C. Incompressible and barotropic perfect fluids.

I. General principles.

16. Preliminary discussion.

We shall begin our detailed considerations of fluid flow with the special but highly important case of **perfect fluids**. Here the stress vector has the simple form **t** = - $p\boldsymbol{n}$, and we have the following equations governing the motion,

$$\frac{d\rho}{dt} + \rho \operatorname{div}\boldsymbol{v} = 0\,, \qquad (16.1)$$

$$\rho\frac{d\boldsymbol{v}}{dt} = \rho\boldsymbol{f} - \operatorname{grad}p\,. \qquad (16.2)$$

In general, one may adjoin to these four equations a fifth (thermodynamical) relation

$$p = p(\rho, T)\,, \qquad (16.3\}$$

where T denotes the absolute temperature. Discussion of this situation is appropriately deferred to the following chapters, while here we consider the elegant theory arising when the pressure and density are **directly related**:

$$p = p(\rho) \quad \text{or} \quad \rho = g(p)\,. \qquad (16.4)$$

A flow in which density and pressure are thus related is called **barotropic**. We observe that Eq. (16.4) may arise from special circumstances in the flow considered, or it may be an inherent property of the fluid itself. In the latter case the fluid is called **piezotropic**; (the distinction between barotropic flow and piezotropic fluid is clarified if we note that every flow of a piezotropic fluid is barotropic, while the converse is not tune, cf. examples below). The special piezotropic fluids for which $\rho = const$ are called **incompressible**.

The following examples of **barotropic** flow may be noted:

1. Air in steady motion in the Mach number range 0 to 0.4. There is less than 8% overall variation of density in this range of Mach numbers, so that for many purposes the density can be supposed to have some appropriate constant value.

2. A gas in isentropic motion. For the case of an ideal gas with constant specific heats we have, in particular,

$$p = N\rho^{\gamma}\,, \quad N, \gamma = const\,.$$

We shall assume in this chapter that the extraneous force $\boldsymbol{f}$ is conservative, $\boldsymbol{f} = -\operatorname{grad}\Omega$, and all results will be stated subject to this condition. It is worthwhile to point out that no further axioms of motion are necessary for the conclusions of this chapter.

The fundamental property which distinguishes barotropic motion is the simple formula of Euler,

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = -\mathrm{grad}\left(\int \frac{dp}{\rho} + \Omega\right), \qquad (16.5)$$

which shows that acceleration is derivable from a potential. The results of this chapter are largely due to the simplifying effect of this single equation.

Plane motion. Axially-symmetric motion. Vector-lines. We conclude this section with a brief summary of these concepts, mainly in order to fix upon a standard terminology.

A motion is called a **plane** flow if, in some rectangular coordinate system $\boldsymbol{x}$ = (x, y, z), the velocities $u = v^1$, $v = v^2$ are functions of x, y only, while $v^3 = 0$. The motion takes place in a series of planes parallel to x y, and is the same in each one. For this reason our attention can be directed entirely at the single plane z = 0. A motion is said to be **axially-symmetric** if, in some cylindrical polar coordinate system $\boldsymbol{x} = (x, y, \theta)$ [1] the velocities at $u = v^1$, $v = v^2$ are functions of x, y only, while $v^3 = 0$. It is obvious that our attention can be confined to the meridian half-plane $\theta = 0$.

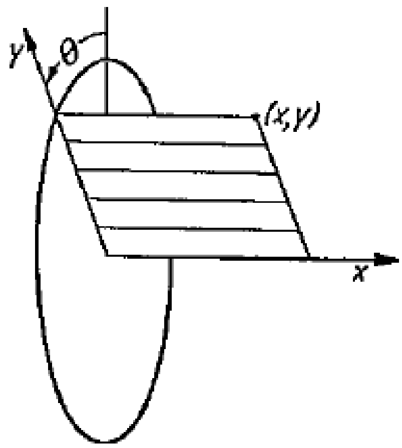

Fig. 2. Coordinates for axially-symmetric motion.

A curve every where tangent to a given continuous vector field is called a **vector-line**. In particular, the vector-lines of the velocity field are ca1led **stream-limes**, and the vector-lines of the vorticity field are called vortex-lines. (It should be noted that streamlines and particle paths are identical in steady motion, but usually not otherwise.) Finally, a motion is said to be **irrotational** if its vorticity field is zero.

17. Convection of vorticity.

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[1] The orientation of coordinates is shown in Fig 2. Instead of the present notation, some authors (notably Lamb and Miln-Thompson) use (x, ϖ, θ). It may be observed that when polar coordinates (r, ϕ) are introduced into the meridian plane, the resulting spatial coordinates (r, ϕ, θ) become spherical polar coordinates.

One of the most important ways of gaining information about a fluid motion is to examine how its vorticity field changes with time. To this end, we shall derive a kinematical identity expressing the rate of change of vorticity in an arbitrary continuous motion. W e begin with the well known vector identity

$$\boldsymbol{v} \cdot \operatorname{grad}\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{v} + \operatorname{grad}\frac{q^2}{2} . \tag{17.1}$$

Taking the curl of Eq. (3.5) and using Eq. (17.1) yields

$$\operatorname{curl}\boldsymbol{a} = \frac{\partial \omega}{\partial t} + \operatorname{curl}(\boldsymbol{\omega} \times \boldsymbol{v}) = \frac{d\boldsymbol{\omega}}{dt} - \boldsymbol{\omega} \cdot \operatorname{grad}\boldsymbol{v} + \boldsymbol{\omega}\operatorname{div}\boldsymbol{v} ,$$

whence by Eq. (5.3) follows the **diffusion equation** of Beltrami[2]:

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{\boldsymbol{\omega}}{\rho} \cdot \operatorname{grad}\boldsymbol{v} + \frac{1}{\rho}\operatorname{curl}\boldsymbol{a} . \tag{17.2}$$

Let us now apply this result to the **barotropic flow** of a perfect fluid. By virtue of Eq. (16.5) we have curl$\boldsymbol{a}$ = 0, so that Eq. (17.2) reduces to

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{\boldsymbol{\omega}}{\rho} \cdot \operatorname{grad}\boldsymbol{v} . \tag{17.3}$$

This celebrated equation[3] governs the convection of vorticity in a barotropic flow. It is a very remarkable fact that Eq. (17.3), considered as a differential equation for $\boldsymbol{\omega}$, can be integrated explicitly. For, introducing a new unknown $\boldsymbol{c}$ by means of[4]

$$\boldsymbol{\omega} = \rho\boldsymbol{c} \cdot \operatorname{Grad}\boldsymbol{x} , \tag{17.4}$$

this being possible since $J \neq 0$, there results after a simple computation

$$\frac{d\boldsymbol{c}}{dt} = 0 , \quad \boldsymbol{c} = \boldsymbol{c}(\boldsymbol{X}) .$$

Setting $t = 0$ in Eq. (17.4) gives $\boldsymbol{\omega_0} = \rho_0\boldsymbol{c}$, and thus

$$\frac{\boldsymbol{\omega}}{\rho} = \frac{\boldsymbol{\omega_0}}{\rho_0} \cdot \operatorname{Grad}\boldsymbol{x} . \tag{17.5}$$

This beautiful result was obtained in 1815 by Cauchy[5], by an entirely different method. Let us note three important consequences of Eq. (17.5).

1. Vortex-lines are* material *lines.

This means simply that a set of particles which composes a vortex-line at one instant will continue to form a vortex-line at later instants (a quite surprising result!). The proof lies in the fact that a direction $d\boldsymbol{x}$ once tangent to a vortex-line is carried by the fluid so that it is always tangent to a vortex-line: If at $t = 0$, $d\boldsymbol{x}$ =

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[2] E. Beltrami: Mem. Ace. Sci. Bologna (1971 to 1873). (Open 2, pp. 202 to 379); especially §6.

[3] Due to E. Nanson: Mess. Math. 3, 120 (1874). In a. slightly different form the result was obtained by Euler, Novi. Comm. Acad. Sci. Petrop. (1762), (Opera Ommia II 12, pp.133 to 168). The special case valid for constant density is usually called Helmholtz's equation.

[4] $\operatorname{Grad} \equiv \dfrac{\partial}{\partial X^\alpha}$

[5] 3 A.-L. Cauchy: Mem. Divers Savants (2) 1, (Oeuvres (1) 1, pp. 5 to 318), especially 1st part, § 4.

$dX = \omega_0 d\tau$, then at any other instant

$$dx = dX \cdot Gradx = \omega_0 \cdot \mathrm{Grad} x d\tau = \frac{\theta_0}{\rho} \omega d\tau , \qquad (17.6)$$

and dx is tangent to a vortex-line. Incidentally, Eq. (17.6) shows that a material arc ds along a vortex-line varies during the motion according to the formula,

$$\frac{\rho}{\omega} ds = \frac{\rho_0}{\omega_0} ds_0 . \qquad (17.7)$$

This discussion will be amplified in Sect. 25.

2. The Lagrange-Cauchy theorem[6].

If a fluid particle or a portion of fluid is initially in irrotational motion, then it will retain this property throughout its entire history. This is an obvious consequence of Cauchy's formula (17.5).

3. In plane flow we have

$$\omega_x = \omega_y = 0 , \quad \omega_z = \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} . \qquad (17.8)$$

ω is thus normal to the flow plane, and $\omega \cdot \mathrm{grad} v = 0$. It follows from Eq. (17.3) that $\omega / \rho = \mathrm{const}$ following each particle, vividly illustrating the results of (1) and (2) above. Similarly, in axially-symmetric flow

$$\omega_x = \omega_y = 0 , \quad \omega_\theta = \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} , \qquad (17.9)$$

(see Sect. 12). It follows from Eq. (17.7) that $\omega / y\rho = \mathrm{const}$ following each particle. This can also be obtained from Eq. (17.3), but not so easily.

18. Bernoullian theorems.

By a Bernoulli equation is commonly meant a first integral of the equations of motion[7]. There are various forms which .this can take, depending on the particular kinematical or dynamical assumptions made about the motion, though in all cases the bask expression

$$H \equiv \frac{1}{2} q^2 + \int \frac{dp}{\rho} + \Omega$$

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[6] J. L. Lagrange: Nouv. Mem. Acad. Berlin (1781). (Oeuvres 4, pp.695 to 748). Lagrange's statement and proof were alike imperfect; a correct formulation and valid proof were given by Cauchy, loc. cit. above. The proof is predicated on the assumption that the motion is continuous in the sense of Sect. 3. Shock waves of course introduce vorticity into an otherwise irrotational motion (Sect. 54), though singular surfaces across which the flow variables themselves are continuous do not (Sect. 51).

[7] Daniel Bernoulli is the discoverer of the general type of theorem to be considered (Hydrodynamica, 1738). This does not mean that he was in possession of the various results which are today called Bernoulli's equation. See the Editor's Introduction to I. Euler, Opera Omnia II 12.

is present. In this section we shall consider the several Bernoulli equations holding for barotropic flow of a perfect fluid.

By means of Eq. (17.1) the fundamental Eq. (16.5) can be written

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{v} = -\mathrm{grad} H \,, \tag{18.1}$$

forming the starting point of the discussion. If it is now assumed that the flow is steady there results the following important

Bernoulli theorem: *Consider the steady barotropic flow of a perfect fluid. Then if* $\boldsymbol{\omega} \times \boldsymbol{v} \equiv 0$ *in the flow region we have*

$$\frac{1}{2} q^2 + \int \frac{dp}{\rho} + \Omega \equiv \mathrm{const} \,.$$

On the other hand, if $\boldsymbol{\omega} \times \boldsymbol{v} \neq 0$ *then there exists in the flow region a set of surfaces*

$$H \equiv \frac{1}{2} q^2 + \int \frac{dp}{\rho} + \Omega = \mathrm{const} \,,$$

each one covered by a network of vortex-lines and streamlines. In particular, H is constant on streamlines. The surfaces H = const may be called ***Lamb surfaces***, after their first investigator[8].

A very similar result may be proved under the weaker assumption that ***merely the vorticity is steady***, rather than the entire flow[9]. Indeed, if $\boldsymbol{\omega} \times \boldsymbol{v} \equiv 0$, then by forming the curl of the relation $\boldsymbol{v} = k\boldsymbol{\omega}$ we find

$$v = \frac{\omega^2}{\boldsymbol{\omega} \cdot \mathrm{curl}\boldsymbol{\omega}} \boldsymbol{\omega} \,,$$

whence the velocity field as well as the vorticity field is steady. This is essentially the case already considered above. When $\boldsymbol{\omega} \times \boldsymbol{v} \neq 0$ the velocity field need not be steady, but instead we have

$$\mathrm{curl} \frac{\partial \boldsymbol{v}}{\partial t} = \frac{\partial \boldsymbol{\omega}}{\partial t} = 0 \,,$$

and so

$$\frac{\partial \boldsymbol{v}}{\partial t} = \mathrm{grad} \chi$$

for some potential function χ . Thus, if $\boldsymbol{\omega} \times \boldsymbol{v} \neq 0$ there exists in the flow region a set of surfaces

$$\frac{1}{2} q^2 + \int \frac{dp}{\rho} + \Omega + \chi = \mathrm{const} \,,$$

each one covered by a network of vortex-lines and streamlines.

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[8] H. Lamb: Proc. Lond. Math. Soc. 9, 91 (1878); also [8], § 165.

[9] This observation seems to be due to Masotti, Rend. Lincei (6) 6, 224 (1927).

It is well known that an irrotational flow is characterized by the existence of a (possibly multiple-valued) ***velocity potential*** $\phi = \phi(\boldsymbol{x}, t)$, such that

$$\boldsymbol{v} = \mathrm{grad}\phi . \tag{18.2}$$

(Many authors define the velocity potential by $\boldsymbol{v} = -\mathrm{grad}\phi$, conforming with the minus sign found in force potentials. Modern usage tends to omit the minus sign as an .unnecessary inconvenience.) The expression (18.2) for $\boldsymbol{v}$ allows an immediate integration of Eq. (18.1), and we thus obtain ***the Bernoulli theorem for irrotational flow***,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} q^2 + \int \frac{dp}{\rho} + \Omega = f(t) . \tag{18.3}$$

For steady flow this reduces to

$$\frac{1}{2} q^2 + \int \frac{dp}{\rho} + \Omega = \mathrm{cons}t . \tag{18.4}$$

Eq. (18.3) and (18.4) constitute complete integrals of the equations of motion; their importance can hardly be overestimated.

19. The stream function

It is possible to introduce a stream function whenever the equation of continuity can be written as the sum of two derivatives. We consider in this section plane flow and axially symmetric flow, though these are by no means the only cases which can be treated. Furthermore we shall assume the fluid to be incompressible, deferring the more elaborate treatment of compressible fluids until later (Sect. 42).

Plane flow.

In this case $\boldsymbol{u} = \boldsymbol{u}(x, y)$, $\boldsymbol{v} = \boldsymbol{v}(x, y)$, $\boldsymbol{w} = 0$, so that the equation of continuity takes the simple form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 .$$

It follows that the line integral $\int u dy - v dx$ from some fixed point (x_0, y_0) to the variable point (x, y) defines a (possibly multiple-valued) function $\psi = \psi(x, y, t)$. Obviously

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{19.1}$$

so that knowledge of ψ determines the complete velocity field. Moreover it is evident that the curves $\psi = \mathrm{const}$ are the streamlines of the field. ψ is called the stream function.

From Eqs. (17.8) and (19.1) we obtain the following important equation

satisfied by ψ, namely

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\omega\,. \tag{19.2}$$

Eq. (19.2) serves to determine ψ when the vorticity magnitude is known. Now for *steady plane flow* the equation of motion (18.1) is equivalent to

$$H=H(\psi)\,,\quad \omega=-\frac{dH}{d\psi}\,.$$

Thus any solution of the equation $\Delta^2\psi=f(\psi)$ provides an example of steady two-dimensional flow; of course in a definite problem one must also take into account boundary conditions on ψ.

In irrotational motion a velocity potential ϕ exists and

$$u=\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}\,,\quad v=\frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}\,.$$

The complex function

$$w=w(z,t)=\phi+i\psi\,,\quad z=x+iy\,,$$

is therefore analytic, a fact which makes plane irrotational motion of an incompressible fluid particularly amenable to exact solution. Concerning this subject to reader is referred to the works of Lamb [8] and Milne-Thomson [10], and to the articles of Berker, Wehausen, and Gilbarg in this Encyclopedia.

Axially symmetric flow.

The development here is entirely analogous to the preceding, and the results need only be sketched. The equation of continuity takes the form

$$\frac{\partial}{\partial x}(yu)+\frac{\partial}{\partial y}(yv)=0\,,$$

[cf. formula (12.11)], hence we can define a stream function $\psi=\psi(x,y,t)$ such that

$$u=\frac{1}{y}\frac{\partial\psi}{\partial y}\,,\quad v=-\frac{1}{y}\frac{\partial\psi}{\partial x}\,. \tag{19.3}$$

In steady flow H and ω are connected by

$$H=H(\psi)\,,\quad \omega=-y\frac{dH}{d\psi}\,;$$

thus any solution of the equation

$$E^2\psi=y^2\,/(\psi)$$

provides an example of steady axially symmetric flow. An interesting case is furnished by Hill's spherical vortex,

$$\psi=\frac{1}{2}Ay^2(a^2-r^2)\,,$$

where $r^2=x^2+y^2$. Here

$$\omega = 5Ay \quad \text{and} \quad H = -5A\psi + \text{const}$$

(cf. [8], p. 245).

20. Intrinsic equations of motion.

Consider a steady plane motion in which s and n denote, respectively, arc length along streamlines and along their orthogonal trajectories. We wish to express the equations of motion in terms of intrinsic derivatives. To this end, it is convenient first to find an intrinsic expression for the divergence of the velocity vector, at the same time illustrating the general technique to be used throughout the section. Consider a rectangular coordinate system with origin at a fixed point P in the flow field and axes oriented along the streamline and orthogonal trajectory at P. Denoting the coordinates by (x', y'), we have

$$\mathrm{div}\boldsymbol{v} = \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = \frac{\partial}{\partial x'}[q\cos(\theta - \theta_P)] + \frac{\partial}{\partial y'}[qdon(\theta - \theta_P)],$$

where θ is the inclination of the velocity vector to the original x-axis. Expanding the above formula and evaluating at P yields

$$\mathrm{div}\boldsymbol{v} = \frac{\partial q}{\partial s} + q\frac{\partial \theta}{\partial n}, \tag{20.1}$$

which is the required formula. Now introducing the curvatures κ and K of the streamlines and their orthogonal trajectories,

$$\kappa = \frac{\partial \theta}{\partial s}, \quad K = \frac{\partial \theta}{\partial n},$$

then by virtue of (20.1) we may write the equation of continuity in the form

$$\frac{\partial}{\partial s}(\rho q) + K\rho q = 0, \tag{20.2}$$

Similarly, using the formula

$$\frac{d}{dt} = q\frac{\partial}{\partial s}, \tag{20.3}$$

we obtain by resolution of Eq. (16.2) along stream1ines and along normals,

$$\rho q\frac{\partial q}{\partial s} = -\frac{\partial p}{\partial s}, \quad \rho q^2\kappa = -\frac{\partial p}{\partial n}. \tag{20.4}$$

Eqs. (20.2) and (20.4) constitute the ***intrinsic equations of steady two-dimensional flow***. A calculation similar to that leading to Eq. (20.1) yields the vorticity formula

$$\omega = -\frac{\partial q}{\partial n} + \kappa q. \tag{20.5}$$

Corresponding equations for axially symmetric flow are easily written down. In particular, we have

$$\frac{\partial}{\partial s}(y\rho q) + Ky\rho q = 0 \tag{20.6}$$

in place of Eq. (20.2), while Eqs. (20.4) and (20.5) remain unchanged.

These equations are easily generalized to three-dimensional flows by using the Frenet formulas

$$\frac{\partial \boldsymbol{s}}{\partial s} = \kappa \boldsymbol{n} \ , \quad \frac{\partial \boldsymbol{b}}{\partial s} = -\tau \boldsymbol{n} \ , \tag{20.7}$$

where $\boldsymbol{s}$, $\boldsymbol{n}$, and $\boldsymbol{b}$ are, respectively, the tangent, normal, and binormal vectors to the congruence of streamlines. We have

$$\text{div}\boldsymbol{v} = \text{div}(\text{q}\boldsymbol{s}) = \frac{\partial q}{\partial s} + q\text{div}\boldsymbol{s} \ .$$

Therefore, if we put $\widetilde{M} = \text{div}\boldsymbol{s}$, the equation of continuity can be written in the form

$$\frac{\partial}{\partial s}(\rho q) + \widetilde{M}\rho q = 0 \ . \tag{20.8}$$

Similarly, setting $\boldsymbol{v} = q\boldsymbol{s}$ in the equation of motion and using Eq. (20.3) yields

$$\rho q \frac{\partial q}{\partial s}\boldsymbol{s} + \rho q^2 \frac{\partial \boldsymbol{s}}{\partial s} = -\text{grad}p \ . \tag{20.9}$$

The second term on the left can be evaluated by one of Frenet's formulas, whence by resolving Eq. (20.9) along the directions $\boldsymbol{s}$, $\boldsymbol{n}$, and $\boldsymbol{b}$ we obtain

$$\begin{cases} \rho q \dfrac{\partial q}{\partial s} = -\dfrac{\partial p}{\partial s} \\ \rho q^2 \kappa = -\dfrac{\partial p}{\partial n} \\ 0 = -\dfrac{\partial p}{\partial b} \end{cases} . \tag{20.10}$$

Eqs. (20.8) and (20.10) are the required ***intrinsic equations of steady motion***.

A formula. for tile vorticity is obtained most simply as follows. We observe that

$$\omega = curl v = \begin{vmatrix} \boldsymbol{s}_P & \boldsymbol{n}_P & \boldsymbol{b}_P \\ \dfrac{\partial}{\partial s} & \dfrac{\partial}{\partial n} & \dfrac{\partial}{\partial b} \\ \boldsymbol{v}\cdot\boldsymbol{s}_P & \boldsymbol{v}\cdot\boldsymbol{n}_P & \boldsymbol{v}\cdot\boldsymbol{b}_P \end{vmatrix} , \tag{20.11}$$

where P is a fixed point in the flow, and (s, n, b) are considered as fixed rectangular coordinates with origin at P (see Fig. 3). Setting $\boldsymbol{v} = q\boldsymbol{s}$ in Eq. (20.11), performing the indicated differentiations and evaluating at P yields the formula

$$\boldsymbol{\omega} = q\left(\boldsymbol{b}\cdot\frac{\partial \boldsymbol{s}}{\partial n} - \boldsymbol{n}\cdot\frac{\partial \boldsymbol{s}}{\partial b}\right)\boldsymbol{s} + \frac{\partial q}{\partial b}\boldsymbol{n} + \left(\kappa q - \frac{\partial q}{\partial n}\right)\boldsymbol{b} \ . \tag{20.12}$$

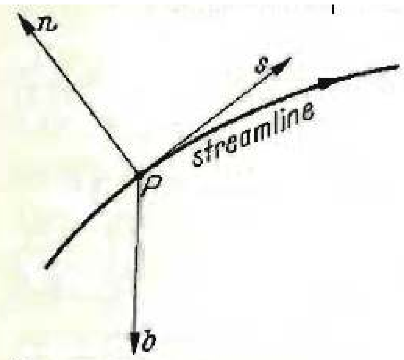

Fig. 3. Intrinsic directions in three-dimensional flow.

In the case of irrotational flow, or, more generally, whenever the velocity field is normal to a one-parameter family of surfaces *S*, the factor $\widetilde{M}$ in Eq. (20.8) can be interpreted as the sum of the principal curvatures (mean curvature) of the equipotential surfaces or the surfaces *S*, which ever the case may be.

The remainder of this chapter is divided into two major sections concerned, respectively, with results peculiar to irrotational motion and to rotational motion. We begin with the simpler and more fully investigated case.

II. Irrotational motion.

21. The assumption of irrotational motion.

22. Principles of irrotational motion.

23. Principles of irrotational motion (continued).

(p.160)

3. Uniqueness.

Consider the problem discussed in the preceding section. We assert that in a simply-connected flow region the fluid motion is completely determined by the motion of the bodies; and by the value of ***U***. Indeed if ϕ_1 and ϕ_2 are the potentials of two flows consistent with the prescribed conditions, then $\phi = \phi_1 - \phi_2$ is the potential of a flow at rest at infinity and satisfying $\frac{\partial \phi}{\partial n} = 0$ at the surfaces of the moving bodies. Assuming the flow region to be simply-connected, we then have $\widetilde{I} = 0$ from Eq. (23.1). It follows that $\mathrm{grad}\phi = 0$ and $\phi = \text{constant}$, that is, the two flows must be identical. If the flow region is not simply-connected, simple examples show the result not to be true. For a discussion of the fluid motion in this case, the reader is referred again to Lamb, in particular §§ 47-55.

The uniqueness theorem proved above is important not only for its own sake, but also because it shows that one cannot in general prescribe more than the

normal fluid velocity at a boundary surface. In particular, an ***adherence condition*** at a rigid surface will usually be incompatible with irrotational fluid motion[10].

4. The d'Alembert paradox.

Consider the force acting on a solid body moving with uniform velocity through a fluid at rest at infinity, or, what comes to the same thing and is more easily computed, the action of a uniform stream on a fixed solid immersed in it. If $\boldsymbol{U}$ denotes the velocity of the uniform stream, then according to Eq. (22.2)

$$\boldsymbol{v} = \boldsymbol{U} + O(r^{-3}) ,$$

where, as in the rest of the paragraph, the order symbol refers to behavior as $r \to \infty$. It follows from Bernoulli's theorem (18.4) that

$$p = p_0 + \frac{1}{2}\rho(U^2 - q^2) = p_0 + O(r^{-3}) ,$$

assuming $\Omega = 0$. Thus by Eq. (10.2), with $\boldsymbol{t} = -p\boldsymbol{n}$ and Σ a large sphere of radius R,

$$\boldsymbol{F} = -\int_\Sigma (p\boldsymbol{n} + \rho\boldsymbol{v}\boldsymbol{v}\cdot\boldsymbol{n})da = -\int_\Sigma (p_0\boldsymbol{n} + \rho\boldsymbol{U}\boldsymbol{U}\cdot\boldsymbol{n})da + O(R^{-1}) .$$

The integral on the right vanishes by virtue of the divergence theorem, and it follows easily that $\boldsymbol{F} = 0$.

This result, which at first glance appears unexpected, is known as d'Alembert's paradox[11]. The difficulty of course lies in the fact that the flow model is much too simplified[12].

The forces on an obstacle in plane flow are slightly more difficult to compute because of the presence of circulation. Let the coordinates be chosen with the positive x-axis in the direction of the uniform stream, so that $\boldsymbol{U} = (U, 0)$. Then setting $\boldsymbol{v} = (u, v)$ we easily derive from Eq. (22.8) and Bernoulli's equation that

$$p = p_0 + \rho U(u - U) + O(r^{-2}) .$$

Hence the force in the direction of the uniform stream is given by

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[10] An extensive discussion of this point is given by Truesdell [26], §37. The present argument, it should be noted, applies also to compressible irrotational motions once the appropriate uniqueness theorem is proved.

[11] J. L. D'Alembert : Opuscules Mathematiques 5 (1768). The result is there proved only for symmetric obstacles, and by the trivial argument involving a symmetric pressure distribution. The same idea had occurred earlier in d'Alembert's work, and the result was also given by Euler in his treatise on gunnery (1745): cf. the Editor's introduction to L. Euler, Opera Omnia (2) 12.

[12] Cf. Lamb [8], §§ 370, 371.

$$X = -\int_\Sigma (p\cos\theta + \rho u \boldsymbol{v}\cdot\boldsymbol{n})ds$$
$$= \rho\int_\Sigma [U(u-U)\cos\theta - (u-U)\boldsymbol{v}\cdot\boldsymbol{n} - U\boldsymbol{v}\cdot\boldsymbol{n}]ds + O(R^{-1})\text{,}$$

where $\boldsymbol{v}\cdot\boldsymbol{n} = u\cos\theta + v\sin\theta$. The sum of the first two terms of the integrand is $O(R^{-2})$ by (22.8). The third term represents outflow and correspondingly its integral vanishes. It follows then that X =0, and again there is no resistance. A similar calculation yields, however, a non-zero value for the lift

$$Y = -\rho\Gamma U\ . \tag{23.2}$$

Remarkably, Y depends on the circulation in a, manner entirely independent of the shape and dimensions of the obstacle. Formula (23.2), obtained independently by Kutta[13] and Joukowsky[14], is the basis for the theory of lift of an airfoil.

We call attention to Kirchhoff's formulas for the force and moment on a solid moving *in an arbitrary way* through a fluid. Although there is not enough space here to consider this theory, the reader will find an interesting discussion in [8], Chap. 6.

24. Kelvin's minimum energy theorem.

Consider flows occupying a bounded simply-connected region v of space, the normal mass-flux being prescribed at each point of the boundary of v, that is,

$$\rho\boldsymbol{v}\cdot\boldsymbol{n} = h \quad \text{prescribed on } v. \tag{24.1}$$

The following criterion characterizes irrotational flow among the totality of incompressible flows satisfying Eq. (24.1).

Kelvin's principle: *Among all motions of an incompressible fluid in v which satisfy Eq. (24.1), the irrotational motion has least kinetic energy.*

The proof given by Kelvin and reproduced by Lamb, § 45, cannot be improved. Kelvin's theorem has a converse, which, for some reason, is seldom stated in works on hydrodynamics, this converse being, in fact, nothing more than a restatement of the classical Dirichlet principle of potential theory. The result can be formulated as follows.

Dirichlet's principle: *Among all irrotational motions in v, the one satisfying*

$$\mathrm{div}\boldsymbol{v} = 0\,,\quad \rho\boldsymbol{v}\cdot\boldsymbol{n} = h \quad \text{on } s \tag{24.2}$$

gives the greatest value to the expression

$$\tilde{J} = -\frac{1}{2}\rho\int_v q^2 dv + \oint_s \phi h da\,; \tag{24.3}$$

here ϕ *is the potential of a competing motion*[15].

[13] W. M. Kutta: Sitzgsber. bayr. Akad. Wiss. (Munch.} 40 (1910).
[14] N. E. Joukowski: Bull. Inst. Aero. Koutchino (St. Petersburg) (1906).
[15] A similar theorem has been given by A. R. Pratelli, Rend. Ist. Lornbardo (3) 17, 484 (1953). Pratelli also considers the possibility of a prescribed distribution of vorticity for the competing motions.

(end of p.161)

III. Rotational motion.