while if **D** =*k***I** the motion is a **combination of pure expansion and rotation**. These results provide a verification of the statements of paragraph 1. On the other hand, if throughout a finite portion of fluid we have $\boldsymbol{\omega} = \boldsymbol{\Omega} = 0$, the relative motion of any element of that portion consists of a **pure deformation**, and is called "**irrotational**". In this case it can be shown that $\boldsymbol{v}$ is everywhere derivable from a **potential** ($\boldsymbol{v} = \mathrm{grad}\phi$), cf. [48], p. 101.

III. Transformation of coordinates.

12. Transformation of coordinates.

We shall here obtain the **equations of continuity and motion in a general curvilinear coordinate system**. For this purpose it is useful to employ the methods of elementary tensor analysis; the reader unfamiliar with this topic will find a lucid discussion in [47], or he may omit the entire section without serious detriment to the rest of the article. Let (x^1, x^2, x^3) be the coordinates of a point in a general curvilinear coordinate system. We set $\boldsymbol{x}$ = (x^1, x^2, x^3) as before, with the understanding, however that $\boldsymbol{x}$ is not a vector. The motion is still represented by equations of the form (3.1), stating the position of the particles at time *t*; for example, in cylindrical polar coordinates motion is represented by the equations

$$r = \chi(\boldsymbol{X}, t)\,,\quad \theta = \phi(\boldsymbol{X}, t)\,,\quad z = \psi(\boldsymbol{X}, t)\,.$$

It is easy to see that the derivatives $\frac{dx^i}{dt}$ of the functions (3.1) form the **contravariant component** of a vector, hence the velocity vector in curvilinear coordinates retains the. form $v^i = \frac{dx^i}{dt}$. We define the **material derivative** of a scalar, vector, or tensor function F by the formula

[21] A.-L. Cauchy: Ex. d'Anal. Phys. Math. **2** (1841), [Oeuvres (2) **12**, pp. 343 to 377]. G. Stokes: Trans.

$$\frac{\delta F}{\delta t} = \frac{\partial F}{\partial t} + v^i F_{,i} , \tag{12.1}$$

where the subscript comma denotes **covariant differentiation**. This definition is clearly consistent with the previous formula (3.6), and furthermore makes the material derivative a tensor quantity. It should be observed that the definition of the material derivative given in Sect. 3 is not generally valid in a curvilinear coordinate system, since for vector or tensor quantities F the expression $\frac{dF}{dt} = \frac{\partial F(\boldsymbol{X},t)}{\partial t}$ does not transform as a tensor. To establish the correct form for the **material derivative** in **material coordinates**, one can proceed as follows. Writing the covariant derivative

$$F_{,i} = \frac{dF}{dx^i} + A_i ,$$

where the A_i denote certain well known expressions involving the Christoffel symbols, we obtain from (12.1) the formula

$$\frac{\delta F}{\delta t} = \frac{\partial F}{\partial t} + v^i \left(\frac{\partial F}{\partial x_i} + A_i \right) = \frac{dF}{dt} + v^i A_i . \tag{12.1a}$$

Eq. (12.1a), which appears also in the theory of parallel translation in differential geometry, clearly shows the difference between $\frac{\delta F}{\delta t}$ and the more naive expression $\frac{dF}{dt}$. The reader should observe, however, that in rectangular coordinates $\frac{\delta F}{\delta t} = \frac{dF}{dt}$; in other words, just as the covariant derivative is the tensor extension of the- ordinary (Cartesian) derivative, so is $\frac{\delta F}{\delta t}$ an extension of $\frac{dF}{dt}$. Finally, it is evident that Eq. (12.1a) could serve as the starting point for the discussion of material derivative, rather than Eq. (12.1). At this point it is convenient to introduce vector notation, the definitions of Sect. 2 being carried over in the obvious way. For example, $\boldsymbol{v}$ will now denote the set of contravariant or covariant components of the velocity vector, whichever is appropriate, and Eq. (12.1) will be written

$$\frac{\delta F}{\delta t} = \frac{\partial F}{\partial t} + \boldsymbol{v} \cdot \operatorname{grad} F .$$

With these preliminaries taken care of, we see that the **equation of continuity** can be written in either: of the invariant forms,

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Cambridge Phil. Soc. **8** (1845). (Papers 1, pp. 75 to 129).

$$\frac{\delta\rho}{\delta t}+\rho\mathrm{div}\boldsymbol{v}=0 \quad \text{or} \quad \frac{\partial\rho}{\partial t}+\mathrm{div}(\rho\boldsymbol{v})=0\,, \tag{12.2}$$

where divergence has its usual tensorial meaning,

$$divb=b^i_{,j}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}b^i)\ .$$

Let the stress tensor be defined in a curvilinear coordinate system by means of its components in rectangular coordinates. Then the relation between the stress vector $\boldsymbol{t}$ and the surface normal $\boldsymbol{n}$ retains the form $\boldsymbol{t}=\boldsymbol{n}\cdot\boldsymbol{T}$, even though the components of **T** are no longer equal to the magnitudes of forces acting upon surface elements. Finally, the **equation of motion** has the invariant form

$$\rho\frac{\delta\boldsymbol{v}}{\delta t}=\rho\boldsymbol{f}+\mathrm{div}\boldsymbol{T}\,, \tag{12.3}$$

where

$$(\mathrm{div}\boldsymbol{T})_i=T^k_{i,k}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}T^k_i)-T^k_j\Gamma^j_{ik}\,. \tag{12.4}$$

It is useful to write out Eqs. (12.2) to (12.4) for an **orthogonal coordinate system**, where the line element has the special form

$$ds^2=(h_1dx^1)^2+(h_2dx^2)^2+(h_3dx^3)^2\,. \tag{12.5}$$

The **equation of continuity** becomes simply

$$\frac{\partial\rho}{\partial t}+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}\rho v^i)=0\,,\quad \sqrt{g}=h_1h_2hl3\,. \tag{12.6}$$

In order to write out Eq. (12.3) we first observe that

$$\boldsymbol{a}=\frac{\delta\boldsymbol{v}}{\delta t}=\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{v}\times\boldsymbol{\omega}+\mathrm{grad}\frac{1}{2}q^2\,, \tag{12.7}$$

[cf. Eq. (17.1)], whence the acceleration can easily be written down in terms of $\boldsymbol{v}$ and $\boldsymbol{\omega}$. The latter is given by the formula

$$\omega^i=\frac{e^{ijk}}{\sqrt{g}}v_{k,j}=\frac{e^{ijk}}{\sqrt{g}}\frac{\partial v_k}{\partial x^j}\,, \tag{12.8}$$

using the fact that $\Gamma^l_{jk}=\Gamma^l_{kj}$. The term div**T** requires more effort because of the fairly complicated form of Eq. (12.4). The Christoffel symbols corresponding to the metric (12.5) are given by

$$\Gamma^i_{ik}=\Gamma^i_{ki}=\frac{1}{h_i}\frac{\partial h_i}{\partial x^k}\,,\quad \Gamma^k_{ii}=-\frac{h_i}{{h_k}^2}\frac{\partial h_i}{\partial x^k}\quad (i\neq k\text{), all others zero,}$$

(i and k unsumrned). Thus after a straightforward calculation,

$$(\mathrm{div}\boldsymbol{T})_i=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}T^k_i)-T^k_k\frac{\partial\log h_k}{\partial x^i}\,, \tag{12.9}$$

(summed on k). The reader should note that this formula is not needed in the case

of a perfect fluid, while for a viscous fluid obeying the Cauchy-Poisson law (Sect. 61) it is usually simpler to obtain the equations of motion without first determining div**T**.

Another method for computing the acceleration may be had from the formula

$$a_i = \frac{\partial v_i}{\partial t} + v^k \left(\frac{\partial v_i}{\partial x^k} - v_k \frac{\partial \log h_k}{\partial x^i} \right), \tag{12.10}$$

proved by the same calculation which led to Eq. (12.9).

In practice, rather than using the covariant or contravariant components of a vector ***b***, it is convenient to use its physical components β_i, defined by

$$\beta_i = h_i b^i = \frac{1}{h_i} b_i \quad (i \text{ unsummed});$$

thus β_i is the magnitude of the projection of ***b*** on the *i*-curve through the point of action of ***b***. The physica.1 components of tensors are similarly defined, but they will not be needed here.

Example: ***cylindrical polar coordinates***. We have in this case

$$ds^2 = dr^2 + (rd\theta)^2 + dz^2 .$$

Letting v_r, v_θ and v_z be the respective physical components of velocity, the equation of continuity (12.6) takes the form

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\left[\frac{\partial}{\partial r}(\rho r v_r) + \frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho r v_z) \right] = 0. \tag{12.11}$$

The acceleration terms in the equation of motion are, from Eq. (12.7) or from Eq. (12.10)

$$\begin{cases} a_r = Dv_r - \dfrac{{v_\theta}^2}{r} \\ a_\theta = Dv_\theta + \dfrac{v_r v_\theta}{r} \\ a_z = Dv_z \end{cases},$$

$$D = \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} .$$

The physical components of div**T** are given in Love's treatise[22] and need not be reproduced here. Finally, the vorticity vector is given by

[22] A. E. H. Love: A Treatise on the Mathematical Theory of Elasticity, 4th edit. Cambridge 1927. See p. 90.

$$\begin{cases} \omega_r = \dfrac{1}{r}\dfrac{\partial v_x}{\partial \theta} - \dfrac{\partial v_\theta}{\partial z} \\ \omega_\theta = \dfrac{\partial v_r}{\partial z} - \dfrac{\partial v_z}{\partial r} \\ \omega_z = \dfrac{\partial v_\theta}{\partial r} - \dfrac{1}{r}\dfrac{\partial v_r}{\partial \theta} + \dfrac{v_\theta}{r} \end{cases} . \tag{12.12}$$

13. Riemannian space.

It may be of interest to consider the nature of the hydrodynamical equations in a Riemannian space given the line element

$$ds^2 = g_{ij} dx^i dx^j$$

in some coordinate system $\boldsymbol{x} = (x^1, \ldots, x^n)$. It is generally not possible to introduce a set of rectangular coordinates, so that one cannot derive suitable "equations of motion" merely by carrying out the steps of the previous work.

Motion in a Riemannian space is represented by a transformation of the form (3.1), although now i runs from 1 to n. We define the velocity vector by $v^i = \dfrac{dx^i}{dt}$, and the material derivative by

$$\frac{\delta F}{\delta t} = \frac{\partial F}{\partial t} + v^i F_{,i} .$$

(This definition is in analogy to the one used in Euclidean space, and also has the property that, should the space be **embedded** in a higher dimensional Euclidean space, as for example a surface in three space, then the material derivative is the surface component of the "natural" material derivative of Euclidean space.)

The equation of continuity is easily derived by the method of Sects. 4 and 5. In this procedure we must replace Eq. (4.2) with

$$\int_{\tilde{V}} \rho(\boldsymbol{x}, t) dV = \int_{\tilde{V}_0} \rho(\boldsymbol{X}, t) \sqrt{g} J dv_0$$

and then make use of the formula

$$\frac{\delta}{\delta t}(\sqrt{g} J) = \sqrt{g} J \operatorname{div} \boldsymbol{v} ,$$

which follows readily from Eq. (3.7). In other respects the argument is exactly as before, the final result being

$$\frac{\delta \rho}{\delta t} + \rho \operatorname{div} \boldsymbol{v} = 0 ,$$

which is exactly the same as Eq. (12.2), but obtained now without recourse to rectangular coordinates.

Deriving appropriate equations of motion involves dynamical considerations

which do not seem adapted to Riemannian space; in particular, it is not evident how to formulate the principle of conservation of momentum. On the other hand, there seems to be no valid objection to taking Eq. (12.3) as a postulate. This done, further considerations will closely parallel corresponding results of ordinary hydrodynamics.

IV. Variational principles.

The wide scope and great success of variational principles in classical dynamics have stimulated many efforts to formulate the laws of continuum mechanics in a similar way. In the following section we shall discuss some of these formulations; the work applies generally to all continuous media, though it is stated only for the motion of fluids. In Sect. 15 we consider some special variational principles which apply to perfect fluids.

14. General fluids.

The variational principle appropriate to a given dissipative system takes a form exactly suited to and dependent on the particular mechanism of dissipation, and is generally not capable of extension in unchanged form to other problems. This fact makes it easy to formulate a variational principle for fluids, but also indicates something of the *a posferiori* nature of the undertaking. The reader will observe that the appropriate variational principle is little more than a reformulation of the equations of motion; it may, however, provide methods for handling constraints otherwise beyond the scope of the original equations.

Let $\delta \boldsymbol{x} = \boldsymbol{\eta}(\boldsymbol{x}, t)$ be a **virtual displacement** of the particles of fluid from their instantaneous position. The vector function $\boldsymbol{\eta}$ is assumed to be finite valued and continuously differentiable; moreover it should conform to any restrictions placed on the fluid position. This latter condition implies, in particular, that $\boldsymbol{\eta}$ should be tangent to any wall bounding the fluid. The virtual work corresponding to a virtual displacement is defined by

$$\delta\widetilde{U} = \delta\widetilde{U}_c - \int_{\widetilde{V}} \boldsymbol{T} : \operatorname{grad} \delta\boldsymbol{x}\, dv\,,$$

where $\widetilde{V}$ is the volume occupied by the fluid, T is a tensor function of position, and

$$\delta\widetilde{U}_c = \int_{\widetilde{V}} \rho \boldsymbol{f} \cdot \delta\boldsymbol{x}\, dv + \oint_{\widetilde{S}} \boldsymbol{t} \cdot \delta\boldsymbol{x}\, da \tag{14.1}$$

is the virtual work done against extraneous force $\boldsymbol{f}$ and surface stresses $\boldsymbol{t}$. The second term in the definition of $\delta\widetilde{U}$ is peculiar to continuum mechanics: it reflects the common observation that deformations of a fluid medium generally

require the expenditure of work against stress forces. We need not assume that $\boldsymbol{T}$ is symmetric, but otherwise a rigid virtual displacement will produce virtual work of deformation. For this reason, it is usual to consider only symmetric stresses $\boldsymbol{T}$. We may now state the fundamental **d'Alembert-Lagrange variational principle**: *A fluid moves in such at way that*

$$\delta\widetilde{U} - \int_{\widetilde{V}} \rho\boldsymbol{a} \cdot \delta\boldsymbol{x} dv = 0\,, \tag{14.2}$$

for all virtual displacements which satisfy the given kinematical conditions.[23] If there are no constraints on the motion, except for wall conditions, it follows in a well known way that

$$\rho\boldsymbol{a} = \rho\boldsymbol{f} + \mathrm{div}\boldsymbol{T} \quad \text{and} \quad \boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{T}\ . \tag{14.3}$$

The first equation holds at all interior points of the motion, the second at "free" surfaces. These are of course just the equations of motion already derived.

Fluid motions on surfaces, or subject to other sorts of constraints, can be handled by the usual techniques of the calculus of variations. The interested reader should consult Hellinger's article in the Encyclopaedia of Mathematical Sciences, in particular §§ 3e, 4c, and 8b.

The **d'Alernbert-Lagrange principle** may be expressed equivalently in the form of **Hamilton's principle**. This is obtained by letting the virtual displacements arise from variations in the paths of the particles. Thus let a set of varied paths be given by $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t; \varepsilon)$, where -1< ε <l, say, and the path $\varepsilon = 0$ is the one to be investigated. If

$$\delta \equiv \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0},$$

then the virtual displacement corresponding to a varied motion is defined by

$$\delta\boldsymbol{x} = \delta\boldsymbol{\varphi} = \left.\frac{d\boldsymbol{\varphi}}{d\varepsilon}\right|_{\varepsilon=0}.$$

We have now the following identity

$$\boldsymbol{a} \cdot \delta\boldsymbol{x} = \frac{d}{dt}(\boldsymbol{v} \cdot \delta\boldsymbol{x}) - \boldsymbol{v} \cdot \frac{d\delta\boldsymbol{x}}{dt} = \frac{d}{dt}(\boldsymbol{v} \cdot \delta\boldsymbol{x}) - \delta\frac{1}{2}q^2\,, \tag{14.4}$$

since δ and d obviously commute. The density of the varied motions is determined by the condition that the mass of fluid corresponding to an arbitrary set of particles shall be the same wherever the particles may be. Mathematically

[23] The statical equivalent of Eq. (14.2), namely that a. continuous medium will be in equilibrium if and only if $\delta\widetilde{U} = 0$ for all virtual displacements, is due to Lagrange (Mécan. Anal. **1** part. Sect. IV. § 1). The extension of this principle to dynamical systems was likewise given by Lagrange, the fundamental idea in his derivation being the application of d'Alembert's principle to the equilibrium condition $\delta\widetilde{U} = 0$ (Mécan. Anal. 2e parts. Sects. I, II}. See the articles of P. Voss (Ency. Math. Wiss. 4, No. 1) and E. Hellinger (Ency. Math. Wiss. 4, No. 30).

this leads to the “continuity condition”

$$\delta\rho = -\rho \operatorname{div} \delta\boldsymbol{x} \tag{14.5}$$

governing the variation of density. To prove Eq. (14.5) we observe that $\delta\boldsymbol{x}$ is the initial velocity in a motion for which ε plays the role of time; thus to obtain Eq. (14.5) we simply replace $\frac{d}{dt}$ and $\boldsymbol{v}$ in the equation of continuity by δ and $\delta\boldsymbol{x}$, respectively. The same reasoning also proves the formula

$$\delta \int \rho F dv = \int \rho \delta F dv . \tag{14.6}$$

Condition (14.5) is also the consequence of assuming, (i) that each varied motion satisfies the equation of continuity, and (ii) that the virtual displacement vanishes at some fixed time. If Eq. (14.4) is multiplied by ρ and integrated over a material volume $\widetilde{V}$, application of formulas (5.7) and (14.6) yields

$$\int_{\widetilde{V}} \rho \boldsymbol{a} \cdot \delta\boldsymbol{x} dv = \frac{d}{dt} \int_{\widetilde{V}} \rho \boldsymbol{v} \cdot \delta\boldsymbol{x} dv - \delta\widetilde{T} , \tag{14.7}$$

where

$$\widetilde{T} = \frac{1}{2} \int_{\widetilde{V}} \rho q^2 dv = \text{kinetic energy.}$$

Finally, by virtue of the d’Alembert-Lagrange principle, Eq. (14.7) can be written in the form

$$\delta\widetilde{T} + \delta\widetilde{U} - \frac{d}{dt} \int_{\widetilde{V}} \rho v \cdot \delta x dv = 0 . \tag{14.8}$$

This equation holds under the condition that the varied motions satisfy the continuity condition (14.5) and conform to external constraints. If Eq. (14.8) is integrated from t_0 to t_1, and if $\delta\boldsymbol{x}$ vanishes at t_0 and t_1, we obtain the so-called **Hamilton's principle**[24]

$$\int_{t_0}^{t_1} (\delta\widetilde{T} + \delta\widetilde{U}) dt = 0 ;$$

each varied motion must satisfy the equation of continuity and external constraints, as well as having $\delta\boldsymbol{x} = 0$ at t_0 and t_1.[25]

15. Perfect fluids.

For an **incompressible perfect** fluid the **d’Alernbert-Lagrange principle** can be formulated in a more elegant fashion, namely, *an incompressible perfect*

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[24] Cf. E. Hellinger: Ency. Math. Wiss. **4**, footnote 61.

[25] Other variational principles which may be mentioned are the principle of least time (Hellinger, §5c) and an interesting energy principle of J. W. Herivel [Proc. Roy. Irish Acad. **56**, 37, 67 (1954)]. Cf. also E. Hoelder: Ber. sachs. Akad. Wiss. (Lpz.), Math-phys. Kl. 97 (1950).

fluid moves in such a way that

$$\delta\widetilde{U}_c - \int_{\widetilde{V}} a \cdot \delta x dv = 0 \qquad (15.1)$$

for all virtual displacements $\delta\boldsymbol{x}$ *which preserve the volume*, or, in other words, *satisfy* $\operatorname{div}\delta\boldsymbol{x} = 0$. T he virtual work $\delta\widetilde{U}_c$ is defined by Eq. (14.1).

According to the theory of **Lagrange multipliers**, this is equivalent to

$$\int_{\widetilde{V}} [\rho(\boldsymbol{a} - \boldsymbol{f}) \cdot \delta\boldsymbol{x} - \lambda \operatorname{div}\delta\boldsymbol{x}] dv - \oint_{\widetilde{S}} \boldsymbol{t} \cdot \delta\boldsymbol{x} da = 0,$$

where λ is a Lagrange multiplier and $\delta\boldsymbol{x}$ is subjected to no side conditions. It follows from an integration by parts that

$$\rho\boldsymbol{a} = \rho\boldsymbol{f} - \operatorname{grad}\lambda \quad \text{and} \quad \boldsymbol{t} = -\lambda\boldsymbol{n}. \qquad (15.2)$$

λ thus becomes the "pressure", one of the principal unknowns of the problem. Eqs. (15.2) together with the continuity condition div$\boldsymbol{v}$ = 0 constitute four equations for the four unknowns $\boldsymbol{v}$ and λ.

For the general case of a compressible perfect fluid, Lagrange took Eq. (15.2) to be the correct equation, where λ is to be considered a "reaction" against the volume changes which are, of course, now permitted.[26] This derivation of a general case from a particular one - by retaining the old equation, but considering the Lagrange multiplier as a new "force of reaction" – Hamel calls the "Lagrange freeing principle". He notes further that the reaction is to depend precisely on the compressibility (i.e., the density) which was before not allowed to vary. This procedure, although. interesting and leading to a correct result, is not entirely convincing - one difficulty becomes evident in the case of gas, where the pressure is a definite thermodynamical variable.

The variational principle (15.1) may be written **in the form of Hamilton's principle** by means of identity (14.5). Thus we have the result: *an incompressible perfect fluid moves in such a way that*

$$\int_{t_0}^{t_1} (\delta\widetilde{T} + \delta\widetilde{U}_c) dt = 0$$

for all variations $\delta\boldsymbol{x}$ *of the motion satisfying*

$$\operatorname{div}\delta\boldsymbol{x} = 0 \quad \text{and} \quad \delta\boldsymbol{x} = 0 \quad \text{at} \quad t = t_0, t_1.$$

Lichtenstein[27] has obtained a similar variational principle for the motion of **compressible** perfect fluids. A certain artificiality in his formulation was noticed

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[26] Cf. [6], pp. 473, 522. A similar method was used by G. Piola [Modena Mem. **24** 1 (1848)] to derive the general equations of continuum mechanics.

[27] L. Lichtenstein [9], Chap. 9.

by Taub[28], who substituted an alternative procedure; the most satisfying form of the principle is, however, due to Herivel[29], and in the following discussion we shall use the latter's formulation.

We begin with the remark that, for a mechanical system whose energy is completely known it should be possible to state Hamilton's principle in the form

$$\int_{t_0}^{t_1} (\delta\widetilde{L} + \delta\widetilde{U}_c)dt = 0\,, \tag{15.3}$$

where the **Lagrangian function** $\widetilde{L}$ is the difference of the kinetic and potential energies. An essential difference between the principle (15.3) and those stated earlier is that (15.3) can be written without a priori knowledge of the equations of motion. Thus this principle provides a way of deriving the equations of motion by a method which is genuinely independent of momentum considerations. Let us apply this to the case of a **gas**.

We suppose the motion takes place without loss of energy through the generation of transfer of heat, or, more precisely, that the specific entropy S of each fluid particle remains constant during the motion,[30]

$$\frac{dS}{dt} = 0\,. \tag{15.4}$$

In this» e of motion the energy is completely known, having the form $\widetilde{T} + \widetilde{I}$, where $\widetilde{T}$ is the **kinetic energy** and $\widetilde{I}$ the **internal energy** of the volume of fluid considered,

$$\widetilde{I} = \int_{\widetilde{V}} \rho E dv\,, \quad E = E(\rho, S) = \text{specific internal energy.}$$

There seems only one reasonable choice for the Lagrangian function, namely $\widetilde{L} = \widetilde{I} - \widetilde{E}$. For this $\widetilde{L}$ we shall now show that Eq. (15.3) leads to the correct equations of motion for a **compressible perfect fluid**.

Let $\delta\boldsymbol{x} = \delta\boldsymbol{x}(\boldsymbol{X}, t)$ be a variation of the path, vanishing at t_0 and t_1. Assuming that the varied motions satisfy the equation of continuity, the variation of density is given by Eq. (14.5). By the same arguments, the variation of entropy must satisfy

$$\delta S = 0\,.$$

From Eqs. (14.6) and (14.5), and since $\left(\frac{\partial E}{\partial \rho}\right)_S = \frac{p}{\rho^2}$, there follows

[28] A. H. Taub [44]. p. 148.

[29] J. W. Herivel: Proc. Cambridge Phil. Soc. **51**, 344 (1955).

[30] The thermodynamical basis for the following work will be found in Sect. 30 and in the first paragraph of Sect. 33.

$$\begin{aligned}\delta\tilde{E} &= \int_{\tilde{V}} \rho\delta E dv = -\int_{\tilde{V}} p \operatorname{div}\delta\boldsymbol{x} dv \\ &= \int_{\tilde{V}} \delta\boldsymbol{x}\cdot \operatorname{grad} p dv - \oint_{\tilde{S}} p\boldsymbol{n}\cdot\delta\boldsymbol{x} da\end{aligned}.$$

$\delta\tilde{T}$ is evaluated by means of Eq. (14.7). We may now conclude in the usual way from Eq. (15.3) and the formulae for $\delta\tilde{T}$, $\delta\tilde{E}$, and $\delta\tilde{U}_c$, that

$$\rho\boldsymbol{a} = \rho\boldsymbol{f} - \operatorname{grad} p \quad \text{and} \quad \boldsymbol{t} = -p\boldsymbol{n}.$$

These are of course the correct equations.[31] We emphasize again that they have been derived from a principle whose statement involved no *a priori* knowledge of their form. This is in contrast to the earlier principle (14.2) and the derivation from it of Eqs. (14.3).

In theoretical mechanics the **energy equation** is a consequence of Hamilton's principle. It is interesting to see that this is also true in the present case. For since

$$\frac{d\tilde{E}}{dt} = \int_{\tilde{V}} \rho \frac{dE}{dt} dv = \int_{\tilde{V}} p \operatorname{div}\boldsymbol{v} dv,$$

we have from Eq. (9.2),

$$\frac{d}{dt}(\tilde{T} + \tilde{E}) = \int_{\tilde{V}} \rho\boldsymbol{f}\cdot\boldsymbol{v} dv + \oint t\cdot\boldsymbol{v} dv,$$

which is the usual statement of **conservation of energy** for a non-heat-conducting media.

In the paper already referred to, Herivel attempted to find the equations of perfect fluids on a variational principle of spatial (Eulerian) type. He was not entirely successful, in that his principle yields as extremals only a subset of the class of flows satisfying the Euler equations. This difficulty was first pointed out by C. C. Lin, who then supplied a correct version of the principle[32]. Consider, in particular, the variational principle,

$$\delta\iint L(v,\rho,S) dv dt = 0, \tag{15.5}$$

where L is the Lagrangian density

$$L = \frac{1}{2}\rho q^2 - \rho(E + \Omega).$$

and the variations of the velocity, density, and entropy are subject to the following constraints,

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[31] The preceding derivation is based on that in Herivel's paper, with, however, certain modifications in the formulation and proof.

[32] Herivel's principle included only the first pair of constraints in Eq. (15.6), the final constraint being due to C. C. Lin (unpublished). Without this additional constraint, isentropic flows could appear as extremals only if they were also irrotational [see Eq. (15.7)].

Conservation of mass: $\dfrac{\partial\rho}{\partial t}+\mathrm{div}(\rho\boldsymbol{v})=0$,

Conservation of energy: $\dfrac{dS}{dt}=0$,

Conservation of the identity of particles: $\dfrac{d\boldsymbol{X}}{dt}=0$, (15.6)

where the vector field $\boldsymbol{X}(\boldsymbol{X}$, t) establishes the initial position of the particle which occupies the position $\boldsymbol{x}$ at time t. We shall now verify that every extremal of the variational principle (15.5) is a flow (Herivel-Lin)[33].

Upon introduction of the Lagrange multipliers $\phi,\beta,\boldsymbol{\gamma}$ the above principle becomes

$$\delta\int\int\left\{L+\phi\left(\frac{\partial\rho}{\partial t}+\mathrm{div}(\rho\boldsymbol{v})\right)-\rho\beta\frac{dS}{dt}-\rho\boldsymbol{\gamma}\cdot\frac{d\boldsymbol{X}}{dt}\right\}dvdt=0,$$

where v, ρ, S and $\boldsymbol{X}$ are now to be varied without restrictions. The separate variations of these quantities now give the following equations

$$\delta v:\qquad \boldsymbol{v}=\mathrm{grad}\phi+\beta\mathrm{grad}S+\mathrm{grad}\boldsymbol{X}\cdot\boldsymbol{\gamma}$$

$$\delta\rho:\qquad \frac{d\phi}{dt}=\frac{1}{2}q^2-I-\Omega,$$

$$\delta S:\qquad \frac{d\beta}{dt}=\left(\frac{\partial E}{\partial S}\right)_\rho=T,$$

$$\delta\boldsymbol{X}:\qquad \frac{d\boldsymbol{\gamma}}{dt}=0. \tag{15.7}$$

With the help of Eqs. (15.6) and (15.6) these equations can be shown to imply Eq. (6.9). Indeed, if we write Eq. (15.7) in the form $\boldsymbol{v}=\sum_x\xi_x\mathrm{grad}\eta_x$, then a straightforward calculation based on Eqs. (3.5) and (3 .6) yields the acceleration formula

$$\boldsymbol{a}+\mathrm{grad}\frac{1}{2}q^2=\sum\left(\xi_x\mathrm{grad}\frac{d\eta_x}{dt}+\frac{d\xi_x}{dt}\mathrm{grad}\eta_x\right). \tag{15.8}$$

But $\dfrac{dS}{dt}=\dfrac{d\boldsymbol{X}}{dt}=\dfrac{d\boldsymbol{\gamma}}{dt}=0$, whence

$$\boldsymbol{a}=-\mathrm{grad}\frac{1}{2}q^2+\mathrm{grad}\frac{d\phi}{dt}+\frac{d\beta}{dt}\mathrm{grad}\mathrm{S}=-\mathrm{grad}\Omega-\frac{1}{\rho}\mathrm{grad}p,$$

where we have used the simple thermodynamic identity $TdS=dI-\dfrac{1}{\rho}dp$.

To complete the discussion, it must still be shown that every flow is an extremal for the Herivel-Lin principle Eq. (15.5) to (15.6). This has been done by

[33] Preliminary results of a similar kind are due to A. Clebsch, J. reine angew. Math. **54**, 293 (1857);

the author of the present article (see Sect. 29A).

It is likely that one can derive the equations of motion for a **viscous fluid** by a variational argument similar to Herivel's. The essential point to be observed is that the energy equation must be postulated as a **side condition** [in Herivel's work, for example, this is reflected in the condition (15.4)]. Without this or some equivalent side condition, it does not appear possible to obtain the equations of motion of a viscous fluid from Hamilton's principle. Thus Millikan[34] has shown that a principle of the type $\delta\int L dv = 0$ where L is a function only of $\boldsymbol{v}$ and grad $\boldsymbol{v}$, cannot represent the steady motion of a viscous incompressible fluid except in certain special cases, namely those investigated in Sect. 75 of this article.[35]

Other variational principles. In addition to the fundamental principles already discussed, there are numerous variational formulations of special. problems in fluid dynamics. At the appropriate place we shall mention some of of these special principles, e.g., Kelvin's minimum energy theorem (Sect. 24), Bateman's principle (Sect-47), the theorem of Helmholtz and Rayleigh (Sect. 75), etc.

56, 1 (1859); and to H. Bateman, Proc. Roy. Soc. Lond., Ser. A **125**, 598 (1929).

[34] C. Millikan: Phil. Mag. (7) **7**, 641 (1929).

[35] Other negative results concerning variational principles yielding the Navier-Stokes equation are due to R. Gerber, Ann. Inst. Fourier (Grenoble) **1**, 157 (1950); J. Math. Pure Appl. **32**, 79 (1950). Cf. also H. lBateman: Phys. Rev. (2) **38**, 815 (1931).