F. Shock waves in perfect fluids.

It is a well-known experimental fact that abrupt changes in pressure and density can occur across surfaces in gas flow. The physical and mathematical reasons for the occurrence of such transition surfaces or shock waves are also well-known and widely discussed in works on gas dynamics. This is not the place to present these arguments[1], but rather we shall assume that the reader is already familiar with the importance of shock waves. This chapter can then be devoted to the fundamental theoretical results of the subject. In particular we shall derive the shock relations, determine some elementary properties of shock waves, and finally consider the structure of shock waves.

54. Shock relations.

Mathematically, a shock wave is a surface $\Sigma = \Sigma(t)$ in the flow region across which one or more of the flow variables $\boldsymbol{v}$, ρ, p and S suffers a jump discontinuity, In order to describe the conditions which hold at a shock surface we begin by assigning the subscript 1 to denote quantities on one side of the shock and the subscript 2 to denote quantities on the other side, (eventually we shall interpret side 1 as the front of the shock and side 2 as the back, but for the moment no such distinction is made). Let $\boldsymbol{n}$ be the unit normal vector to the shock surface, directed towards side 2, and let G be the speed of advance of the surface in that direction. Then the following equations hold relating the flow variables on the two sides of Σ:

$$\left.\begin{aligned} [\rho U] &= 0 \\ [\rho U\boldsymbol{v} + p\boldsymbol{n}] &= 0 \\ [\rho U(\tfrac{1}{2}q^2 + E) + p\boldsymbol{v}\cdot\boldsymbol{n}] &= 0 \\ [\rho US] &\geq 0 \end{aligned}\right\} . \qquad (54.1)$$

In these equations $U = \boldsymbol{v}\cdot\boldsymbol{n} - G$ is the relative normal flow velocity on Σ, and the brackets denote the jump of a quantity across the discontinuity front, i.e.,

$$[f] = f_2 - f_1 .$$

The first three of conditions (54.1) express, respectively, conservation of mass, momentum, and energy as the fluid crosses Σ[2]. The final condition is a

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[1] See [17], §§ 50, 117. and the article of H. Cabannes in this Handbuch, vol. IX.

[2] The concept of a discontinuity surface and the first two conditions above are due originally to G. Stokes, Phil Mag. (3) 33, 349 (1848). Stokes' remarks concerning these topics still make interesting reading: "These conclusions certainly seem sufficiently startling; yet a still more extraordinary result…"; "the result, however, is so strange…";etc.

The fact that energy should be conserved across a shock wave is implicit in an investigation of Rankine [Phil Trans. Roy. Soc. Lond. 160, 277 (1870)], and was later given precise form by Hugoniot

consequence of postulate (33.5) governing the change of entropy of a material volume[3]. The above equations are usually obtained from considerations more or less independent of the basic postulates of Chaps. B and D, so that it is perhaps worthwhile to indicate here how they can be obtained directly from these postulates.

We assume in what follows that the shock surface Σ separates two states of continuous flow, that the flow variables have definite limiting values on either side of Σ, and that these limiting values are different for at least one of the flow variables. Consider now a volume V moving with the fluid particles, and let Σ divide $\widetilde{V}$ into two parts. According to Eq. (5.2)

$$\frac{d}{dt}\int_V \rho dv = 0\,. \tag{54.2}$$

In analogy with the derivation in Chap. B, we should like to use Eq. (4.1) to evaluate the left hand side of Eq. (54.2). Because of possible discontinuities of both the density and velocity across Σ, it is necessary to use Eq. (4.1) in a modified form, namely,

$$\frac{d}{dt}\int_V f dv = \int_{V'}\left(\frac{df}{dt} + f\mathrm{div}\boldsymbol{v}\right)dv + \int_\Sigma [fU]da\,. \tag{54.3}$$

In this equation, which is easily inferred from Eq. (4.2), the designation Σ as the area of integration naturally refers only to the portion of Σ inside V. From Eqs. (54.2) and (54.3), together with the equation of continuity, we obtain

$$\int_\Sigma [\rho U]da = 0\,.$$

Since the portion of Σ over which this integration takes place may be arbitrarily small, it follows in the usual way that the integrand is zero, proving Eq. (54.1). The remaining equations of (54.1) are derived similarly from Eqs. (6.1), (33.3), and (33.5); of course, in these laws it is assumed that $\boldsymbol{t} = -p\boldsymbol{n}$ and $\boldsymbol{q} = 0$. (It is of related interest to study the shock relations which hold at a discontinuity surface

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[J. Ecole Polytech. Cahier 57, 1 (1887); Cahier 58, 1 (1889)].

[3] That the entropy should increase across a shock was first pointed out by G. Zemplen, C. R. Acad, Sci., Paris 141, 710 (1905). Zemplen's remark, which today seems obvious enough, is actually far from a triviality. After the time when his note was published, both Kelvin and Raleigh were of the opinion that $[S]=0$ at a shock surface. They therefore had serious doubts as to the validity of the shock wave hypothesis as a model for the behavior of real gases, since the first three of Eqs. (54.1) are incompatible with the presumed truth of $[S]=0$. Even Stokes had joined this view! It should be added that the continental hydrodynamicists generally did not entertain the doubts of the English school, and that, in any case, the correct principles were more or less universally understood by 1915, (strangely enough, however, Lamb persists in the earlier misconception).

in a viscous fluid motion, especially since certain of the results are quite remarkable[4]. Such a study must be mainly of theoretical importance, however, since there does not appear to be any experimental situation which would require viscous shocks for its explanation.)

The remainder of this section is concerned with certain elementary consequences of the shock relations. In order to facilitate the discussion it is convenient to consider separately the two cases

I. $U_1 = U_2 = 0$;

II. Neither U_1 nor U_2 is zero.

In the first case no fluid crosses Σ so that it can hardly be considered a true shock. Such a discontinuity surface in fact moves with the gas and separates two zones of different density (and temperature); but the pressure and normal velocity are the same on both sides. We shall exclude this simple and relatively unimportant case from the further discussion. The second case is that of a genuine shock wave. Here it can be assumed without loss of generality that

$$U_1 > 0, \quad U_2 > 0, \tag{54.4}$$

for the other possibility can be converted into this one simply by relabeling the sides 1 and 2 of Σ (consequently reversing the direction of $\boldsymbol{n}$). The geometrical significance of Eq. (54.4) is that the fluid enters the shock surface on side 1 and emerges on side 2.

Now let $\boldsymbol{v}_t$ denote the (vector) projection of $\boldsymbol{v}$ onto the shock surface. By means of some simple reductions making use of the fact that U_1 and U_2 are positive we may rewrite Eq. (54.1) in the elegant form

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[4] The basic work on this subject is due to P. Duhem, Recherches sur l'hydrodynamique, Ann. Toulouse (2) (1901-1903). reprintcd Paris 1903-1904. Unfortunately, Duhem's paper is quite difficult to read due to its shear bulk, its somewhat outdated notation, and an altogether remarkable profusion of symbols. In view of this fact, and since some of Duhem's results (see below) have an inherent interest quite apart from possible applications, it seems desirable to reexamine Duhem's work. This the author has done, with particular regard to the following two of Duhem's results: (A) no singular surface of zero order (shock wave) can exist in a viscous fluid, and (B) singular surfaces of order one (Sect. 51) in a non-viscous heat-conducting fluid propagate with the Newtonian speed of sound. Our conclusions relative to these statements are as follows: (1) shock waves, across which all of the fundamental conservation laws hold, *are* possible in viscous fluids; (2) if, however, it is postulated that [v] = [T] = 0 across any discontinuity surface in a viscous flow, then only contact discontinuities are possible (this is the true content of Duhem's first result); (3) in a non-viscous, heat-conducting fluid it *is* possible for a singular surface to propagate at the Newtonian sound speed. A complete discussion of these results will be published in the Journal of Mathematics and Mechanics.

$$\left.\begin{array}{c}[\rho U]=0\\ [\rho U^2+p]=0,[\boldsymbol{v}_t]=0\\ [\frac{1}{2}U^2+I]=0\\ [S]\geq 0\end{array}\right\},\tag{54.5}$$

where I is the specific enthalpy, $I=E+p/\rho$. When the motion is steady and the shock surface consequently stationary, we have from Eqs. (54.5-3) and (54.5-4),

$$[\frac{1}{2}q^2+I]=0\,.\tag{54.5a}$$

Thus Bernoulli's equation (38.3) holds for steady flow even when shock fronts intervene in the motion.

It is useful to exhibit Eqs. (54.5) in several other forms where the thermodynamic variables stand separated. To this end, let us introduce the mass flux m across the shock

$$m=\rho_1 U_1=\rho_2 U_2\,.\tag{54.6}$$

Then from Eqs. (54.5-2) and (54.5-1),

$$\begin{aligned}p_2-p_1&=\rho_1 U_1^2-\rho_2 U_2^2\\ &=m(U_1-U_2)\\ &=m^2(\tau_1-\tau_2)\\ &=U_1U_2(\rho_2-\rho_1)\end{aligned},\tag{54.7}$$

(τ denotes specific volume). From the first, second, and third forms respectively of the right hand side of Eq. (54.7) we derive

$$(p_2-p_1)(\tau_2+\tau_1)=U_1^2-U_2^2,\tag{54.8}$$

and

$$\frac{p_2-p_1}{\tau_2-\tau_1}=-m^2\,,\tag{54.9}$$

$$\frac{p_2-p_1}{\rho_2-\rho_1}=U_1U_2\,.\tag{54.10}$$

Finally from Eqs. (54.8) and (54.5-4) follows

$$(p_2-p_1)(\tau_2+\tau_1)=2(I_2-I_1)\,.\tag{54.11}$$

The important Eq. (54.11) was first obtained by Hugoniot, although the result for ideal gases was already known to Rankine. Eq. (54.11) determines all the possible thermodynamical state (p_2,τ_2) which may be reached across a shock wave from an initial state (p_1,τ_1). Another form of (54.11) is

$$(p_2+p_1)(\tau_2-\tau_1)=2(E_2-E_1)\,.$$

An important property of shock waves is that they introduce vorticity into an otherwise irrotational flow. The usual reason advanced for this statement is that energy (more precisely, the stagnation enthalpy H) is conserved across a shock

while entropy is introduced. This argument is restricted to steady motion. Recently, however, several papers have appeared in which the unsteady flow situation is analysed and in which the same conclusion is reached[5].

There is a remarkably simple formula for the vorticity introduced behind a shock when the flow in front is uniform, namely[6]

$$\boldsymbol{\omega} = \frac{(\tau_1 - \tau_2)^2}{\tau_2} \boldsymbol{n} \times \operatorname{grad}_\Sigma m\,, \tag{54.12}$$

where $\operatorname{grad}_\Sigma$ denotes the surface gradient operator, and $m = \rho U$. The proof of Eq. (54.12) is too long to include here, but it is worthwhile to notice several consequences. First, it is apparent that $\boldsymbol{\omega}$ is tangential to the shock surface. Also, in plane or axially-symmetric steady flow Eq. (54.12) reduces simply to

$$\omega = \frac{(\rho_1 - \rho_2)^2}{\rho_1 \rho_2} K v_t\,, \tag{54.13}$$

where K is the curvature of the shock line and v_t is the tangential component of velocity[7]. It is seen from Eq. (54.13) that vorticity is introduced at every point of a shock front where its curvature is non-zero and its inclination not normal to the uniform stream. Furthermore, although the entropy introduced by a shock front is of third order in shock strength, the vorticity is of second order.

In Sect. 56 we shall obtain several further properties of the shock transition in an arbitrary perfect fluid. In particular it will be shown that the entropy change across a shock front is of third order in the shock strength $\tau_1 - \tau_2$. Thus, considering a sequence of shock fronts whose strengths tend to zero, we have

$$\lim U_1^2 = \lim U_2^2 = \lim \frac{p_2 - p_1}{\rho_2 - \rho_1} = c^2\,.$$

In other words, ***the speed of advance of an infinitely weak shock front relative to the fluid is precisely the speed of sound***. This result offers another justification for calling c the speed of sound.

55. Shock relations for an ideal gas.

For an ideal gas with constant specific heats,

$$I = \frac{c^2}{\gamma - 1} = \frac{\gamma}{\gamma - 1} p \tau\,.$$

This allows the Hugoniot relation (54.11) to be written in the useful form[8]

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[5] W. D. Hayes: J. Fluid Mech. 2, 595 (1957). R. P. Kanwal: Arch. Rational Mech. Anal. 1, 225 (1958).

[6] M. J. Lighthill: J. Fluid Mech. 2, 1 (1957). W. D. Hayes: J. Fluid Mech. 2, 595 (1957).

[7] C. Truesdell: J. Aeronaut. Sci. 19, 826 (1952).

[8] Other forms of the Hugoniot relation for ideal gases are

$$\left(p_2 + \frac{\gamma-1}{\gamma+1} p_1\right)\left(\tau_2 - \frac{\gamma-1}{\gamma+1}\tau_1\right) = \frac{4\gamma}{(\gamma+1)^2} p_1\tau_1 . \tag{55.1}$$

Eq. (55.1) determines all possible end states (p_2, τ_2) which can be reached across a shock wave from the initial state (p_1, τ_1). The locus of these end states in the (p, τ)-plane is a rectangular hyperbola with asymptotes

$$p = -\frac{\gamma-1}{\gamma+1} p_1 , \quad \tau = \frac{\gamma-1}{\gamma+1}\tau_1 ;$$

this curve is called the ***Hugoniot curve***, The condition $S_2 \geq S_1$ requires us to choose only that portion of the hyperbola above the point (p_1, τ_1), whence

$$\frac{\gamma-1}{\gamma+1}\tau_1 < \tau_2 < \tau_1 \quad \text{or} \quad \rho_1 < \rho_2 < \frac{\gamma+1}{\gamma-1}\rho_1 ;$$

in other words the increase in density across a shock wave cannot be arbitrarily great. The adiabatic through (p_1, τ_1) and the Hugoniot curve through that point are shown in Fig. 12; these curves have a second order contact.

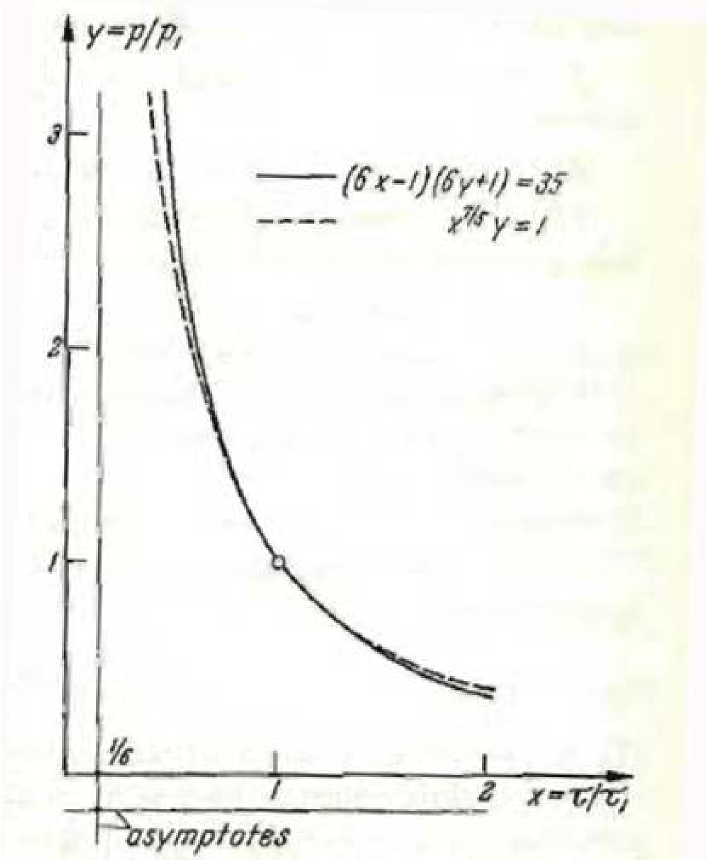

Fig. 12. Hugoniot curve (solid line) and adiabatic curve (broken line) for $\gamma = \frac{7}{5}$. The asymptotes of the Hugoniot curve are indicated.

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\rho_2 - (\gamma-1)\rho_1}{(\gamma+1)\rho_1 - (\gamma-1)\rho_2} \quad \text{and} \quad \frac{p_2 - p_1}{\rho_2 - \rho_1} = \gamma \frac{p_1 + p_2}{\rho_2 + \rho_1}$$

The fluid state in front of the shock wave, together with the shock speed G, suffices for the complete determination of the state behind the shock. Specifically, introducing the "Mach numbers"

$$M_1 = \frac{U_1}{c_1}, \quad M_2 = \frac{U_2}{c_2},$$

we have for the ideal gas,

$$\left.\begin{aligned} \frac{U_2 - U_1}{U_1} &= \frac{\tau_2 - \tau_1}{\tau_1} = \frac{2}{\gamma+1}\frac{1-M_1^2}{M_1^2} \\ \frac{p_2 - p_1}{p_1} &= \frac{2\gamma}{\gamma+1}(M_1^2 - 1) \\ \frac{T_2 - T_1}{T_1} &= \frac{2(\gamma-1)}{(\gamma+1)^2}\frac{(\gamma M_1^2+1)(M_1^2-1)}{M_1^2} \\ 1 - M_2^2 &= \frac{M_1^2 - 1}{1 + \frac{2\gamma}{\gamma+1}(M_1^2-1)} \end{aligned}\right\}. \tag{55.2}$$

It is sufficient to prove the first of these, for the others then follow readily. From Eqs. (54.9) and (54.6)

$$\begin{aligned} p_2 - p_1 &= m^2(\tau_1 - \tau_2) \\ &= \gamma p_1 M_1^2\left(1 - \frac{\tau_2}{\tau_1}\right) \end{aligned}. \tag{55.3}$$

It is a simple matter to eliminate the quantity $(p_2 / p_1 - 1)$ between Eqs. (55.1) and (55.3), and thus to obtain Eq. (55.2-1).

The entropy increase through a shock wave is given by

$$\frac{S_2 - S_1}{c_v} = \log\left(\frac{p_2}{p_1}\right)\left(\frac{\tau_2}{\tau_1}\right)^{\gamma}. \tag{55.4}$$

This may be brought into a somewhat different form when the flow is .steady. For by Eq. (54.5a) the stagnation enthalpy is the same in front of the shock as behind the shock, hence the same holds for the stagnation temperature, Thus, using the fact that entropy is constant along streamlines, we find that

$$\frac{p_2\tau_2^{\gamma}}{p_1\tau_1^{\gamma}} = \left(\frac{p_{02}}{p_{01}}\right)^{1-\gamma}\left(\frac{T_{02}}{T_{01}}\right)^{\gamma} = \left(\frac{p_{02}}{p_{01}}\right)^{1-\gamma}, \tag{55.5}$$

where the subscript zeros denote stagnation quantities. With the help of Eq. (55.5) we can therefore write Eq. (55.4) in the alternate form

$$S_2 - S_1 = R\log\left(\frac{p_{01}}{p_{02}}\right). \tag{55.6}$$

Since the critical speed q_* and the critical enthalpy I_* are also unchanged across a shock front [because of Eq. (54.5a) and the fact that $I = c^2/(\gamma-1)$], the above reasoning also serves to prove the useful chain of equalities,

$$\frac{p_{01}}{p_{02}} = \frac{\rho_{01}}{\rho_{02}} = \frac{p_{*1}}{p_{*2}} = \frac{\rho_{*1}}{\rho_{*2}} = \frac{Q_{*1}}{Q_{*2}} . \tag{55.7}$$

The ratio (55.7) is tabulated as the final column in Table 1 of Sect 37.

Finally, since $p_2 / p_1 > 1$ we must have $M_1 > 1$ according to Eq. (55.2a). This implies $U_1 > c_1$; that is, ***the relative normal flow velocity in front of a shock front is supersonic***. Conversely $M_2 < 1$ and ***the normal velocity behind a shock front is subsonic with respect to the shock front***[9]. These properties of a shock front are also true for a general gas, as we shall see in the following section, For steady flow, by Eq. (55.2-1) and Bernoulli's equation,

$$\begin{aligned} v_{1n}v_{2n} &= U_1 U_2 = \frac{\gamma-1}{\gamma+1}U_1^2 + \frac{2}{\gamma+1}c_1^2 = \frac{\gamma-1}{\gamma+1}(q_{\max}^2 - v_t^2) \\ &= q_*^2 - \frac{\gamma-1}{\gamma+1}v_t^2 \end{aligned} .$$

This interesting formula is due to Prandtl. For normal shocks it reduces to

$$q_1 q_2 = c_*^2 ,$$

56, Basic properties of the shock transition.

In this section we shall establish four important properties of the gas state on either side of a shock front, namely

I. The increase of entropy across a shock front is of third order in the shock strength $(\tau_1 - \tau_2)$.

II. Shocks are compressive, that is $p_2 > p_1$, $\tau_2 < \tau_1$.

III. The normal flow velocity relative to a shock front is supersonic at the front side, subsonic at the back side.

IV. The fluid state in front of a shock wave, together with the relative normal speed U_1, completely determines the state behind the shock wave.

For ideal gases with constant specific heats, properties (II) through (IV) were shown in the preceding section. It is remarkable that the same properties hold for arbitrary gases, if only very mild assumptions are made concerning their thermodynamic properties[10]. Specifically, we require of the gas that:

1. Its thermodynamical state Z is uniquely determined by the pressure and specific volume;

2. $$\left(\frac{\partial p}{\partial \tau}\right)_S < 0 , \quad \left(\frac{\partial^2 p}{\partial \tau^2}\right)_S > 0 .$$

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[9] Some geometric consequences of this for the shock line-characteristic line configuration are given in [17], p. 305.

[10] First noticed by H. Bethe and H. Weyl; see H. Weyl: Comm. Pure Appl. Math. 2, 103 (1949).

[It is tacitly assumed that all points in the first quadrant of the (p,τ)-plane define possible thermodynamic states *Z*. If (1) and (2) do not hold over the whole quadrant, our argument still applies (with only slight changes) in any ***convex*** region where (1) and (2) are valid.] Before giving the formal proof of I to IV we observe that our assumptions imply that adiabatics in the (p,τ)-plane are convex curves with negative slope (Fig. 13). Now it can be shown that (1) and (2) imply furthermore that $\left(\frac{\partial S}{\partial p}\right)_\tau$ must be either everywhere positive or everywhere negative. We shall suppose

$$\left(\frac{\partial p}{\partial \tau}\right)_\tau > 0^{11}, \qquad (56.1)$$

this being the case encountered in practice [actually only a few cases are known in which Eq. (56.1) is violated, the best example being liquid water below 4° centigrade; in any case, the alternative to Eq. (56.1) would make little difference]. Because of Eq. (56.1), adiabatics in the (p,τ)-plane corresponding to higher values of entropy lie above and to the right of adiabatics with lower entropy.

Proof of I to IV.

For fixed (p_1,τ_1) let us set

$$H(p,\tau) = 2(I - I_1) - (p - p_1)(\tau + \tau_1)\,. \qquad (56.2)$$

The locus $H = 0$ in the (p,τ)-plane is the Hugoniot curve for the state $Z_1 = (p_1,\tau_1)$; it will be denoted by $\widetilde{H}$. Any state $Z_2 = (p_2,\tau_2)$ which can be reached across a shock wave from the front state Z_1 must lie on $\widetilde{H}$. The differential of the function $H(p,\tau)$ is given by

$$dH = 2TdS - [(p - p_1)d\tau + (\tau_1 - \tau)dp]\,, \qquad (56.3)$$

which follows at once from $dI = TdS - \tau dp$. Along the Hugoniot curve $dH = 0$, hence

$$dS = 0 \quad \text{along} \quad \widetilde{H} \quad \text{at} \quad Z_1\,.$$

Again, by forming the second and third differentials of *H* and evaluating at Z_1, we find

$$d^2S = 0\,,\quad 2Td^3S = dpd^2\tau - d\tau d^2p\,,$$

along $\widetilde{H}$ at Z_1. Taking τ as independent variable now gives

$$dS = d^2S = 0 \quad \text{and} \quad d^3S > 0 \quad \text{for} \quad d\tau < 0\,.$$

This proves assertion I[12].

Another treatment of this topic was recently given by R. D. Cowan, J. Fluid Mech. 3, 531 (1958).

[11] These assumptions will be recognized as precisely those postulated in the earlier discussion of thermodynamical principles, see Eqs. (30.7) and (37.6).

[12] In fact, we have the following development of $S - S_1$ along the Hugoniot curve

We now set

$$r = \frac{p - p_1}{\tau - \tau_1};$$

r is the slope of the straight line joining Z_1 to Z. In terms of r, Eq. (56.3) takes the form

$$dH = 2TdS + (\tau_1 - \tau)^2 \, dr \,. \tag{56.4}$$

Now the adiabatic $\widetilde{U}$ through Z_1 is convex, so that r increases as we trace $\widetilde{U}$ from left to right. Thus by Eq. (56.4)

$$dH > 0 \quad \text{as we trace } \widetilde{U} \text{ from left to right.}$$

Since $H = 0$ at Z_1, it follows that $H < 0$ on the upper branch of $\widetilde{U}$ while $H > 0$ on the lower branch (see Fig. 13). If $\widetilde{R}$ denotes a ray r = const through Z_1, then $dH = 2TdS$ along $\widetilde{R}$.

Consider a ray $\widetilde{R}$ which is directed so that it intersects $\widetilde{U}$ only at Z_1. Then by virtue of Eq. (56.1) dS is always positive or always negative as we follow $\widetilde{R}$ away from Z_1. Since $dH = 2TdS$, such a ray cannot contain any points where $H = 0$. Next let $\widetilde{R}$ intersect $\widetilde{U}$ at a point Z_A on the upper branch of $\widetilde{U}$. Because all adiabatic curves are convex it is geometrically evident that as we follow $\widetilde{R}$ away from Z_1, S first increases from Z_1 to Z_0 (see Fig. 13), and decreases from then on. Since $dH = 2TdS$, H also increases and then

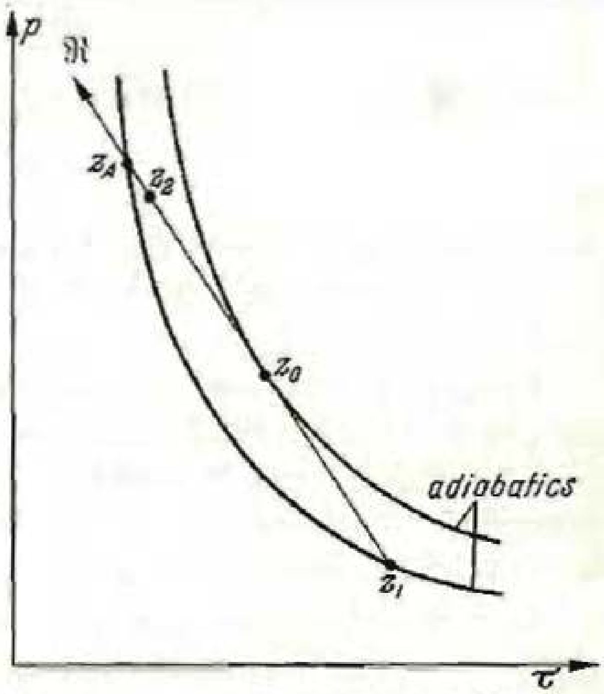

Fig. 13. (p, τ) diagram for the Hugoniot curve.

$$S - S_1 = \frac{1}{12T_1}\left(\frac{\partial^2 p}{\partial \tau^2}\right)_{S,1}(\tau_1 - \tau)^2 + \cdots$$

decreases as we follow $\widetilde{R}$ away from Z_1. But $H < 0$ when we arrive at Z_A, so that there must be exactly one point Z_2 on $\widetilde{R}$ between Z_0 and Z_A where $H = 0$. The entropy at this point is obviously greater than the entropy at Z_1.

Finally, if $\widetilde{R}$ intersects $\widetilde{U}$ on the lower branch, the same procedure shows that $\widetilde{R}$ can contain at most one point where $H = 0$. Such a point Z have to lie *below* $\widetilde{U}$ (recall that $H > 0$ on the lower branch of $\widetilde{U}$), and would therefore have $S < S_1$.

The Hugoniot curve is thus shown to be a simple line through Z_1 with $S > S_1$ on the upper branch and $S < S_1$ on the lower. According to the shock condition (54.5-5) only states Z on the upper branch can be reached across a shock wave from the front state Z_1. Since $p > p_1$ and $\tau < \tau_1$ on this branch, assertion II is proved.

To prove III, note from Fig. 13 that for a shock front joining the front state Z_1 and the back state Z_2,

$$\left(\frac{\partial p}{\partial \tau}\right)_{S,Z_2} < r < \left(\frac{\partial p}{\partial \tau}\right)_{S,Z_1} . \tag{56.5}$$

Since $\left(\frac{\partial p}{\partial \tau}\right)_S = -\rho^2 c^2$ and [see Eq. (54.6)]

$$r = -m^2 - -\rho^2 U^2 ,$$

the two inequalities (56.5) are equivalent, respectively, to $c_2 > U_2$ and $U_1 > c_1$. That is, in front of the shock the relative normal speed is supersonic while in back it is subsonic.

Finally, if the front state Z_1 and the relative normal speed U_1 are prescribed, then to find the back state we need only intersect the upper branch of the Hugoniot contour with the ray $\widetilde{R}$ of slope $r = -\rho_1^2 U_1^2$. There is ***exactly one*** such intersection (provided of course $U_1 > c_1$; otherwise there is no shock possible). The value U_2 is then found from $U_2 = (\rho_1 U_1)\tau_2$

In conclusion note that r decreases monotonically as Z moves away from Z_1 along the upper branch of the Hugoniot contour. Moreover as r decreases, S must increase because of Eq. (56.4): that is, ***for a given thermodynamic state in front of a shock wave, the greater the normal speed*** U_1***, the greater the change in entropy across the shock***. The entropy increase across a detached bow wave in supersonic flight, for example, is greatest at the central streamline and monotonically decreases as one proceeds out along the wave.

57. The shock layer.

In real gases, the passage of a particle through a shock front is not an

instantaneous process in which the particle suddenly finds itself confronted with the new state behind the shock, but rather it involves a rapid transition from the front state to the back state through a narrow region, the shock layer. In this region the motion cannot be described adequately by perfect fluid theory, and therefore there is some question as to the validity of the preceding derivation of the Rankine-Hugoniot conditions. Considerable interest has thus been focused on the shock layer, and its structure has been widely studied. These investigations have yielded an increased understanding of the nature of shock waves, some information concerning the thickness of a shock layer, and an alternative justification of the Rankine-Hugoniot conditions. Moreover the comparison of the. theoretical results with experiment provides a crucial testing ground for the validity of the Navier-Stokes equation. For mathematical reasons the problem has been considered mainly in the simplest case of one-dimensional steady motion, but this is in many respects the prototype of all shock phenomena.

The problem of shock structure in one-dimensional motion includes two essential phases: first, proving that the differential equations of viscous fluids actually admit a solution of the general type desired (the velocity profile should be of the form shown in Fig. 14), and second, describing the shock profile with particular emphasis on its thickness. The first of these problems, after being studied inconclusively by Rayleigh[13],, was solved by Von Mises[14] and Gilbarg[15]. The second problem - quantitative description of the profile - involves fairly difficult numerical computation and therefore lies mostly outside the scope of this article.

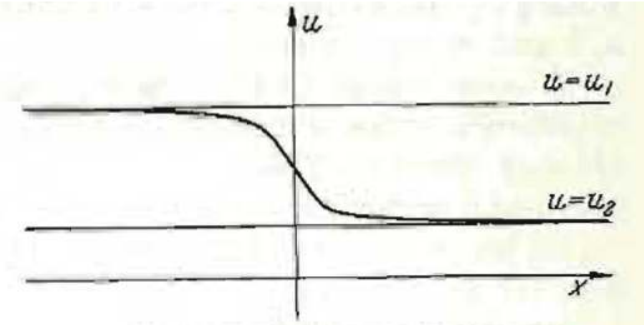

Fig. 14. Velocity profile in a shock layer.

Since the work below is based on the equations of continuum mechanics, it is only fair to point out certain objections to the applicability of these equations. It is claimed, first, that since the thickness of a shock layer is of the order of a few molecular mean free paths, therefore any approach by continuum mechanics is *a*

[13] Lord Rayleigh: Proc. Roy. Soc. Lond., Ser. A 84. 247 (1910); see also G. I. Taylor: Proc. Roy. Soc. Lond., Ser. A 84, 371 (1910).
[14] R. von Mises: J. Aeronaut. Sci. 17, 551 (1950). The shock layer in three-dimensional flow is discussed by Ludford, Quart. Appl. Math. 10, 1 (1952).
[15] D. Gilbarg: Amer. J. Math. 73, 256 (1951).

priori invalid[16]; and second, that continuum mechanics predicts too small values for shock thicknesses (bearing out the former criticism). The second objection has been completely negated by the work of Gilbarg and Paolucci[17], who have shown that if account is taken of the temperature dependence of viscosity and heat conductivity - effects only partially considered by most earlier investigators, then the Navier-Stokes equation provides at least as good values for shock thickness as does kinetic theory, values, moreover, which are in acceptable agreement with recent experiments[18]. Finally, the first objection, upon due reflection, can hardly be considered convincing[19]. For these reasons we definitely do not believe it outmoded to use continuum methods in studying the shock layer.

The mathematical theory of the shock layer, so far as it falls under the scope of continuum mechanics, is based on the equations of one-dimensional steady motion, namely

$$\left.\begin{aligned} &\frac{d}{dx}(\rho u) = 0 \\ &\rho u \frac{du}{dx} = \frac{dT_{xx}}{dx} \\ &\rho u \frac{dE}{dx} = T_{xx}\frac{du}{dx} - \frac{dq}{dx} \end{aligned}\right\}, \tag{57.1}$$

[cf. Eqs. (5.4), (6.7), and (33.4)]. From the following chapter, Sects. 61 and 63, we draw the constitutive formulae

$$\left.\begin{aligned} &T_{xx} = -p + (\lambda + 2\mu)\frac{du}{dx} \\ &q = -\kappa \frac{dT}{dx} \end{aligned}\right\}. \tag{57.2}$$

Eqs. (57.1) can be integrated once without difficulty, and with the help of Eq.(57.2) this integrated form may be written

$$\rho u = m \tag{57.3}$$

and

$$\left.\begin{aligned} (\lambda + 2\mu)\frac{du}{dx} &= p + m(u-a) \\ \kappa \frac{dT}{dx} &= m[E - \frac{1}{2}(u-a)^2 + b] \end{aligned}\right\} \tag{57.4}$$

where a, b and m are constants. Here $p = p(\rho, T)$ and $E = E(\rho, T)$ can be considered known functions of u and T, by virtue of Eq. (57.3).

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16 The most recent statements to this effect will be found in [43], p. 126 and [23], p. 550.

17 D. Gilbarg and D. Paolucci: J. Rational Mech. Anal. 2, 617 (1953).

18 F. S. Sherman: NACA Tech. Note 3298 (1955); see also the comprehensive review of G. N. Patterson [34], Chap. 4.

19 A. E. Puckett and H. J. Stewart [Quart. Appl. Math. 7, 457 (1950)] conclude that the Navier-Stokes equation should apply except in certain exceptional situations, e.g., where dissociation or condensation effects are important, in highly rarified gases, or when the stagnation temperature is very high. See also the penetrating remarks of C. Truesdell concerning the relative validity of methods based on continuum mechanics and on kinetic theory: J. Rational Mech. Anal. 2. 678 (1954); 5, 55

A solution $u=u(x)$, $T=T(x)$, of the system (57.4) is called a ***shock layer*** if, as x tends respectively to $-\infty$ and $+\infty$ the point $Z(u,T)$ tends to finite "endvalues" $Z_1(u_1,T_1)$ and $Z_2(u_2,T_2)$, with $u_2<u_1$. It is easy to see that for a shock layer to exist the endvalues must reduce the right members of Eqs. (57.4) to zero, or m other words,

$$\left.\begin{aligned} \rho_1 u_1 = \rho_2 u_2 = m \\ p_1 + m u_1 = p_2 + m u_2 = am \\ E_1 - \frac{1}{2}(u_1-a)^2 = E_2 - \frac{1}{2}(u_2-a)^2 = b \end{aligned}\right\}. \qquad (57.5)$$

These conditions are equivalent to the Rankine-Hugoniot conditions (54.5); therefore ***a shock layer joining two states Z_1 and Z_2 can occur only if Z_1 and Z_2 are allowable initial and final states, respectively, of a normal shock of an ideal fluid having the same equations of state as the given fluid***. Conversely, if states Z_1 and Z_2 satisfy the Rankine-Hugoniot conditions, then to find a shock layer joining Z_1 and Z_2 one must solve the differential equations (57.4) where the values a, b and m are determined from Eq. (57.5).

A consequence of the above result may be mentioned in passing: ***the speed of advance of an infinitely weak shock layer relative to the fluid is precisely the speed of sound*** c. This is the last of our series of justifications for calling c the speed of sound.

In the sequel we shall specialize to ideal gases with constant specific heats[20]. Eqs. (57.4) then take the form

$$\left.\begin{aligned} (\lambda+2\mu)\frac{du}{dx} = m(RT/u + u - a) \\ \kappa\frac{dT}{dx} = m[c_v T - \frac{1}{2}(u-a)^2 + b] \end{aligned}\right\}. \qquad (57.6)$$

[These equations admit an exact solution when

$$\frac{c_p(\lambda+2\mu)}{\kappa} = 1 .$$

Indeed in this case if Eq. (57.6-1) is multiplied by u and added to Eq. (57.6-2) there results

$$(\lambda+2\mu)\frac{d}{dt}\left(c_p T + \frac{1}{2}u^2\right) = m\left(c_p T + \frac{1}{2}u^2 - \frac{1}{2}a^2 - b\right),$$

whence

$$c_p T + \frac{1}{2}u^2 = \frac{1}{2}a^2 + b = \text{const} .$$

(1956)].

[20] For a discussion of the case of a general gas, see D. Gilbarg, Amer. J. Math. 73, 256 (1951).

Using Eq. (57.7) to eliminate T from Eq. (57.6-1) leads obviously to

$$(\lambda + 2\mu)\frac{du}{dx} = \text{const}\frac{(u-u_1)(u-u_2)}{u}$$

which is easily integrated if $\lambda + 2\mu$ is constant, and numerically integrable[21] if $\lambda + 2\mu = f(T)$.]

We now show, following Gilbarg[22], that ***there exists exactly one shock layer joining any two allowable endstates*** Z_1 ***and*** Z_2. In order to do this, it is useful to visualize the direction field of Eqs. (57.6) in the $Z(u,T)$ -plane. This has the form shown in Fig. 15, where the parabolic curves have the equations

$$\begin{cases} L(u,T) = RT/u + u - a = 0 \\ M(u,T) = c_c T - \frac{1}{2}(u-a)^2 - b = 0 \end{cases},$$

necessarily intersecting at Z_1 and Z_2. The problem is to find an integral curve ("shock curve") joining Z_1 and Z_2. Now Z_1 and Z_2 are singular points of Eq.(57.6) so that to determine the nature of the direction field in the neighborhood of these points it is necessary to examine the roots of the "characteristic" equation

$$\begin{vmatrix} \frac{1}{\lambda+2\mu}\frac{\partial L}{\partial u} - \delta & \frac{1}{\lambda+2\mu}\frac{\partial L}{\partial T} \\ \frac{1}{\kappa}\frac{\partial M}{\partial u} & \frac{1}{\kappa}\frac{\partial M}{\partial T} - \delta \end{vmatrix} = 0. \tag{57.8}$$

It is easily found that the roots of Eq. (57.8) are real and of equal sign at Z_1, real and of opposite sign at Z_2. Hence the former point is a node and the latter a saddle[23], as indicated in Fig. 15. If we follow backwards along the integral curve entering Z_2 from the right, then we never cross the curves L =0 or M =0 (the arrows are directed so that this cannot occur). We must therefore approach the other singular point Z_1 as $x \to -\infty$. This proves the existence of a shock layer joining Z_1 and Z_2, and the uniqueness of this layer follows by a similar argument.

[21] 2 L.H. Thomas: J. Chem. Phys. 12, 449 (1944). M. Morduchow and P. Libby: J. Aeronaut. Sci. 16, 674 (1949). R. Von Mises: J. Aeronaut. Sci. 17, 551 (1950). L. Meyerhoff: J. Aeronaut. Sci. 17, 775 (1950). A. E. Puckett and H. L. Stewart: Quart. Appl. Math. 7, 457 (1953).

[22] D. Gilbarg: Amer. J. Math. 73,256 (1951).

[23] Cf. E. A. Coddington and N. Levinson: Theory of Ordinary Differential Equations, Chap. 15, New

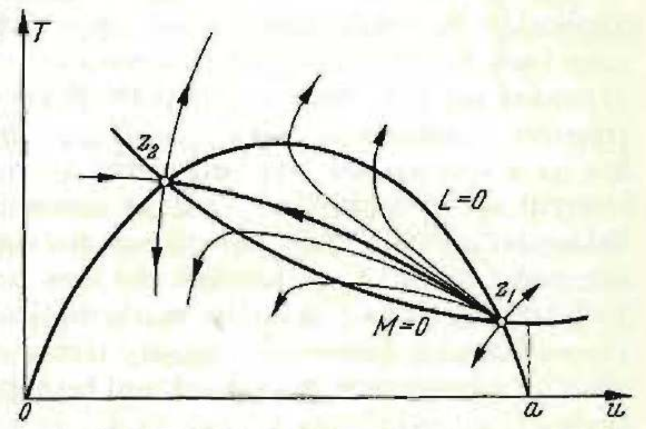

Fig. 15. Field of integral curves and the shock curve in the Z-plane.

(The method just outlined provides a practical procedure for the numerical determination of shock profiles, cf. Gilbarg and Paolucci.)

It remains only to verify that for small κ and $\lambda+2\mu$ the shock profile has qualitatively the form shown in Fig. 14. with the transition region arbitrarily narrow. Rather than give a formal proof of this fact, it will be sufficient to make it plausible. First of all, the monotone decrease of u is obvious from the fact that $\frac{du}{dx}<0$ along the entire shock curve. Now suppose we wish 90%, say, of the change in u to occur in an interval of width less than ε. In other words, we wish the "shock curve" to cross the greater part of the distance between Z_1 and Z_2 with x-variation less than ε. A glance at Eq. (57.6) shows that this will indeed occur if $\lambda+2\mu$ and κ are sufficiently small whether they are variable or not.

Further discussion of Eq. (57.6) is facilitated by the substitution $v=u^2$. The resulting equation may be treated geometrically with some ease, and simple and valuable inequalities for the shock thickness can be obtained (cf. the paper of von Mises). A comprehensive treatment of the shock layer from a more physical point of view has recently been given by Lighthill[24].

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[24] M. J. Lighthill: Surveys in Applied Mechanics, p.250. Cambridge 1956.

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14. R. von Mises: J. Aeronaut. Sci. 17, 551 (1950).

R. von Mises, On the Thickness of a Steady Shock Wave, J. Aeronaut. Sci. 17, No. 9, 551-554 (1950).