E. The perfect gas.

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I. General principles.

35. Preliminary discussion.

The dynamical equations governing the motion of a perfect fluid have been derived in Part B; they are

$$\frac{d\rho}{dt} + \rho \operatorname{div}\boldsymbol{v} = 0\,, \tag{35.1}$$

and

$$\rho\frac{d\boldsymbol{v}}{dt} = \rho\boldsymbol{f} - \operatorname{grad}p\,. \tag{35.2}$$

To these is to be added the energy equation (34.3) of Part D. Since tangential stresses are neglected in the definition of a perfect fluid, it is logical to suppose in addition that $\boldsymbol{q}$ = 0. [Indeed, according to kinetic theory viscosity and heat conduction arise from similar mechanisms (molecular impact), and are of the same order of magnitude; thus if one is to be neglected, we should also neglect the other.] Accordingly we set $\Phi = 0$ and $\boldsymbol{q}$ = 0 in Eq. (34.3), which then becomes simply

$$\frac{dS}{dt} = 0\,. \tag{35.3}$$

The system of equations (35.1) to (35.3) is completed by a thermodynamical equation of state relating p, ρ, and S, namely

$$p = f(\rho, S)\,. \tag{35.4}$$

When the fluid is an ideal gas with constant specific heats, Eq. (35.4) has the particular form

$$p = e^{S/c_v}\rho^{\gamma}\,,\quad \gamma = \text{const} > 1\,. \tag{35.5}$$

Ordinarily, however, we shall not specify the form of.Eq. (35.4) beyond requiring that it be compatible with the general thermodynamic considerations of Sect. 30. By Eq. 00.7) this involves the condition $\left(\frac{\partial p}{\partial \rho}\right)_S > 0$, thus allowing us to ***define*** the thermodynamic variable c,

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_S} = \sqrt{\frac{\partial f}{\partial \rho}}\,. \tag{35.6}$$

In the sequel we shall call a fluid satisfying Eqs. (35.1) to (35.4) a ***perfect gas***. Also, as is almost universal in dealing with the dynamics of gases, we shall neglect the effect of the external force f in Eq. (35.2).

Perhaps the most striking single feature of the system (35.1) to (35.4) is the propagation of pressure waves with finite velocity. This intuitive fact can be given quantitative form by the following procedure; Supposing that the pressure wave is of "small" amplitude, all flow quantities may be considered as perturbations from the rest state v = 0, $p = p_0$, $\rho = \rho_0$, $S = S_0$. Upon neglect of squares of small quantities, Eqs. (35.1) and (35.2) become

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} \boldsymbol{v} = 0 \\ \rho_0 \dfrac{\partial \boldsymbol{v}}{\partial t} = -\operatorname{grad} p \end{cases} .$$

Elimination of $\boldsymbol{v}$ between these equations yields

$$\nabla^2 p = \frac{\partial^2 \rho}{\partial t^2} .$$

On the other hand, from Eqs. (35.3) and, (35.4) there follows

$$\frac{\partial^2 p}{\partial t^2} = c_0^2 \frac{\partial^2 \rho}{\partial t^2} .$$

The equation governing small pressure disturbances is then[1]

$$c_0^2 \nabla^2 p = \frac{\partial^2 p}{\partial t^2} ,$$

and according to well-known theory it follows that disturbances are propagated with the speed c_0. This semi-heuristic analysis justifies calling c the ***speed of sound***[2]. Later (Sects. 51 and 54) we shall give a more rigorous approach to this topic.

It should be observed that c is not a constant, but a thermodynamic variable, depending on the state of the fluid. Indeed, for an ideal gas[3]

$$c = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma R T} , \qquad (35.7)$$

where R is the gas constant for a unit mass of fluid ($= 2.87 \times 10^6$ for air, and $= 83.1 \times 10^6 /$ molecular weight, for a pure gas in the C. G. S. system). The accuracy of formula (35.7) can be estimated by computing c for air at 0° Centigrade (273.16 absolute). We find, using $\gamma = 1.40$ and the above value of R, that c = 331.3 m/sec, in remarkable agreement with the observed value. This agreement is, in fact, important among the reasons for accepting perfect fluid

[1] This equation goes back to Euler, Mem. Acad. Sci. Berlin (1759), (Opera Omnia (3) 1, pp.428-507).

[2] The first theoretical formula for the speed of sound was found by Newton, Principia Mathematica, Lib. II, Sect. VIII, Props. 48, 49. Euler improved upon Newton's argument and derived the wave equation as an alternative approach to the subject. Since the concept of adiabatic changes was not then known, the final outcome of Euler's work, like that of Newton, was the formula $c^2 = p/\rho$. The reconciliation of this theory with fact is due to Laplace, who pointed out that the temperature as well as the pressure rises in sudden compression. Laplace's remark appears to have been first published in a paper of Biot, [Bull. Soc. Phil. Paris, 3, 116 (1802)]. In this paper the temperature rise is, however, still looked upon as an empirical fact. Some years later Laplace explained the rise as due to the adiabatic nature of sound transmission, and so found the formula $c^2 = \gamma p/\rho$ with γ denoting the ratio of specific heats [Ann. Chim. Phys. 3, 238 (1816)] The general formula $c^2 = (\partial p / \partial \rho)_S$ is, of course, the work of a later period.

[3] This formula holds whether or not the specific heats are constant, for by Eq. (30.6)

$$\left(\frac{\partial p}{\partial \rho}\right)_S = \gamma \left(\frac{\partial p}{\partial \rho}\right)_T = \frac{\gamma p}{\rho} .$$

theory as an accurate account of the motion of gases.

From the concept of a finite speed of sound, one arrives directly at the well-known picture of subsonic and supersonic flows. Although the distinction will occur naturally in the subsequent analysis, nevertheless it is worthwhile to examine the situation here in an avowedly heuristic manner. Consider the case of subsonic steady motion - for example, the uniform level flight of an airplane. Here a pressure signal travels forward from the plane at sound velocity minus flight velocity, relative to the plane, whereas a signal travels backward at sound velocity plus flight velocity. Every point in space is therefore reaches by a signal, provided the flight has proceeded from an infinitely remote point. By contrast, in supersonic flight it is seen that all effect is restricted to a cone proceeding backward from the nose of the plane, the angle of the cone with respect to its axis being $\arcsin(c/q)$; (this picture will have to be amended when we consider disturbances of finite amplitude, i.e., shock waves)[4].

36. Dynamical similarity.

In this section it is assumed that the reader knows the usual engineering treatment of dynamical similarity, and desires instead a discussion of the fundamental mathematical principles involved. The idea of dynamical similarity, it may be remarked, is due originally to Stokes. In his paper on the motion of a pendulum in a retarding fluid medium[5], not only is the notion of dynamical similarity formulated for the first time, but there even appears the combination of flow parameters now known as the Reynolds number.

Two perfect fluid motions are said to be ***dynamically similar*** if they are related by equations

$$\boldsymbol{v} = U\boldsymbol{v}', \quad \rho = R\rho', \quad p = Pp', \tag{36.1}$$

and

$$\boldsymbol{x} = D\boldsymbol{x}', \quad t = Tt' \tag{36.2}$$

where *U, R, P, D,* and *T* are similarity constants[6]. We shall show that these constants must be related in definite ways, or, in other words, that certain parameters must be the same for each flow[7]. Making the substitutions (36.1) and (36.2) in the equation of continuity we find

$$\frac{R}{T}\frac{\partial \rho'}{\partial t'} + \frac{RU}{D}\operatorname{div}'(\rho'\boldsymbol{v}') = 0 .$$

4 An excellent introduction to the theory of compressible perfect fluids will be found the first chapter of [43]. General textbooks which can be recommended are [17], [19], [21] and [23].

5 G. Stokes: Trans. Cambridge Phil. Soc. 9, 8 (1850). (Papers 3. pp. 1-141.)

6 In the usual engineering treatments of dynamical similarity both motions are reduced to the same "dimensionless" flow. We find this procedure logically less appealing than the one adopted here.

7 The conclusion is, of course, that mere geometric similarity of two flow regions does not guarantee dynamical similarity of the flows.

Since the "primed" flow is also a solution of the equation of continuity, this implies

$$T = D/U, \tag{36.3}$$

[an exception occurs when the motion is steady, but then Eq. (36.2-2) need not be considered]. With the help of Eq. (36.3) the remaining flow equations become

$$\rho' \frac{d\mathbf{v}'}{dt'} = -\frac{P}{RU^2} \operatorname{grad}' p' \quad \text{and} \quad \frac{dS}{dt'} = 0 .$$

From the first of these we derive simply

$$\frac{RU^2}{P} = 1 , \tag{36.4}$$

Now if the equation of state of the "primed" flow is $p' = g(\rho', S')$, then

$$f(R\rho', S) = Pg(\rho', S') . \tag{36.5}$$

Forming the material derivative of each side and using the fact that both S and S' are constant following particles, after cancellation of a factor there results

$$Rc^2 = Pc'^2 . \tag{36.6}$$

It follows now from Eqs. (36.4) and (36.6) that ***in order for two flows to be dynamically similar the local Mach[8] number M = q/c must be the same at corresponding points of each flow.***

The condition just stated is not the only one necessary for dynamical similarity. Equally important, the equations of state must be such that Eq. (36.5) reduces to a relation not involving ρ'. For example, in case of an ideal gas Eq. (36.5) reduces to

$$S - S' = c_v \log PR^{-\gamma} ,$$

which serves to determine the entropy of the "primed" flow.

In application of the above results (for example, in wind tunnel experiments) the external geometry of two flows is made similar and the reduced velocities $\mathbf{v}/c$ and $\mathbf{v}'/c$ are made to agree ***at one point*** *P*. Under these circumstances and provided that Eq. (36.5) can be satisfied, dynamically similar flows are mathematically possible. Whether dynamically similar flows actually occur or not is another question, one which rests squarely on the unique dependence of the flows on the conditions prescribed at *P*. At least in the case of subsonic flow past an obstacle, such uniqueness has been proved for conditions prescribed in the uniform stream (cf. Sect. 46). In actual wind tunnel tests, of course, so many factors intervene that the question whether dynamical similarity occurs must be answered at least partly on the basis of the particular experimental situation.

It has been shown that dynamically similar flows are possible for an ideal gas. Conversely, ***if two gases allow dynamically similar flows for arbitrary values of***

[8] A survey of the original work of Mach in fluid mechanics and a discussion of the origin of the terms

P and R then each has an equation of state of the form

$$p = \sigma(S)\rho^m \tag{36.7}$$

with the same constant m[9]. This result shows how special a gas must be in order to possess useful properties of dynamical similarity. It is singularly fortunate that the common gases more or less accurately obey (36.7).

To prove the assertion, let us follow the path of a given particle, so that both S and S' in Eq. (36.5) will be constant. The unprimed flow being given once and for all, S is a fixed number which we can suppress for the moment in all relations; S', on the other hand, will be a function of the parameters P and R, or conversely

$$P = P(R, S')\,. \tag{36.8}$$

Substituting Eq. (36.8) into Eq. (36.5) yields

$$g(\rho', S') = \frac{f(R\rho')}{P(R,S')} = \frac{f(\rho')}{P(1,S')}\,, \tag{36.9}$$

since R may be varied independently of S'. The second equality in Eq. (36.9) can be written in the form

$$\frac{P(R,S')}{P(1,S')} = \frac{f(R\rho')}{f(\rho')} = \frac{f(R)}{f(1)}\,,$$

the final equality following since the left hand side is independent of ρ'. Hence

$$f(R\rho') = \text{const} f(R)/(\rho')\,, \tag{36.10}$$

and this in turn implies[10] that

$$f(\rho) = \text{const}\rho^m\,.$$

Therefore

$$f(\rho, S) = \sigma(S)\rho^m\,,$$

and our assertion is proved.

The above analysis does not provide a complete list of necessary conditions for dynamically similar motion if viscosity, thermal conductivity, extraneous forces, boundary conditions, etc., are important factors in determining the motion. The influence of some of these quantities is dealt with in Sect. 66.

Dimensional analysis and dynamical similarity.

Some of the results described above can be arrived at by simple dimensional analysis. Thus, since the equation of motion are dimensionally consistent, it follows that the dimensionless ratio q/c must be the same at corresponding points of two dynamically similar flows. More generally, ***if one assumes the existence of dynamically similar flows and their unique dependence on the flow variables***

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which bear his name will be found in an article by J. Black, J. Roy. Aero. Soc. 54, 371 (1950).

[9] A similar result for isentropic flows is given by Birkhoff [16], p.112.

[10] Differentiate (36.10) with respect to R and set $R = 1$.

at some reference point *P*, then. using standard procedures of dimensional analysis one sees that any dimensionless quantity connected with the flows must be a function solely of the Mach number at *P*. The italicized assumptions are, however, just the crux of the matter: without them the dimensional analysis cannot be, logically defended[11]. It is at this point that the theory of dynamical similarity is of value, the analysis in the preceding paragraphs serving to determine ***necessary and sufficient conditions for the existence of dynamically similar flows***. For reason such as these Goldstein[12] says, in commenting on an investigation of dynamical similarity, "so we recover by a somewhat clearer and more logical method, the results obtained earlier by considerations of dimensions".

This is not to say that dimensional analysis is un important, for there are many instances where the above assumptions are verified, or where nothing else avails; the technique of dimensional analysis should, however, be supplemented whenever possible by a consideration of the full equations governing the phenomena in question.

II. Energy, entropy, and vorticity.

37. Bernoulli's equation.

38. The Crocco-Vazsonyi equation.

39. Isentropic, isoenergetic, and irrotational steady flow.

40. Diffusion of vorticity.

III. Special methods in two-dimensional flow.

41. Intrinsic equations.

42. The stream function.

43. The hodograph method.

44. Special solutions.

IV. Subsonic potential flow.

In this subchapter we treat some of the important theoretical questions concerning steady subsonic potential flow of a perfect gas.

45. General principles.

The basic equations of steady potential flow are Bernoulli's equation

[11] In ordinary applications of dimensional analysis the basic assumptions are, unfortunately, almost never stated. A critical discussion of these points, together with a number of examples, may be found in Chap. III of reference [16]: in particular, it is there emphasized (pp.92-93) that Pi Theorem, which is fundamental result of dimensional analysis, is never the point of objection, but rather that the difficulties lie in the assumption upon which the dimensional analysis rests.

[12] [42], p. 112.

$$\frac{1}{2}q^2 + \int \frac{dp}{\rho} = \text{const}\,, \tag{45.1}$$

Where $p = f(\rho)$ is given, and the equation of continuity. Since

$$\boldsymbol{v} = \text{grad}\phi$$

the latter can be written in the form

$$\text{div}(\rho\,\text{grad}\phi) = 0\,,$$

or in tensor notation,

$$(\rho\phi_{,i}),i = 0\,,\quad v^i = \phi_{,i}\,. \tag{45.2}$$

If the differentiations in Eq. (45.2) are carried out, the result can be expressed in the form

$$c^2\nabla^2\phi - v^i v^j \phi_{,ij} = 0\,. \tag{45.3}$$

This celebrated differential equation is of course valid whether or not the flow is subsonic. For two-dimensional flow it takes the explicit form

$$(c^2 - u^2)\phi_{xx} - 2uv\phi_{xy} + (c^2 - v^2)\phi_{yy} = 0\,. \tag{45.4}$$

Besides the two forms (45.2) and (45.3) of the potential equation, we have also

$$p_{v^i v^j}\phi_{,ji} = 0$$

this being connected with the fact that ϕ solves a certain variational problem (Sect. 47, Appendix).

Mathematically, Eq. (45.3) is a second order quasi-linear differential equation for the velocity potential; its type is determined by the nature of the quadratic form $c^2\xi_i\xi_i - v^i v^j \xi_i\xi_j$ where ξ is an arbitrary real vector. Considering from here on only subsonic flow, this quadratic form is positive definite [$> (c^2 - q^2)\xi^2$ in fact], and accordingly Eq. (45.3) is an ***elliptic*** partial differential equation. Its mathematical treatment is not easy because the coefficients c^2 and $v^i v^j$ are functions of the velocity field. There are, nevertheless, a number of important results.

1. A velocity maximum cannot occur in the interior of the fluid.

To see this, let us differentiate Eq. (45.3) with respect to *x* to obtain

$$c^2\nabla^2 u - v^i v^j u_{,ij} + D^i u_{,i} = 0\,, \tag{45.5}$$

where *u* is the *x*-component of velocity and the D^i are certain coefficients whose exact form need not be given. For an everywhere subsonic flow Eq. (45.5) is an elliptic differential equation (see above) for *u*. According to a theorem of E.

Hopf[13], non-constant solutions of an elliptic differential equation of the form (45.5) cannot take on interior maximum values. Therefore u does not take on an interior maximum, and, by the same reasoning, neither docs v or w. The conclusion is now a consequence of Kirchhoff's reasoning [8], § 37.

2. Flow past an obstacle; development at infinity.

We consider for definiteness a plane flow, uniform at infinity, impinging on a fixed obstacle and in continuous motion around the obstacle. Let the velocity at infinity be $\boldsymbol{U}$, directed (for simplicity) along the x-axis. The flow being assumed subsonic, the Mach number M_∞ is less than 1. The asymptotic behavior of the flow as $x \to \infty$ is given by the following formulas:

$$\left.\begin{aligned} \phi &= \boldsymbol{U}\cdot\boldsymbol{x} + C + \frac{\Gamma}{2\pi}\arctan(\beta\tan\theta) + O(r^{-1+\varepsilon}) \\ \boldsymbol{v} &= \boldsymbol{U} + \frac{\rho\Gamma}{2\pi}\frac{(-y,x)}{x^2+\beta^2 y^2} + O(r^{-2+\varepsilon}) \end{aligned}\right\}. \qquad (45.6)$$

In these formulas (r,θ) are ordinary polar coordinates in the plane, $\beta = \sqrt{1-M_\infty^2}$, Γ is the circulation, and ε is a positive number which can be taken as near zero as one cares. If there is an outflow of strength Λ from a source in the finite part of the plane, an additional term

$$\frac{\Lambda}{4\pi\beta}\log(x^2+\beta^2 y^2)$$

must be added to Eq. (45.6), and a corresponding term to Eq. (45.6-2). A complete expansion for ϕ can be given when the pressure-density relation is analytic, namely,

$$\phi = \boldsymbol{U}\cdot\boldsymbol{x} + \frac{\Delta}{2\pi\beta}\log r + \frac{\Gamma}{2\pi}\arctan(\beta\tan\theta) + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{nm}(\theta)\left(\frac{\log r}{r}\right)^n\left(\frac{1}{r}\right)^m , \qquad (45.7)$$

unless both Δ and Γ are zero, in which case

$$\phi = \boldsymbol{U}\cdot\boldsymbol{x} + \sum_{n=0}^{\infty} a_n(\theta) r^{-n} , \qquad (45{,}7')$$

These series are uniformly and absolutely convergent for large enough r, and can be differentiated term by term at will. The resulting series have similar convergence properties.

The history of these expansions is worth recounting. At an early stage the form of the initial terms was surmised from the fact that, for large r, the equation for ϕ becomes "nearly"

[13] E. Hopf: Sitzgsber. preuss. Akad. Wiss. 147 (1927).

$$(1-M_\infty^2)\phi_{xx}+\phi_{yy}=0\,.$$

Making the transformation $x\to\sqrt{1-M_\infty^2}\,\xi$ gives Laplace's equation; we are thus led to Eq. (45.6) by analogy with the known expansions for the incompressible case. This treatment, although highly suggestive, is no proof that Eq. (45.6) really holds; moreover, the rest of the expansion remains unknown. Bateman[14] was the first to guess the true form of the complete expansion, though he included terms $(\log r^n)/r^p$ with $n > p$ which actually do not occur. Next, Bergman[15] obtained a development of ϕ in terms of the hodograph variables, and Ludford[16] performed the calculations necessary to transform Bergman's development into physical variables. The result of this work was formula (45.7). One step remained, however, namely, to show that the flow potential actually had the asymptotic behavior supposed by Bergman as the basis for his development. This was done by Finn and Gilbarg[17], who gave an independent proof of the necessary formula (45.6). The reader will see from the above outline that formula (45.7) involves exceedingly deep considerations, beginning with Bergman(s theory of singularities of solutions of linear analytic elliptic differential equations, and concluding with the (not simple) demonstration of Eq. (45.6).

One of the major applications of Eq. (45.6) is in the proof of the Kutta-Joukowsky lift formula for flows of compressible fluid. Gilbarg and Finn[18] have recently pointed out that all that is really necessary for this proof is the estimate

$$\boldsymbol{v}=\boldsymbol{U}+o(r^{-1/2})\,. \tag{45.8}$$

Eq. (45.8) is relatively simpler than Eqs. (45.6) or (45.7), and we can outline its **proof**.

Following Gilbarg and Finn, let us make the change of variables $x\to\sqrt{1-M_\infty^2}\,\xi$ in Eq. (45.4), yielding an equation of the form

$$A\phi_{\xi\xi}+2B\phi_{\xi y}+C\phi_{yy}=0\,, \tag{45.9}$$

where A, $C\to 1$ and $B\to 0$ as (ξ, y) tends to infinity. From Eq. (45·9) the velocity "components" $\bar{u}=\phi_\xi$ and $v=\phi_y$ satisfy the equations

$$\begin{cases}-Cv_y=A\bar{u}_\xi+2B\bar{u}_y\\ \quad v_\xi=\bar{u}_y\end{cases}.$$

[14] H. Bateman: Proc. Nat. Acad. Sci. U.S.A. 24, 246 (1938).
[15] S. Bergman: Trans. Amer. Math. Soc. 62, 452 (1947).
[16] G. Ludford: J. Math. Phys. 30, 117 (1952). An explicit development for flow past a circle is given by I. Imai, J. Phys. Soc. Japan 8, 537 (1953).
[17] R. Finn and D. Gilbarg: Comm. Pure Appl. Math. 10, 23 (1957).

Now, outside a circle Σ of suitably large radius A, B, C differ from their limiting values by less than ε. Hence

$$C(v_\xi \bar{u}_y - v_y \bar{u}_\xi) = A\bar{u}_\xi^2 + 2B\bar{u}_\xi \bar{u}_y + C\bar{u}_y^2 \geq (1-2\varepsilon)(\bar{u}_\xi^2 + \bar{u}_y^2) ,$$

outside Σ. Similarly

$$A(v_\xi \bar{u}_y - v_y \bar{u}_\xi) \geq (1-2\varepsilon)(v_\xi^2 + v_y^2) ,$$

and so

$$\bar{u}_\xi^2 + \bar{u}_y^2 + v_\xi^2 + v_y^2 \leq 2K(v_\xi \bar{u}_y - v_y \bar{u}_\xi) \tag{45.10}$$

outside Σ, where

$$K = \frac{1+\varepsilon}{1-2\varepsilon} .$$

From inequality (45.10), which in mathematical terms means that

$$w = v + i\bar{u}$$

is a quasi-conformal mapping, it can be shown that[19]

$$|v|, |\bar{u} - \beta U| \leq \text{const}\left|\frac{R}{r'}\right|^\mu , \quad r' > 2R ,$$

where $\mu = K - \sqrt{K^2 - 1}$, R is the radius of Σ, and $r' = \sqrt{\xi^2 + y^2}$. Returning to the original variables gives

$$|v|, |u - U| \leq \text{const}\left|\frac{R}{r}\right|^\mu , \quad r > 2R/\beta ,$$

and Eq. (45.8) is an immediate consequence provided ε is chosen so that $\mu > 1/2$.

3. Flow past an obstacle: Force formulas.

It is a remarkable fact that the familiar results for flow of an incompressible fluid,

$$X = 0 , \quad Y = -\rho \Gamma U ,$$

hold also for subsonic flow past an obstacle. This may be inferred almost exactly as in the original proofs of the Kutta-Joukowsky theorem [8], § 370b. Indeed, from Bernoulli's equation we have

$$\begin{aligned} p &= P_\infty + \left(\frac{dp}{dq}\right)_\infty (q - q_\infty) + \cdots \\ &= p_\infty - \rho_\infty U(u - U) + o(r^{-1}) \end{aligned}$$

using Eq. (45.8). Hence by the force formula (10.2),

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[18] R. Finn and D. Gilbarg: Trans. Amer. Math. Soc. 88, 375 (1958).

[19] R. Finn and J. Serrin: Trans. Amer. Math. Soc. 89, 1 (1958).

$$X = -\oint (p\cos\theta + \rho u \boldsymbol{v}\cdot\boldsymbol{n})ds = -U\oint \rho u \boldsymbol{v}\cdot\boldsymbol{n} ds + o(1),$$

since $\boldsymbol{v}\cdot\boldsymbol{n} = u\cos\theta + v\sin\theta$. Since the outflow integral vanishes, it follows in the usual way that $X = 0$. The treatment of the lift formula is the same. The above proof should be compared with the analogous one in Sect. 23, No. 4.

The d' Alembert paradox for three-dimensional flows has been proved by Finn and Gilbarg[20] using the asymptotic formula $\boldsymbol{v} = \boldsymbol{U} + o(r^{-2})$.

46. Existence and uniqueness theorems.

As in the previous paragraphs, we consider plane steady flow past a profile. The basic problem is to determine the fluid motion when (1) the conditions in the uniform stream and the circulation are prescribed or (2) the conditions in the uniform stream are prescribed and the circulation is determined by the Kutta-Joukowsky condition. In both cases a solution should be proved to exist, it should be unique, and a procedure for computation should be given. All of these problems have been attacked more or less successfully in recent years. We shall outline the results concerning existence and uniqueness, but a discussion of the numerical computation of compressible flows is beyond the scope of this article. The reader interested in this field may consult recent textbooks in gas dynamics (e.g., [23], [25], [40], and [43]) from which further references may be obtained.

The best results on the problem of uniqueness have been obtained by Finn and Gilbarg, By using the asymptotic formulas (45.6) and certain integral identities somewhat analogous to the kinetic energy formula (23.1) they have proved the following theorem. ***A plane subsonic potential flow past a smooth profile is uniquely determined by conditions in the uniform stream and by the circulation.***

A plane subsonic potential flow past a profile with a sharp trailing edge is uniquely determined by conditions in the uniform stream.

These results have the following significance in the theory of dynamical similarity. Suppose two ideal gases having the same ratio of specific heats are in continuous subsonic potential flow past geometrically similar profiles. Suppose also that both flows have the same Mach number M_∞ and the same circulation ratio Γ/U (the last condition can be dropped if the profiles have a sharp trailing edge). Then the flows are dynamically similar.

It would be of some theoretical interest to know if Γ/U is an increasing function of the Mach number in flows past a profile with a sharp trailing edge. The lift coefficient $C_L = Y/(1/2\cdot\rho U^2)$ would then be an increasing function of

[20] R. Finn and D. Gilbarg: Acta Math. 98. 265 (1957).

M_∞.

The mathematical problem of existence, although extremely difficult, has been largely settled (for plane flows) by the work of Frankl and Keldysh, Shiffman, and Bers. For definiteness the problem is phrased for a *fixed* Bernoulli equation (45.1), a fixed profile, and a fixed velocity direction at ∞; the prescribed conditions are then the Mach number M_∞ and, in case of a smooth profile, the circulation Γ. The basic result is a follows:

For a given smooth profile and a given Bernoulli equation there is a region m the (M_∞, Γ) plane, including the origin, such that for any point in this region there exists a unique subsonic flow past the profile with these values of M_∞ and Γ. Moreover, as the point (M_∞, Γ) approaches the boundary of this region the maximum local Mach number of the corresponding flow approaches 1.

For a given profile with a sharp trailing edge there is a number $\hat{M}$, such that for every M_∞ in the interval $0 \leq M_\infty < \hat{M}$ there exists a unique subsonic flow past the profile with this speed at infinity. Moreover, as $M \to \hat{M}$, the maximum local Mach number approaches 1.

The existence proofs are too technical to be more than cursorily described here. Frankl and Keldysh use (essentially) an iteration procedure, and establish the existence of flows only for "sufficiently small" M_∞. Shiffman's proof uses direct methods in the calculus of variations (starting with the Bateman-Kelvin principle), combined with an ingenious device which allows one always to conclude the existence of a minimizing extremal. Bers' proof may be described broadly as being function-theoretic (as in the treatment. of plane flow of an incompressible fluid), but a whole arsenal of analysis comes into play. In addition to establishing the existence of subsonic flows, Bers proves their continuous dependence on M_∞, on the shape of the profile, and on the form of the speed-density relation. In spite of the highly difficult mathematical work in these papers, they fill a gap long felt by applied mathematicians and aerodynamicists[21].

The problem of existence and uniqueness of three-dimensional subsonic flows has been studied recently by Gilbarg and Finn[22]. Their comprehensive results include a complete solution of the uniqueness problem, an asymptotic expansion

[21] Other theoretical work of an allied nature will be found in the following papers: L. Bers: Comm. Pure Appl. Math. 7, 79 (1952); D. Gilbarg: J. Rational Mech. Anal. 2, 233 (1953): D. Gilbarg and M. Shiffman: J. Rational Mech. Anal. 3, 209 (1954); C. Loewner: J. Rational Mech. Anal. 2, 537 (1954) and an article on the critical Mach number in Studies in Mathematics and Mechanics, presented to R. von Mises, New York, 1954: J. Serrin: J. Math. Phys. 33, 27 (1954).

[22] D. Gilbarg and R. Finn: Acta Math. 98, 265 (1957).

of the potential at infinity, and the existence of subsonic flows when the local Mach number does not exceed 0.53. Moreover, a remarkable theorem of Nash[23] may lead to the solution of the existence problem in comparable generality to that for plane flows.

47. Variational principles in gas dynamics.

In this section we consider some ***maximum*** and ***minimum*** principles in steady subsonic flow. The motivation for these principles is the desire to extend Kelvin's minimum energy theorem to gas dynamics, and also the need for practical methods to. compute compressible flows. Our discussion does not include results such as Herivel's theorem (Sect. 45) because these are not true minimization principles. On the other hand, Herivel's discovery that $\hat{I} - \hat{O}$ is the . appropriate Lagrangian function in Hamilton's principle may serve as a partial motivation for the choice of integrands below.

Consider now a finite region v with boundary s. Let a normal mass-flux h be prescribed on s subject to the condition

$$\text{Outflow} = \oint_s h\,da = 0\,.$$

We are interest in setting up a variational problem. which will characterize irrotational motion by the minimization of some function of the velocity field. For this purpose it is convenient to define for each velocity field a corresponding "density" field. This we do by means of the Bernoulli equation[24]

$$\frac{1}{2}q^2 + \int \frac{dp}{\rho} = \text{const}\,. \tag{47.1}$$

The flow $(\rho, \boldsymbol{v})$ thus determined will in general satisfy neither the equation of continuity nor the equation of motion. We shall say that v is subsonic if $q < c = \sqrt{dp/d\rho}$. With these preliminaries understood we may proceed to the result in question.

The Bateman-Kelvin principle[25].

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[23] J. Nash: Proc. Nat. Acad. Sci. U.S.A., 43, 754 (1957).

[24] If one wishes, (47.1) can be considered as a "side condition" for the following variational problems. The constant of integration and the constant on the right hand side of Eq. (47.1) are supposed fixed throughout the ensuing discussion.

[25] H. Bateman: Proc. Nat. Acad. Sci. U.S.A. 16, 816 (1930). The present formulation as a minimum problem is based on the results of a paper of P. E. Lush and T. M. Cherry: Quart. J. Mech. Appl. Math. 9, 6 (1956). It is unlikely that any of these authors noted the strict analogy with Kelvin's theorem, for they were primarily concerned with the two-dimensional problem. Another formulation of the Bateman principle is given by E. Hoelder, Math. Nachr. 4, 366 (1950).

Consider the variational problem, to minimize the integral

$$\hat{I}(\boldsymbol{v}) = \int_v (p + \rho q^2) dv$$

among all subsonic velocity fields which satisfy the equation of continuity and have prescribed mass-flux h ***on*** s. ***Then***

$$\hat{I}(\boldsymbol{v}) = \text{minimum}$$

if and only if v is irrotational.

It is clear that the minimizing flow field provides a dynamically possible, isentropic, irrotational flow satisfying the prescribed boundary conditions. It is this fact which makes the Bateman principle especially valuable. There is a converse proposition, also due in essence to Bateman.

The Bateman-Dirichlet principle[26].

Consider the variational problem, to maximize the integral

$$\hat{I}(\boldsymbol{v}) = \int_v p dv + \oint_s \phi h da$$

among all subsonic velocity fields $\boldsymbol{v} = \text{grad}\phi$***. Then***

$$\hat{I}(\boldsymbol{v}) = \text{maximum}$$

if and only if $\text{div}(\rho \boldsymbol{v}) = 0$ ***and*** $\rho \boldsymbol{v} \cdot \boldsymbol{n} = h$ ***on*** s.

Now an extremal for one of these problems is also an extremal for the other. Moreover, as we shall see later, an extremal is unique if it exists at all. It follows therefore that these problems either possess a common extremal or none at all. In the former case, we see by application of the divergence theorem that

$$\hat{I}_{\min} = \hat{I}_{\max} = \int_v (p + \rho q^2) dv\ ,$$

(first noticed by Lush and Cherry). The possibility that there may be no extremal arises from the fact that there may be no irrotational subsonic flow satisfying the given boundary conditions. Shiffman[27] has shown how to modify the variational problems so that they will always possess an extremal.

In order to prove the Bateman-Kelvin principle, we set $\boldsymbol{Q} = \rho \boldsymbol{v}$ and note from the Bernoulli equation that $\boldsymbol{v}$, and hence also $\hat{I}$, may be considered a

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As a motivation for the choice of integrand, we observe the identity

$$p + \rho q^2 = \frac{1}{2}\rho q^2 - \rho E + \rho(\frac{1}{2}q^2 + I) = \frac{1}{2}\rho q^2 - \rho(E + \text{const});$$

thus since E is defined only up to an arbitrary constant, $\hat{I} = \hat{T} - \hat{P}$.

[26] H. Bateman: Cf. footnote 1. Although Bateman discovered the integrand *p,* the additional surface integral necessary to make a true maximum principle was found only recently by Lush and Cherry (footnote 1). The present formulation is slightly more general than that of Lush and Cherry.

[27] M. Shiffman: J. Rational Mech. Anal. 1, 605 (1952).

function of $\boldsymbol{Q}$. (There are, of course, two speeds corresponding to each value of *Q,* see Sect. 37· We naturally choose that one corresponding to subsonic flow.) The advantage of using $\boldsymbol{Q}$ instead of $\boldsymbol{v}$ as the fundamental variable is that $\boldsymbol{Q}$ is subject to simpler conditions than $\boldsymbol{v}$, namely just

$$\mathrm{div}\boldsymbol{Q} = 0 .$$

Now consider a vector field $\boldsymbol{Q}$ which minimizes $\hat{I}$, and let $\boldsymbol{Q}^* = \boldsymbol{Q} + \varepsilon\boldsymbol{\zeta}$ be a competing field. Expanding $\hat{I}(\boldsymbol{Q}^*)$ in powers of ε yields

$$\hat{I}(\boldsymbol{Q}^*) = \hat{I}(\boldsymbol{Q}) + \varepsilon\delta\hat{I} + \varepsilon^2\hat{H} ,$$

where

$$\delta\hat{I} = \int_v \boldsymbol{v}\cdot\boldsymbol{\zeta}dv ,$$

$$\hat{H} = \frac{1}{2}\int_v \frac{(\tilde{\boldsymbol{v}}\cdot\boldsymbol{\zeta})^2 + (\tilde{c}^2 - \tilde{q}^2)\zeta^2}{\tilde{\rho}(\tilde{c}^2 - \tilde{q}^2)}dv ;$$

here the tildes denote evaluations at some intermediate *Q;*. the comutation of $\delta\hat{I}$ and $\hat{H}$ is simplified by setting $p + \rho q^2 = F(Q^2)$ and noting that $F' = (2q)^{-1}$, $F'' = [4\rho^3(c^2 - q^2)]^{-1}$.

Since $\hat{H} \geq 0$ by the condition of subsonic flow, the extremal $\boldsymbol{Q}$ is characterized by $\delta\hat{I} = 0$. We must show that this condition IS equivalent to $\mathrm{curl}\boldsymbol{v} = 0$.

First, if curl $\boldsymbol{v}$ *=0,* then $\boldsymbol{v} = \mathrm{grad}\phi$ and

$$\int_v \boldsymbol{v}\cdot\boldsymbol{\zeta}dv = \int_v \mathrm{div}(\phi\boldsymbol{\zeta})dv = 0$$

where we have rnade use of the conditions $\mathrm{div}\boldsymbol{\zeta} = 0$ 0 and $\boldsymbol{\zeta}\cdot\boldsymbol{n} = 0$ on *s*. On the other hand if $\delta\hat{I} = 0$ for all admissible variations $\boldsymbol{\zeta}$ then $\boldsymbol{v}$ must be irrotational. For suppose $\mathrm{curl}\boldsymbol{v} \neq 0$ at some point *P*. Then we can find a vector $\boldsymbol{A}$ which vanishes outside the immediate neighborhood of *P* and has the property

$$\int_v \boldsymbol{A}\cdot\mathrm{curl}\boldsymbol{v}dv \neq 0 .$$

But this implies $\delta\hat{I} \neq 0$ for the admissible variation $\boldsymbol{\zeta} = \mathrm{curl}\boldsymbol{A}$. This contradiction proves that $\mathrm{curl}\boldsymbol{v} = 0$.

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V. Supersonic flow and characteristics.

The theory of unsteady flow and supersonic flow of a compressible fluid is based largely on the. existence of characteristic curves or surfaces in the flow field. Indeed, our insight into the behavior of gases at supersonic speeds is due largely to our understanding of the characteristic field, and most of the available solutions have been obtained from this knowledge.

48. The nature of characteristics.

Consider for the moment a general flow of a compressible fluid. Let Σ be a 3-dimensional manifold in the four-dimensional (x, t)-space occupied by the flow, or, in less abstract terms, let Σ be a ***moving*** surface in the three-dimensional physical space. Suppose now that the values of the flow variables $\boldsymbol{v}$, ρ, p, and S are known on Σ. Then by means of Eqs. (35.1) to (35.4) we can, in general, uniquely determine their derivatives at points of Σ. The process is a familiar one: the derivatives tangential to Σ are known because the flow variables on Σ are known, while the remaining ***normal*** derivatives can be obtained by considering Eqs. (35.1) to (35.4) at points of Σ as a system of linear equations with these derivatives as the unknowns. It may happen, however, that Eqs. (35.1) to (35.4) do not thus determine the normal derivatives, and in this case Σ is called a ***characteristic manifold***. The physical significance of a characteristic manifold lies in the fact that it is only along such a manifold that two solutions may be "**tangent**", or in other words, discontinuities in the derivative of a solution can only appear on characteristic manifolds. It may also be remarked that characteristic manifolds play an important role in the propagation of disturbances in a flow field. as is apparent from the above discussion.

In the following two sections we shall consider the special case of steady flow[28]. The characteristic manifolds then do not involve the variable t, and hence are curves in plane flow and surfaces in spatial flow. During the discussion it will become apparent that **characteristics *do not* occur in steady subsonic flow**, streamlines of the motion being excepted. Thus until Sect. 51 our considerations will be directed mainly to supersonic flows. An alternative treatment of characteristic manifolds is given in Sect. 51. serving to complement the present discussion..

The elegant theory of characteristic curves in one-dimensional unsteady flows is not covered in this article, since a number of excellent treatments are available.

49. Steady plane flow.

Let C be a curve in a region of steady plane flow, defined parametrically by the equation $\boldsymbol{x} = \boldsymbol{x}(\sigma)$ where $\boldsymbol{x} = (x, y)$ and σ is arclength on C. The values of the flow variables on C may be expressed in the form

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[28] Characteristic manifolds for a general three-dimensional nonsteady flow are treated in [21]. pp. 112-116. This book also contains a useful discussion of characteristic manifolds for an arbitrary system of first order partial differential equations (pp. 103-112).

$$\boldsymbol{v} = \boldsymbol{v}(\sigma), \quad p = p(\sigma), \quad \rho = \rho(\sigma), \quad S = S(\sigma). \tag{49.1}$$

We ask, under what conditions on C will the functions (49.1) not determine the derivatives of $\boldsymbol{v}, p, \rho$ and S on C?

We have the following set of conditions for the determination of these derivatives

$$\left.\begin{aligned} \boldsymbol{v}\cdot\text{grad}\rho + \rho\,\text{div}\boldsymbol{v} = 0 \\ \rho\boldsymbol{v}\cdot\text{grad}\boldsymbol{v} + \text{grad}p = 0 \\ \boldsymbol{v}\cdot\text{grad}S = 0 \end{aligned}\right\},$$

where $p = f(\rho, S)$ is the given equation of state. These equations can easily be reduced to the form

$$\left.\begin{aligned} \rho\boldsymbol{v}\cdot\text{grad}\boldsymbol{v} + \text{grad}p = 0 \\ \rho c^2\text{div}\boldsymbol{v} + \boldsymbol{v}\cdot\text{grad}p = 0 \end{aligned}\right\}, \tag{49.2}$$

it is this set of equations which we shall actually use. A simple artifice now suggests itself for determining the derivatives of $\boldsymbol{v}$ and p on C. We orient the coordinates so that the y-axis is tangential to C at some point P; then the y-derivatives will be known at P, while it is only the x-derivatives which must be found. For these we have the system of linear equations

$$\left.\begin{aligned} \rho u u_x + p_x &= -\rho v u_y \\ \rho u v_x &= -\rho v v_y - p_y \\ \rho c^2 u_x + u p_x &= -\rho c^2 v_y - v p_y \end{aligned}\right\}. \tag{49.3}$$

$$\boxed{\begin{pmatrix} \rho u & 0 & 1 \\ 0 & \rho u & 0 \\ \rho c^2 & 0 & u \end{pmatrix}\begin{pmatrix} u_x \\ v_x \\ p_x \end{pmatrix} = \begin{pmatrix} -\rho v u_y \\ -\rho v v_y - p_y \\ -\rho c^2 v_y - v p_y \end{pmatrix}}$$

The condition that C be a characteristic is that these equations ***not*** determine u_x, v_x and p_x.

Equating to zero the determinant of the coefficients of Eqs. (49.3) yields

$$\rho^2 u(u^2 - c^2) = 0.$$

That is, ***a curve C is a characteristic if and only if at each point on C either*** (1) ***the normal velocity component is zero***, or (2) ***the normal velocity component has magnitude equal to the local sound speed***. Characteristics of the first kind are obviously nothing more than the streamlines of the motion. The occurrence of streamlines in this role is certainly not surprising and is relatively unimportant in the theory.

Characteristic curves satisfying the second condition evidently occur only in supersonic flow. In a fixed coordinate system (x, y) let us introduce the two direction fields

$$\left.\begin{aligned} \frac{dy}{dx} = \tan(\theta + A) \\ \frac{dy}{dx} = \tan(\theta - A) \end{aligned}\right\}, \tag{49.4}$$

where θ is the velocity inclination and A is the (local) ***Mach angle***,

$$\sin A = \frac{c}{q} = \frac{1}{M}\,.$$

Then the curves C_+ satisfying the first of Eqs. (49.4), the curves C_- satisfying the second of Eqs. (49.4), together with the streamlines form the set of characteristics of two-dimensional steady flow. There are three characteristics, therefore, passing through each point of a supersonic flow region; we note that the C_+ and C_- characteristics are bisected by the streamline (Fig. 7a). One may interpret the $C_\pm$ characteristics as the paths of infinitesimal (steady-state) disturbances in the flow; equally important, however, is the interpretation of these characteristics as curves along which two solutions may be "tangent ".

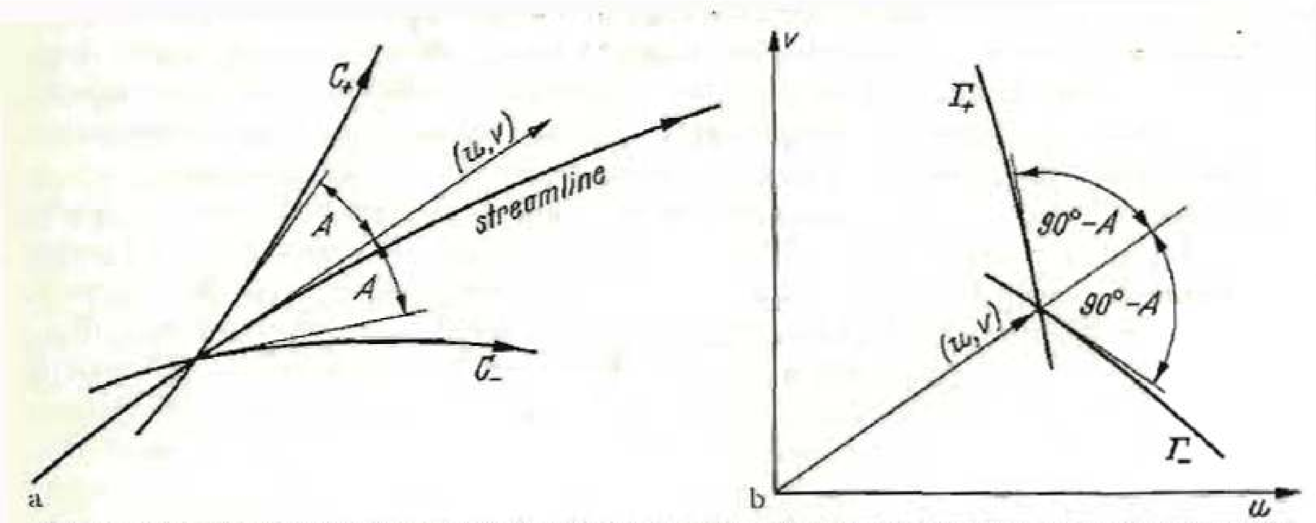

Fig. 7a and b. (a) Characteristic curves in the physical plane. (b) The corresponding curves in the hodograph plane.

The flow variables cannot be entirely arbitrary along a characteristic curve. Indeed, on the streamline characteristics we have the evident conditions

$$S = \text{const},\ H = \text{const},$$

expressing constancy of entropy and "energy". On. a. Mach line (i.e., a $C_\pm$ characteristic) the vanishing of the determinant of coefficients in Eq. (49.3) similarly places a compatibility condition on the right hand members, namely

$$\begin{vmatrix} -\rho v u_y & 0 & 1 \\ -\rho r v v_y - p_y & \rho u & 0 \\ -\rho c^2 v_y - v p_y & 0 & u \end{vmatrix} = 0\,,$$

or, after cancellation of a non-zero factor,

$$\rho c^2 v_y - \rho u v u_y + v p_y = 0\,. \qquad (49.5)$$

$$\boxed{\begin{aligned} & -\rho v u_y \cdot \rho u \cdot u - 1 \cdot \rho u \cdot (-\rho c^2 v_y - v p_y) \\ & = -\rho^2 u^2 v u_y + \rho^2 c^2 u v_y + \rho u v p_y \\ & = \rho u(-\rho u v u_y + \rho c^2 v_y + v o_y) \end{aligned}}$$

This equation is of course expressed in the specially oriented coordinate system which was introduced at the beginning of the discussion. Using now the condition that $|u| = c$ on a Mach line, we may write Eq. (49.5) in the form

$$\rho(uv_y - vu_y) \pm \cot A p_y = 0\,, \tag{49.6}$$

where the + and - signs refer, respectively, to the C_+ and C_- characteristics. Eq. (49.6) may in turn be written

$$\rho q^2 \dot{\theta} \pm \cot A \dot{p} = 0\,, \tag{49.7}$$

the dot denoting differentiation with respect to "y" at a point on the characteristic, or with respect to σ along the length of the characteristic.

In irrotational isentropic flow the streamlines no longer appear as characteristics (the conditions H = const, S = const give additional information sufficient to determine flow derivatives on a streamline). Moreover. using the Bernoulli equation (37.3) we can eliminate p from Eq. (49.7), thus obtaining the important relation

$$\frac{d\theta}{dq} = \pm\frac{\cot A}{q} = \pm\frac{\sqrt{M^2 - 1}}{q}\,, \tag{49.8}$$

holding along a Mach line. The right hand side is of course a function of q alone. Consequently along any characteristic in the physical plane, we have the following simple relation between the speed and flow angle:

$$\theta = \pm\int \frac{\sqrt{M^2 - 1}}{q}\, dq = \pm r(q) + \text{const}\,. \tag{49.9}$$

Otherwise expressed, ***the image of a*** $C_\pm$ ***characteristic in the hodograph plane belongs to a one-parameter family of fixed curves***. It is customary to call the image of a C_+ characteristic a Γ_+ curve, and the Image of a C_- characteristic a Γ_- curve. (Since the $\Gamma_\pm$ curves are characteristics of the hodograph equations (43.2), they are also called ***hodograph characteristics***.)

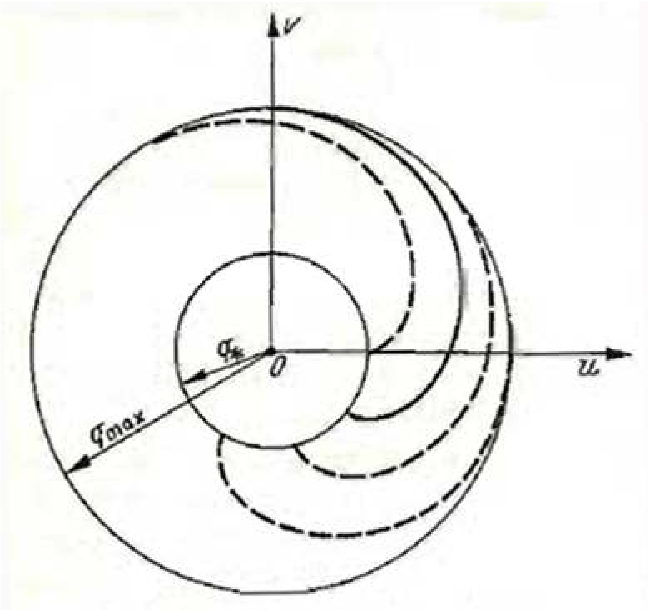

Fig. 8. Γ_+ **curves in the hodograph plane.**

A considerable amount of information concerning the geometry of the $\Gamma_\pm$

curves can be deduced. To begin with, we observe that the function $r(q)$ is monotonically increasing for $q_* \le q < q_{\max}$. Moreover, for $q = q_*$ we have $M = 1$ and $d\theta / dq = 0$, while for $q = q_{\max}$ we have $M = \infty$ and $d\theta / dq = \infty$. The curvature of the Γ_+ curves is given by

$$\frac{q^2 + 2q'^2 - qq''}{(q^2 + q'^2)^{3/2}} = \frac{a}{c\sqrt{M^2 - 1}} \;, \quad ' = \frac{d}{d\theta}$$

see Eq. (37.7), and is therefore always positive. Consequently, ***the*** Γ_+ ***curve in the (u, v)-plane are counter-clockwise spirals between the fixed circle*** $q = q_*$ ***and*** $q = q_{\max}$; ***all of them can be obtained by rotating any one about the origin*** (Fig. 8).

A similar result obviously holds for the Γ_- curves. Finally by elementary geometry Eq. (49.8) is seen to be equivalent to

$$\frac{dv}{du} = -\cot(\theta \mp A) \,. \qquad (49.10)$$

From Eqs. (49.4) and (49.10) it follows that ***the direction of the C-characteristics of one kind are perpendicular to the*** Γ ***characteristics of the other kind*** (Fig. 7). The above results are valid for any gas, irrespective of its equation of state.

It can be shown that a discontinuity in grad$\boldsymbol{v}$ which appears across a characteristic in isentropic, irrotational flow will satisfy a Riccati equation along that characteristic[29]. It follows that such discontinuities are uniquely determined and are not zero along a whole characteristic if they are known to be different from zero at any point of the characteristic. It should be pointed out that these remarks on the propagation of discontinuities in grad$\boldsymbol{v}$ do not apply to discontinuities in $\boldsymbol{v}$ itself. Discontinuities in the functions themselves are propagated as "shocks" in quite a different manner (Part F).

In the special case of an ideal gas with constant specific heats, the Γ characteristics are epicycloids generated by a circle of radius $\frac{1}{2}(q_{\max} - q_*)$ rolling on the circle $q = q_*$. One advantage of an ideal gas is that a single characteristic diagram will serve, whatever the reference state may be. For a general gas, on the contrary, the hodograph characteristics are essentially different for different entropy and energy levels of the reference state. This is no particular disadvantage theoretically, but for the purposes of numerical computation it is a decisive reason for employing the ideal gas law.

Axially-symmetric flow.

The procedure applied above to deduce the characteristic equations for plane flow

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[29] Joh. Nitsche: J. Rational Mech. Anal. 2, 291 (1953).

can be used similarly for axially-symmetric flow. Because of the changed form of divd$\boldsymbol{v}$ we find that in place of Eqs. (49.3) there arises the slightly different set of equations,

$$\left.\begin{aligned} \rho u u_x + p_x &= -\rho v u_y \\ \rho u v_x &= -\rho v v_y - p_y \\ \rho c^2 u_x + u p_x &= -\rho c^2 v_y - v p_y - \rho q c^2 \frac{\sin\theta}{y} \end{aligned}\right\},$$

where θ is the inclination of the velocity vector to the axis of revolution. It follows that the characteristic curves in the flow half-plane are given exactly as before. The characteristic condition (49.7) is, however, replaced by

$$\dot{\theta} \pm \left(\frac{\cot A}{\rho q^2} \dot{p} + \frac{\sin A \sin\theta}{y} \right) = 0\,, \tag{49.11}$$

as is seen by following the preceding calculation.

In isentropic irrotational flow Eq. (49.11) may be written

$$\frac{d}{d\sigma}(r \mp \theta) = \frac{\sin A \sin\theta}{y}\,. \tag{49.12}$$

The hodograph image of a Mach line is therefore not one of a fixed family of curves as in two-dimensional flow. This fact contributes to the difficulty of any exact treatment of axially-symmetric flow.

50. Steady irrotational flow in three dimensions.

51. ???

VI. Special topics

52. Transonic flow.

A gas flow is called transonic if the motion is part subsonic and partly supersonic. A number of transonic flow problems have been treated in the literature in recent years, but limitations of space make it impossible to deal here with more than one aspect of the situation. The problem chosen for discussion is of considerable physical interest, and also exhibits a remarkable mathematical behavior which has provoked much comment.

Consider the steady plane flow of a perfect gas past a fixed profile, the flow being uniform at infinite distances. We have noted the existence of a uniquely everywhere subsonic flow past the profile when the Mach number M_∞ lies in certain range $0 \le M_\infty < \widetilde{M}$; moreover, as M_∞ approaches $\widetilde{M}$ the maximum local Mach number in the flow approaches 1. Now when M_∞ increases beyond $\widetilde{M}$ it is observed experimentally that local supersonic zones develop on the sides of the profile, and, after further increase of M_∞, shock waves occur in the supersonic zones. The Mach number M_{sw} at which shock waves first appear is

not too well-defined, but at all events satisfies $\widetilde{M} < M_{sw} < 1$

Now by using the hodograph method it is possible to construct exact transonic flows past profiles[30]. Although this result is of considerable importance, it does not guarantee that a solution exists for an arbitrary profile shape; furthermore, no hodograph solution has ever provided a continuous transition from subsonic to transonic flow for a ***given fixed*** profile. These problems might be expected to yield to further analysis but for the difficulty that once a continuous transonic flow past a fixed profile is set up no mechanism is known which will explain to eventual breakdown of the flow as M_∞ is increased[31], the attractive limiting line hypothesis having been shown false[32]. A reason for these apparent anomalies may be gained from the following important theorem of Nikolsky and Taganov:

If in a continuous potential flow there is a local supersonic zone adjacent to arc of the flow boundary, this arc must be strictly convex[33].

Leaving the proof until later, we see plainly from this result that, should transonic flow exist, it could always be destroyed by the slightest variation the profile, namely any variation putting a straight or concave arc into the supersonic region[34]. This result leads naturally to the following assertion, made independently by Frankl, Guderley, and Busemann[35], and finally proved in 1957 by C. S. Morawetz[36]: ***the problem of continuous transonic flow past a fixed profile is badly set in the theory of perfect fluids***.

The explanation for the observed phenomena seems therefore to involve the effect of viscosity in a boundary layer. Presumably the edge of the boundary layer makes the adjustments necessary to accommodate a non-viscous flow outside the boundary layer; as M_∞ increases, the edge of the boundary layer develops a very large curvature at some point and a shock line enters the flow. This surmise

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[30] M. J. Lighthill [35], p.251. T. M. Cherry: Phil. Trans. Roy. Soc. Lond., Ser. 245, 583 (1953).

[31] This statement refers only to breakdown mechanisms lying in the domain of perfect fluid theory; when viscosity is taken into account the situation may be somewhat different; see below.

[32] K. O. Friedrichs: Comm. Pure Appl. Math. 1, 287 (1948); a simpler proof is due to I. Kolodner and C. S. Morawetz: Comm. Pure Appl. Math. 6, 97 (1953); cf. also A. M. Well: Quart. Appl. Math. 12, 343; 13, 337 (1955).

[33] A. A. Nikolsky and G. I. Taganov: Prikl. Mat. Mek., USSR. 10, 481 (1946), [English translation, NACA Tech. Mem. 1213 (1949)].

[34] Using a method somewhat similar to that of Nikolsky and Taganov, H. Johnson (Masters thesis, U. of Minn.) has shown that an axially-symmetric transonic flow is likewise unstable to small variations of the boundary.

[35] F. I. Frankl: Prikl. Mat. Meck. USSR, 11, 192 (1947), [English translation, NACA Tech. Mem. 1251 (1949)]; G. Guderley: Wright Field Rep. No. F-TR-1171-ND; and A. Busemann: J. Aeronaut. Sci. 16, 337 (1949).

[36] C. S. Morawetz: Comm. Pure Appl. Math. 10, 107 (1957); 11, 129 (1958).

is supported by the result of Friedrichs quoted earlier. Lin has conjectured that the transonic flow problem can be solved for *analytic* convex profiles; If this is the case then the boundary laver argument just given provides a sort of existence theorem for transonic flows past a profile.

According to the discussion of the preceding paragraph we may expect continuous inviscid transonic flow to exist *outside* the boundary layer. Some information on the possible location of local supersonic zones can then be inferred from the Nikolsky-Taganov theorem: for example, the first appearance of sonic speed and the first appearance of a shock line must occur on convex portions of the profile boundary. The theorem applies also to local supersonic zones in a plane nozzle, and can be used to determine the location of the sonic point in transonic flow past a wedge.

The proof of the Nikolsky-Taganov theorem follows in three parts.

1. ***Let*** C^* ***be a sonic arc in the flow region, separating a zone of subsonic flow from a zone of supersonic flow. Then the inclination of the velocity vector decreases monotonically as*** C^* ***is traversed so that the supersonic zone is to the right*** (Fig. 9).

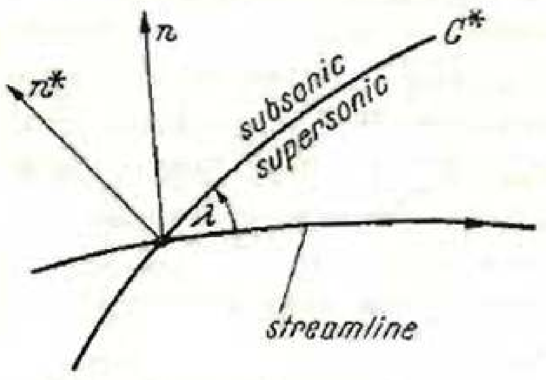

Fig. 9. Geometry of the sonic line C*.

For let s and s* denote, respectively, arc length on the streamlines and on C^*. Then (see Fig. 9)

$$\frac{\partial \theta}{\partial s^*} = \frac{\partial \theta}{\partial s}\cos\lambda + \frac{\partial \theta}{\partial n}\sin\lambda\,. \qquad (52.1)$$

Now on C^*, $M = 1$ and $q = q^* =$ const. Therefore by Eq. (52.1) and the intrinsic Eqs. (41.4),

$$\frac{\partial \theta}{\partial s^*} = \frac{1}{q_*}\frac{\partial q}{\partial n}\cos\lambda = \frac{\cos^2\lambda}{q_*}\frac{\partial q}{\partial n^*}$$

Since n^* is measured into the subsonic zone we have $\frac{\partial q}{\partial n^*} \le 0$; it follows that

$\frac{\partial \theta}{\partial s^*} \le 0$, and our assertion is proved.

According to this lemma there cannot exist a bounded supersonic region in the interior of a flow: If this occurred, then a complete traversal of the boundary would contradict the single-valuedness of θ.

2. ***A local supersonic zone D adjacent to the flow boundary has a simply covered image in the hodograph plane***.

For consider a C_- characteristic in the supersonic zone emanating from a point *P* on the profile. This characteristic ends at a point *A* on the sonic line, for it surely does not return to the profile. Let *Q* be a variable point on *AP,* and let *R* the sonic point on the C_+ characteristic through *Q* (see Fig. 10). In the hodograph plane, the image points *A', R', B'* have the order indicated since $\frac{\partial \theta}{\partial s^*} \le 0$. Thus as *Q* goes from *A* to *P*, *R'* moves from *A'* to *B'*, and *Q'* moves from *A'* to *P'*. Now letting *A* vary the length of the sonic line, it is easily seen that the hodograph image of *D* is simply covered.

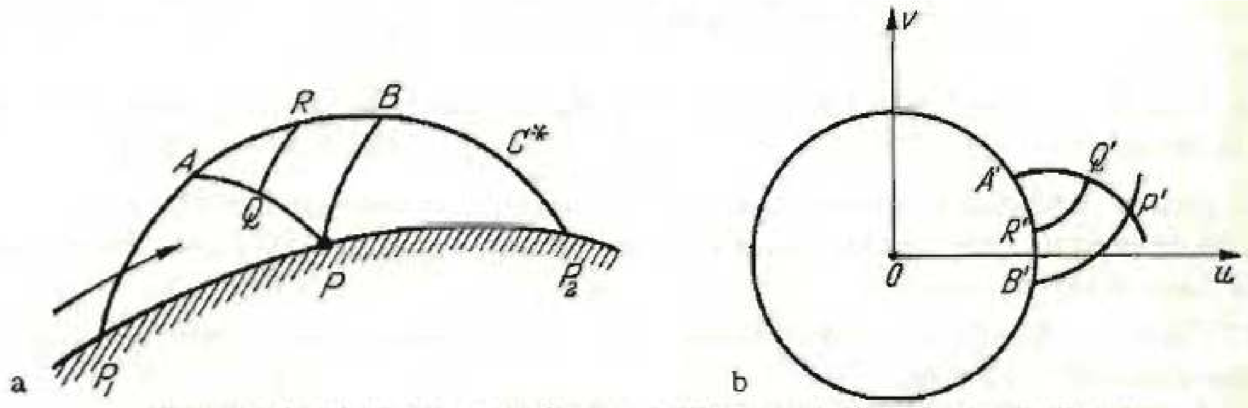

Fig. 10a and b. (a) Local supersonic zone. (b) The corresponding characteristic diagram.

The above argument also proves that the hodograph image of the supersonic zone must lie between the Γ_+ characteristic emanating from P'_2 and the Γ_- characteristic emanating from P'_1. Thus the maximum speed $\bar{q}$ attainable in a local supersonic zone is given by

$$r(\bar{q}) = \frac{\theta_1 - \theta_2}{2} ,$$

where $\theta = \pm r(q)$ is the equation of the characteristic curves. For thin profiles the angles θ_1 and θ_2 are very nearly equal, so that high speeds in a local supersonic zone in continuous flow are impossible.

The reader will note that the results of 1, 2, and 3 apply equally well to the

supersonic region P_1BP in Fig. 11. Since such configurations are observed to be completely stable, we have another example of the predominant role which the boundary layer plays in smoothing out profile variations in transonic flows.

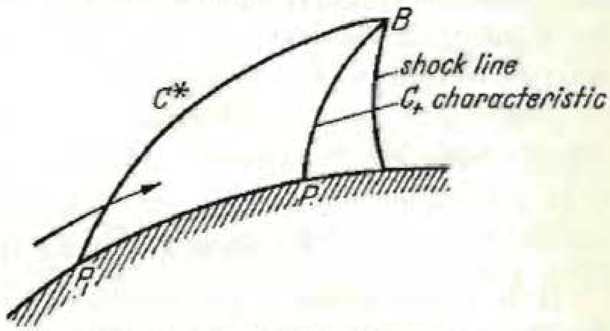

Fig. 11. Local supersonic zone bounded by a shock line.

53. Elimination of the pressure and density from the equations of motion.

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End of Chap. E.