CHAPTER 1. MANIFOLDS

1.1 Elementary properties of manifold

The nicest example of a metric space is Euclidean *n*-space R^n, consisting of all *n*-tuples $x = (x^1, \ldots, x^n)$ with each $x^i \in R$, where R is the set of real numbers. Whenever we speak of R^n as a metric space, we shall assume that it has the "usual metric"

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (y^i - x^i)^2} \;,$$

unless another metric is explicitly suggested. For n = 0 we will interpret R^0 as the single point $0 \in R$.

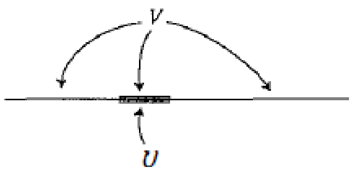
A manifold is supposed to be "locally" like one of these exemplary metric spaces R^n. To be precise, a manifold is a metric space *M* with the following property:

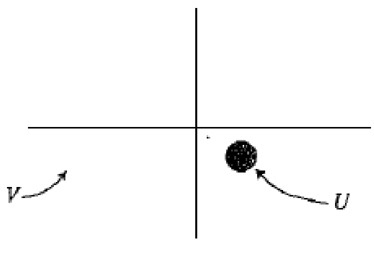
If $x \in M$, then there is some neighborhood U of x and some integer $n \geq 0$ such that U is homeomorphic to R^n.

The simplest example of a manifold is, of course, just R^n itself; for each $x \in R^n$ we can take *U* to be all of R^n. Clearly, R^n supplied with an equivalent metric (one which makes it homeomorphic to R^n with the usual metric), is also a manifold. Indeed, a hasty recollection of the definition shows that anything homeomorphic to a manifold is also a manifold -- the specific metric with which *M* is endowed plays almost no role, and we shall almost never mention it.

[If you know anything about topological spaces, you can replace "metric space" by "topological space" in our definition; this new definition allows some pathological creatures which are not metrizable and which fail to have other properties one might carelessly assume must be possessed by spaces which are locally so nice. Appendix A contains remarks, supplementing various chapters, which should be consulted if one allows a manifold to be non-metrizable.]

The second simplest example of a manifold is an **open ball** in R^n; in this case we can take *U* to be the entire open ball since an open ball in R^n is



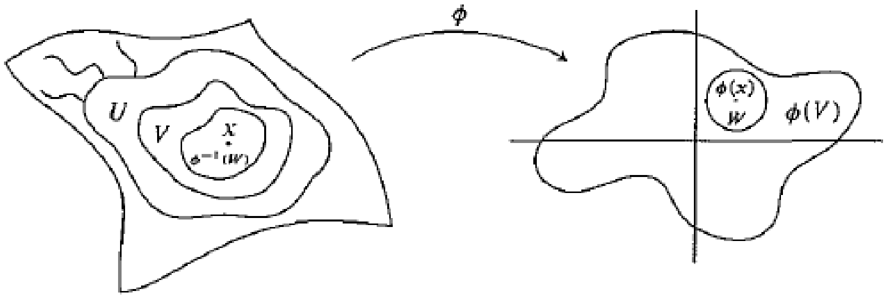


homeomorphic to R^n. This example immediately suggests the next: any open subset V of R^n is a manifold -- for each $x \in V$ we can choose U to

be some open ball with $x \in U \subset V$. Exercising a mathematician's penchant for generalization, we immediately announce a proposition whose proof is left to the reader: An open subset of a manifold is also a manifold (called, quite naturally, an ***open submanifold*** of the original manifold).

The open subsets of R^n already provide many different examples of manifolds (just how many is the subject of Problem 24), though by no means all. Before proceeding to examine other examples, which constitute most of this chapter, some preliminary remarks need to be made.

If x is a point of a manifold M, and U is a neighborhood of x (U contains some open set V with $x \in V$) which is homeomorphic to R^n by a



homeomorphism $\phi : U \to R^n$, then $\phi(V) \subset R^n$ is an open set containing $\phi(x)$. Consequently, there is an open ball W with $\phi(x) \in W \subset \phi(V)$. Thus $x \in \phi^{-1}(W) \subset V \subset U$. Since $\phi : U \to R^n$ is continuous, the set $\phi^{-1}(W)$ is open *in* V, and thus open in M; it is, of course, homeomorphic to W, and thus to R^n. This complicated little argument just shows that we can always choose the neighborhood U in our definition to be an open neighborhood.

With a little thought, it begins to appear that, in fact, V *must* be open. But to prove this, we need the following theorem, stated here without proof.[1]

1. THEOREM. *If $U \subset R^n$ is open and $f : U \to R^n$ is one-one and continuous, then $f(U) \subset R^n$ is open.* (It follows that *$f(V)$ is open for any open $V \subset U$, so f^{-1} is continuous, and f is a homeomorphism.*)

Theorem 1 is called "***Invariance of Domain***", for it implies that the property of being a "domain" (a connected open set) is invariant under one-one continuous maps into R^n. The proof that the neighborhood U in our definition must be open is a simple deduction from Invariance of Domain, left to the reader as an easy exercise (it is also easy to see that if Theorem 1

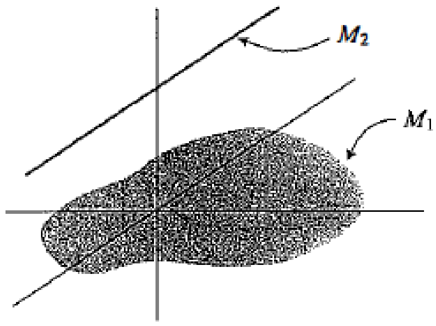
[1] All proofs require some amount of machinery. The quickest routes are probably provided by Vick, *Homology Theory* and Massey, *Singular Homology Theory*. An old-fashioned, but pleasantly geometric, treatment may be found in Newman, *Topology of Plane Sets*.

were false, then there would be an example where the U in our definition was not open).

We next turn our attention to the integer n appearing in our definition. Notice that n may depend on the point x. For example, if $M \subset R^3$ is

$$\begin{aligned} M &= \{(x,y,z): z=0\} \cup \{(x,y,z): x=0, z=0\} \\ &= M_1 \cup M_2 \end{aligned},$$

then we can choose $n = 2$ for points in M_1 and $n=1$ for points in M_2. This



example, by the way, is an unnecessarily complicated device for producing one manifold from two. In general, given M_1 and M_2, with metrics d_1 and d_2, we can first replace each d_i with an equivalent metric $\overline{d}_i$ such that $\overline{d}_i(x,y) < 1$ for all $x, y \in M_i$; for example, we can define

$$\overline{d}_i = \frac{d_i}{1+d_i} \quad \text{or} \quad \overline{d}_i = \min(d_i, 1)\,.$$

Then we can define a metric d on $M = M_1 \cup M_2$ by

$$d(x,y) = \begin{cases} \overline{d}_i(x,y) & \text{if there is some i such that } x, y \in M_i \\ 1 & \text{otherwise} \end{cases}$$

(We assume that M_1 and M_2 are disjoint; if not, they can be replaced by new sets which are). In the new space M, both M_1 and M_2 are open sets. If M_1 and M_2 are manifolds, M is clearly a manifold also. This construction can be applied to any number of spaces -- even uncountably many; the resulting metric space is called the ***disjoint union of the metric spaces*** M_i. A disjoint union of manifolds is a manifold. In particular, since a space with one point is a manifold, so is any discrete space M, defined by the metric

$$d(x,y) = \begin{cases} 0\_\text{if}\_x = y \\ 1\_\text{if}\_x \neq y \end{cases}.$$

Although different n's may be required at different points of a manifold M, it would seem that only one n can work at a given point $x \in M$. For the proof of this intuitively obvious assertion we have recourse once again to Invariance of Domain. As a first step, we note that R^n is not homeomorphic to R^m when $n \neq m$, for if $n>m$, then there is a one-one continuous map from R^m into a non-open subset of R^n. The further deduction, that the n of our definition is unique at each $x \in M$, is left to the reader. This unique n is called the ***dimension*** of M at x. A manifold has ***dimension*** n or is n-***dimensional*** or is an n-***manifold*** if it has dimension n at each point. It is convenient to refer to the manifold M as M^n when we want to indicate that M has dimension n.

Consider once more a discrete space, which is a 0-dimensional manifold. The only compact subsets of such a space are finite subsets. Consequently, an uncountable discrete space is not σ-compact (it cannot be written as a countable union of compact subsets). The same phenomenon occurs with higher-dimensional manifolds, as we see by taking a disjoint union of uncountably many manifolds homeomorphic to R^n. In these examples, however, the manifold is not connected. We will often need to know that this is the only way in which σ-compactness can fail to hold.

2. THEOREM. *If X is a connected, locally compact metric space, then X is σ-compact.*

PROOF For each $x \in X$ consider those numbers $r > 0$ such that the closed ball

$$\{y \in X : d(x, y) \leq r\}$$

is a compact set (there is at least one such $r > 0$, since X is locally compact). The set of all such $r > 0$ is an interval. If, for some x, this set includes all $r > 0$, then X is σ-compact, since

$$X = \bigcup_{n=1}^{\infty} \{y \in X : d(x, y) \leq n\}.$$

If not, then for each $x \in X$ define $r(x)$ to be one-half the least upper bound of all such r.

The triangle inequality implies that

$$\{y \in X : d(x_1, y) \leq r\} \subset \{y \in X : d(x_2, y) \leq r + d(x_1, x_2)\}$$

so that

$$\{y \in X : d(x_1, y) \leq r - d(x_1, x_2)\} \subset \{y \in X : d(x_2, y) \leq r\}$$

which implies that

(1) $$r(x_1) \geq r(x_2) - \frac{1}{2} d(x_1, x_2).$$

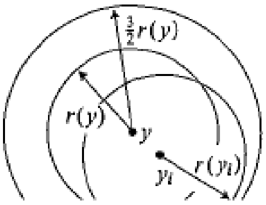
Interchanging $x1$, and $x2$ gives

(2) $$|r(x_1) - r(x_2)| \leq \frac{1}{2} d(x_1, x_2)$$

so the function $r : X \to R$ is continuous. This has the following important consequence. Suppose $A \subset X$ is compact. Let A' be the union of all closed

balls of radius $r(y)$ and center y, for all $y \in A$. Then A' is also compact. The proof is as follows.

Let $z_1, z_2, z_3, \ldots$ be a sequence in A'. For each i there is a $y_i \in A$ such that z_i is in the ball of radius $r(y_i)$ with center y_i. Since A is compact, some subsequence of the y_i, which we might as well assume is the sequence itself, converges to some point $y \in A$. Now the closed ball B of radius $(3/2)r(y)$ and center y is compact. Since $y_i \to y$ and since the function r is continuous, eventually the closed balls



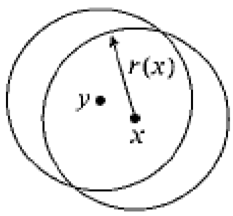
$$\{y \in X : d(y, y_i) \le r(y_i)\}$$

are contained in B. So the sequence z_i is eventually in the compact set B, and consequently some subsequence converges. Moreover, the limit point is actually in the closed ball of radius $r(y)$ and center y (Problem 10). Thus A' is compact.

Now let $x_0 \in X$ and consider the compact sets

$$A_1 = \{x_0\},$$

$$A_{n+1} = A_n{}'.$$



Their union A is clearly open. It is also closed. To see this, suppose that x is a point in the closure of A. Then there is some $y \in A$ with $d(x,y) < \frac{2}{3} r(x)$.

By (1),

$$\begin{aligned} r(y) &\ge r(x) - \frac{1}{2} d(x,y) \\ &> r(x) - \frac{1}{2} \cdot \frac{2}{3} r(x) = \frac{2}{3} r(x) \\ &> d(x,y) \end{aligned}$$

This shows that if $y \in A_n$, then $x \in A_n{}'$, so $x \in A$.

Since X is connected, and $A \ne \phi$ is open and closed, it must be that X=A, which is σ-compact. (QED)

1.2 Examples of manifold

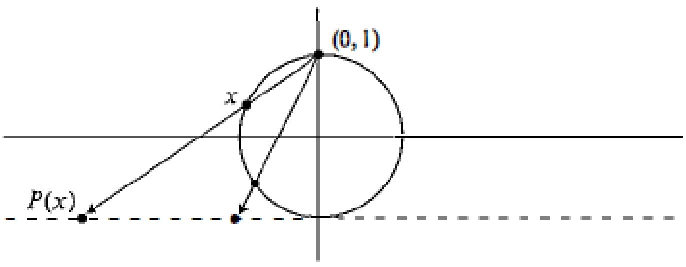
After this hassle with point set topology, we present the long-promised

examples of manifolds. The only connected I-manifolds are the line R and the circle, or 1-dimensional sphere, S^1, defined by

$$S^1 = \{x \in R^2 : d(x, y) = 1\}.$$

The function $f:(0,2\pi) \to S^1$ defined by $f(\theta) = (\cos\theta, \sin\theta)$ is a homeomorphism; it is even continuous, though not one-one, on $[0,2\pi]$. We will often denote the point $(\cos\theta, \sin\theta) \in S^1$ simply by $\theta \in [0,2\pi]$. (Of course, it is always necessary to check that use of this notation is valid.) The function $g:(-\pi,\pi) \to S^1$, defined by the same formula, is also a homeomorphism; together with f it shows that S^1 is indeed a manifold.

There is another way to prove this, better suited to generalization. The projection P from the point (0,1) onto the line $R \times \{-1\} \subset R \times R$, illustrated



in the above diagram, is a homeomorphism of $S^1 - \{(0.1)\}$ onto $R \times \{-1\}$: this is proved most simply by calculating $P: S^1 - \{(0,1)\} \to R \times \{-1\}$ explicitly. The point (0,1) may be taken care of similarly, by projecting onto $R \times \{1\}$, or it suffices to note that S^1 is "homogeneous" -there is a homeomorphism taking point into any other (namely, an appropriate rotation of R^2). Considerations similar to these now show that the n-sphere

$$S^n = \{x \in R^{n+1} : d(x,0) = 1\}$$

is an n-manifold. The 2-sphere S^2, commonly known as "the ***sphere***", is our first example of a compact 2-manifold or surface.

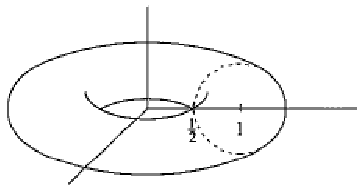
From these few manifolds we can already construct many others by noting that if M_i are manifolds of dimension n_i (i = 1, 2), then $M_1 \times M_2$ is an $(n_1 + n_2)$ -manifold. In particular

$$\underbrace{S^1 \times \cdots \times S^1}_{n\_times}$$

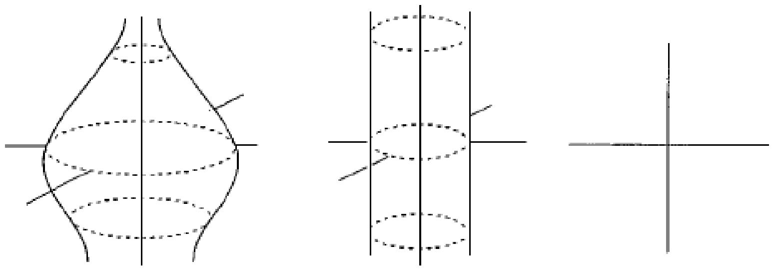
is called the n-***torus***, while $S^1 \times S^1$ is commonly called "***the torus***". It is

obviously homeomorphic to a subset of R^4, and it is also homeomorphic to a certain subset of R^3 which is what most people have in mind when they speak of "the torus": This subset may be obtained by revolving the circle

$$\{(0,y,z)\in R^3 : (y-1)^2+z^2=1/4\}$$



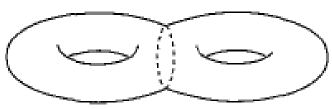
around contained in $\{(0,y,z)\in R^3 : y>0\}$. The resulting surface, called a ***surface of revolution***, has components homeomorphic either to the torus or to the cylinder $S^1\times R$, the latter of which is also homeomorphic to the annulus, the region of the plane contained between two concentric circles.



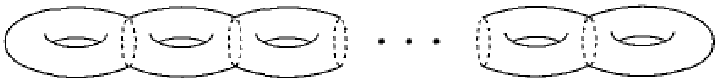
The next simple compact 2-manifold is the ***2-holed torus***. To provide a more explicit description of the 2-holed torus, it is easiest to begin with a

"**handle**", a space homeomorphic to a torus with a hole cut out; more precisely, we throw away all the points on one side of a certain circle, which remains in our handle, and which will be referred to as the boundary of the

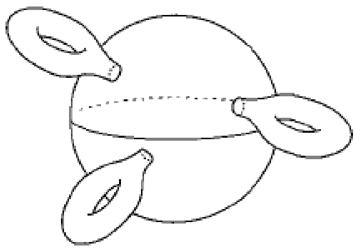
handle. The 2-holed torus may be obtained by piecing two of these together; it is also described as the disjoint union of two handles with corresponding



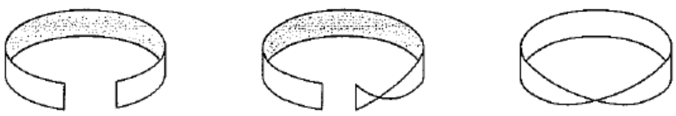
points on the boundaries “identified”.



The n-holed torus may be obtained by repeated applications of this procedure. It is homeomorphic to the space obtained by starting with the disjoint union of n handles and a sphere with n holes, and then identifying points on the boundary of the ith handle with corresponding points on the ith boundary piece of the sphere.

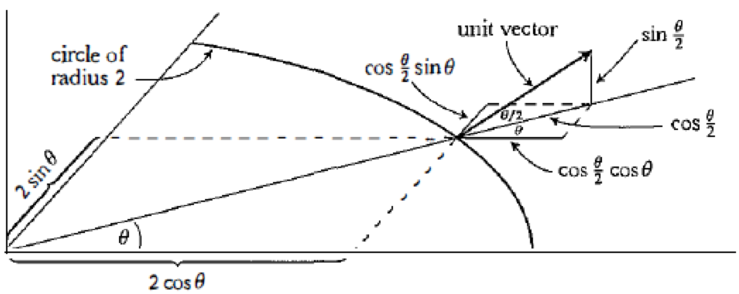


There is one 2-manifold of which most budding mathematicians make the acquaintance when they still know more about paper and paste than about metric spaces-the famous ***Möbius strip***, which you "make" by giving a strip of paper a half twist before pasting its ends together. This can be described

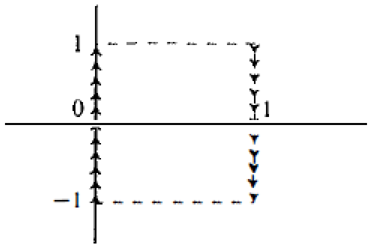


analytically as the image in R^3 of the function $f:[0,2\pi]\times(-1,1)\to R^3$ defined by

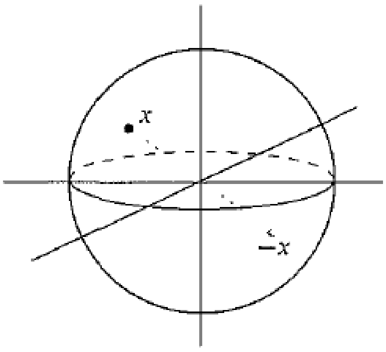
$$f(\theta,t)=\left(2\cos\theta+t\cos\frac{\theta}{2}\cos\theta, 2\sin\theta+t\cos\frac{\theta}{2}\sin\theta, t\sin\frac{\theta}{2}\right)$$



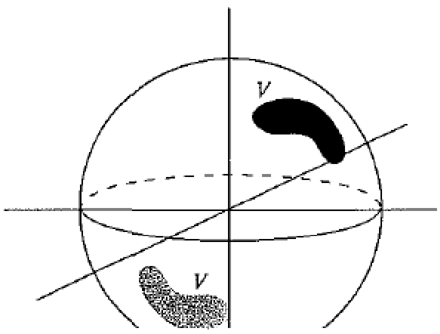
If we define f on $[0,2\pi]\times[-1,1]$ instead, we obtain the Möbius strip with a boundary; as investigation of the paper model will show, this boundary is homeomorphic to a circle, not to two disjoint circles. With our recently introduced terminology, the Möbius strip can also be described as $[0,1]\times(-1,1)$ with $(0, t)$ and $(1, -t)$ "identified".



We have not yet had to make precise this notion of "identification", but our next example will force the issue. We wish to identify each point $x \in S^2$ with its antipodal point $-x \in S^2$. The space which results, the projective plane, P^2, is a lot harder to visualize than previous examples; indeed, there is no subset of R^3 which represents it adequately.



The precise definition of P^2 uses the same trick that mathematicians always use when they want two things which are not equal to be equal. The ***points*** of P^2 are defined to be the sets $\{p, -p\}$ for $p \in S^2$. We will denote this set by $[p] \in P^2$, so that $[-p] = [p]$. We thus have a map $f : S^2 \to P^2$ given by $f(p) = [p]$, for which $f(p) = f(q)$ implies $p = \pm q$. We will postpone for a while the problem of defining the metric



giving the distance between two points $[p]$ and $[q]$, but we can easily say what the open sets will turn out to be (and this is all you need to know in order to check that P^2 is a surface). A subset $U \subset P^2$ will be open if and only if $f^{-1}(U) \subset S^2$ is open. This just means that the open sets of P^2 are of the form $f(V)$ where $V \subset S^2$ is an open set with the additional important property that if it contains *p* it also contains *-p*.

In exactly the same way, we could have defined the points of the Möbius strip *M* to be

$$\text{all points } (s,t) \in (0,1) \times (-1,1)$$

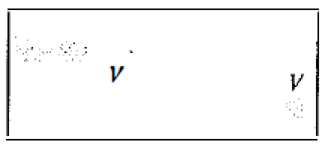
together with

$$\text{all sets } \{(0,t),(1,-t)\}, \text{ denoted by } [(0,t)] \text{ or } [(1,-t)]$$

There is a map $f : [0,1] \times (-1,1) \to M$ given by

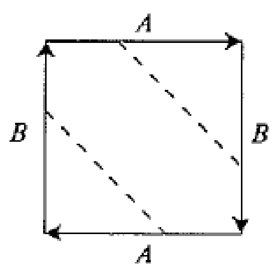
$$f((s,t)) = \begin{cases} (s,t)\text{ if } s \neq 0,1 \\ [(s,t)]\text{ if } s = 0 \text{ or } 1 \end{cases},$$

and $U \subset M$ is open if and only if $f^{-1}(U) \subset [0,1] \times (-1,1)$ is open, so that the open sets of *M* are of the form *f*(*V*) where *V* is open and contains (*s*, - *t*) whenever it contains (*s*, *t*) for *s* = 0 or 1.

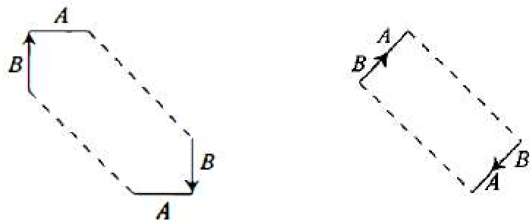


To get an idea of what P^2 looks like, we can make things easier for ourselves by first throwing away all points of S^2 below the (*x*, *y*)-plane, since they are identified with points above the (*x*, *y*)-plane anyway. This leaves the upper hemisphere (including the bounding circle), which is homeomorphic to the disc

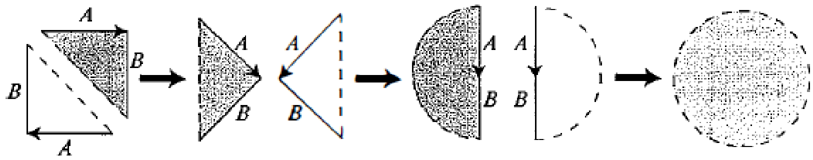
$$D^2 = \{x \in R^2 : d(x,0) \leq 1\},$$



and we must identify each $p \in S^1$ with $-p \in S^1$. Squaring things off a bit, this is the same as identifying points on the sides of a square according to the scheme shown below (points on sides with the same label are identified in such a way that the heads of the arrows are identified with each other). The dotted lines in this picture are the key to understanding P^2. If we distort the region between them a bit we see that the front part of B followed by the back part of A, at the upper left, is to be identified with the same thing at the lower right, in reverse direction; in other words, we obtain a Möbius strip with a boundary (namely, the dotted line, which is a single circle). If this Möbius

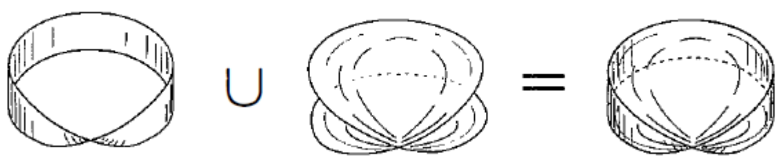


strip is removed, we are left with two pieces which can be rearranged to form something homeomorphic to a disc. The projective plane is thus obtained



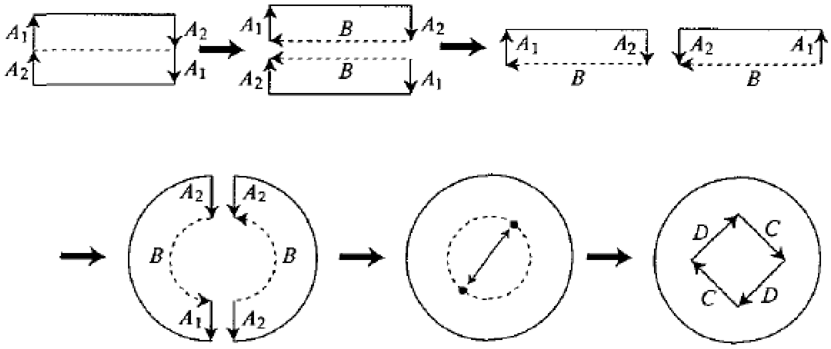
from the disjoint union of a disc and a Möbius strip with a boundary, by identifying points on the boundary and points on the boundary of the disc, both of which are circles. Thus to make a model of P^2 we just have to sew a circular piece of cloth and a cloth Möbius strip together along their edges. Unfortunately, a little experimentation wil1 convince you that this cannot be done (without having the two pieces of doth pass through each other).

The subset of R^3 obtained as the union of the Möbius strip and a disc, although not homeomorphic to P^2, can still be described mathematically in

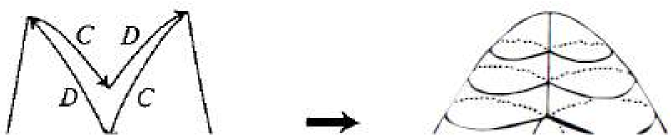


terms of P^2. There is dearly a continuous function $f: P^2 \to R^3$ whose image is this subset; moreover; although f is not one-one, it is **locally one-one**, that is, every point $p \in P^2$ has a neighborhood U on which f is one-one. Such a function f is called a ***topological immersion*** (the single word "**immersion**" has a more specialized meaning, explained in Chapter 2). We can thus say that P^2 can be topologically immersed in R^3, although not **topologically imbedded** (there is no homeomorphism f from P^2 to a subset of R^3). In R^4, however, with an extra dimension to play around with, the disc can be added so as not to intersect the Möbius strip.

Another topological immersion of P^2 in R^3 can be obtained by first immersing the Möbius strip so that its boundary circle lies in a plane; this can be done in the following way. The figures below show that the Möbius strip may be obtained from an annulus by identifying opposite points of the inner circle. (This is also obvious from the fact that the Möbius strip is the



projective plane with a disc removed.) This inner circle can be replaced by a quadrilateral. When the resulting figure is drawn up into 3-space and the appropriate identifications are made we obtain the "**cross-cap**". The cross-cap together with the disc at the bottom is a topologically immersed



P^2.

The one gap in the preceding discussion is the definition of a metric for P^2. The missing metric can be supplied by an appeal to Problem 3-1, which will later be used quite often, and which the reader should peruse sometime before reading Chapter 3. Roughly speaking, it shows that things like P^2, which ought to be manifolds, are. (Those who know about topological spaces will recognize it as a disguised case of the Urysohn Metrization Theorem.) For the present, however, we will obtain our metric by a trick that simultaneously provides an imbedding of P^2 in R^4. Consider the function $f: S^2 \to R^4$ defined by

$$f(x, y, z) = (yz, xz, xy, x^2 + 2y^2 + 3z^2).$$

Clearly $f(p) = f(-p)$. We maintain that $f(p) = f(q)$ implies that $p = \pm q$. To prove this, suppose that $f(x,y,z) = f(a,b,c)$. We have, first of all

$$(1) \qquad \begin{cases} yz = bc \\ xz = ac \\ xy = ab \end{cases}.$$

If $a, b, c \neq 0$, this leads to

$$(2) \qquad \begin{cases} y = \dfrac{bx}{a} \\ z = \dfrac{cx}{a} \end{cases}.$$

Now

$$\begin{aligned}(x+y+z)^2 &= x^2+y^2+z^2+2(xy+xz+yz) \\ &= 1+2(xy+xz+yz)\end{aligned},$$

so we also have

$$(x+y+z)^2 = (a+b+c)^2,$$

hence

(3) $$a+b+c = \pm(x+y+z).$$

Using (2), this gives

$$a+b+c = \pm x\left(1+\frac{b}{c}+\frac{c}{a}\right) = \pm x\left(\frac{a+b+c}{a}\right),$$

so $x = \pm a$. Similarly, we obtain $y = \pm b$, $z = \pm b$, with the same sign (which comes from (3)) holding for all three equations. In this case we have proved our contention without even using the fourth coordinate of *f*. Now suppose *a* = 0. If $x \neq 0$, then (1) would immediately give $y = z = 0$, so that

$$(x, y, z) = (\pm 1, 0, 0).$$

But $y = z = 0$ implies (by (1) again) that $bc = 0$, so $b = 0$ or $c = 0$ and

$$(a, b, c) = (0, \pm 1, 0) \quad \text{or} \quad (0, 0, \pm 1).$$

These equations clearly contradict

$$x^2 + 2y^2 + 3z^2 = a^2 + 2b^2 + 3c^2.$$

Thus x = 0 also, and we have

(4) $$yz = bc,$$

(5) $$2y^2 + 3z^2 = 2b^2 + 3c^2,$$

(6) $$\begin{cases} y^2 + z^2 = 1 \\ b^2 + c^2 = 1 \end{cases}.$$

But (6) implies that

$$2y^2 + 3z^2 = 2y^2 + 3(1-y^2) = 3 - y^2,$$

and similarly for *b* and *c*, so (5) gives

$$3 - y^2 - 3 - b^2$$

(7) $$y = \pm b.$$

Now (4) gives

(8) $$z = \pm c$$

(this holds even if $y = b = 0$, since then $z, c = \pm 1$). Clearly, (4) also shows that the same sign holds in (7) and (8), which completes the proof.

Since $f(p) = f(q)$ precisely when $p = \pm q$, we can define $\bar{f} : P^2 \to R^4$ by

$$\bar{f}([p]) = f(p) .$$

This map is one-one and we can use it to define the metric in P^2:

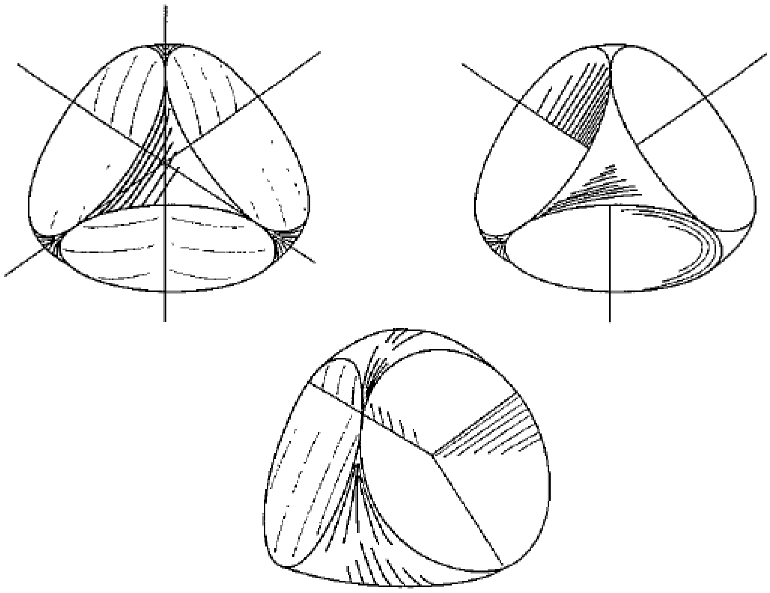
$$\bar{d}([p],[q]) = d(\bar{f}([p]), \bar{f}([q]) = d(f(p), f(q)) ,$$

Then one can check that the open sets are indeed the ones described above.

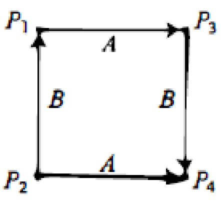
By the way, the map $g : P^2 \to R^3$ defined by the first 3 components of f,

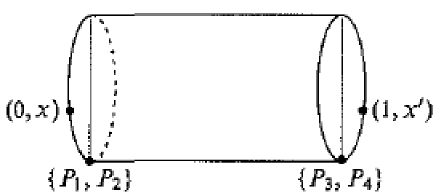
$$g([x, y, z]) = (yz, xz, xy)$$

is a topological immersion of P^2 in R^3. The image in R^3 is Steiner's "***Roman surface***".

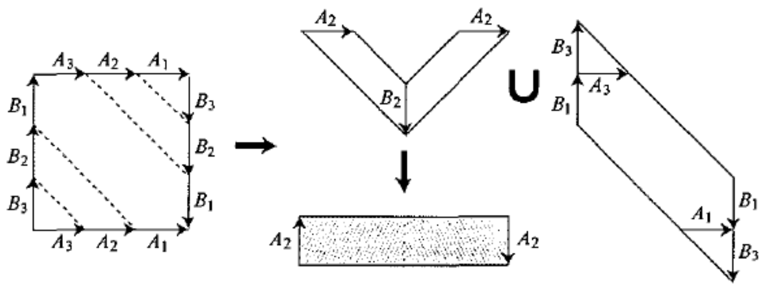


With the new surface P^2 at our disposal, we can create other surfaces in t he same way as the n-holed torus. For example, to a handle we can attach a projective space with a hole cut out, or, what amounts to the same thing, a Möbius strip. The closest we can come to picturing this is by drawing a cross-cap sticking on a torus. We can also join together a pair of projective planes with holes cut out, which amounts to sewing two Möbius strips together along their boundary. Although this can be pictured as two cross-caps joined together, it has a nicer, and famous, representation. Consider the surface obtained from the square with identifications indicated below; it may also be obtained from the cylinder $[0,1] \times S^1$ by identifying

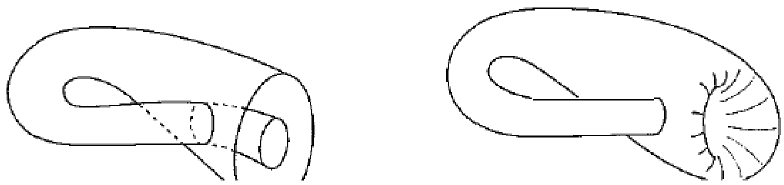




$(0, x) \in [0,1] \times S^1$ with (1, x'), where x' is the reflection of x through a fixed diameter of the circle. Notice that the identifications on the square force P_1, P_2, P_3, and P_4 to be identified, so that the set $\{P_1, P_2, P_3, P_4\}$ is a single point of our new space. The dotted lines below, dividing the sides into thirds, form a single circle, which separates the surface into two parts, one of which is shaded.



Rearrangement of the two parts shows that this surface is precisely two Möbius strips with corresponding points on their boundary identified. The description in terms of $[0,1] \times S^1$ immediately suggests an immersion of the surface. Turning one end of the cylinder around and pushing it through itself orients the left-hand boundary so that (0, x) is directly opposite (1, x'), to which it can then be joined, forming the "***Klein bottle***".



Examples of higher-dimensional manifolds will not be treated in nearly such detail, but, in addition to the family of n-manifolds S^n, we will mention the related family of "projective spaces". ***Projective n-space*** P^n is defined as the col1ection of all sets $\{p, -p\}$ for $p \in S^n$. The description of the open sets in P^n is precisely analogous to the description for P^2. Although these spaces seem to form a family as regular as the family S^n, we will see later that the spaces P^n for even n differ in a very important way from the same spaces for odd n.

One further definition is needed to complete this introduction to manifolds. We have already discussed some spaces which are not manifolds only because they have a "boundary", for example, the Möbius strip and the disc. Points on these "boundaries" do not have neighborhoods homeomorphic to R^n, but they do have neighborhoods homeomorphic to an important subset of R^n. The (***closed***) ***half-space*** H^n is defined by

$$H^n = \{(x^1, \ldots, x^n) \in R^n : x^n \geq 0\}.$$

A ***manifold-with-boundary*** is a metric space M with the following property:

If $x \in M$, then there is some neighborhood U of x and some integer $n \geq 0$ such that U is homeomorphic to either R^n or H^n.

A point in a manifold-with-boundary cannot have a neighborhood homeomorphic to both R^n and H^n (Invariance of Domain again); we can therefore distinguish those points $x \in M$ having a neighborhood homeomorphic to H^n. The set of all such x is called the ***boundary*** of M and is denoted by ∂M. If M is actually a manifold, then $\partial M = \phi$. Notice that if M is a subset of R^n, then ∂M is not necessarily the same as the boundary of M in the old sense (defined for any subset of R^n); indeed, if M is a manifold-with-boundary of dimension < n, then all points of M will be boundary points of M.

If manifolds-with-boundary are studied as frequently as manifolds, it becomes bothersome to use this long designation. Often, the word "manifold" is used for "manifold-with-boundary". A manifold in our sense is then called "non-bounded"; a non-bounded compact manifold is called a "***closed manifold***". We will stick to the other terminology, but will sometimes use "bounded manifold" instead of "manifold-with-boundary".