CHAPTER 7 DIFFERENTIAL FORMS

We turn our attention once more to tensor fields, but we will be concerned with a special kind of tensor field, the discussion of which requires some more algebraic preliminaries.

Let V be an n-dimensional vector space over R, An element $T \in T^k(V)$ is called ***alternating*** if

$$T(v_1,\ldots,v_i,\ldots,v_j,\ldots,v_k)=0 \ \text{ if } \ v_i = v_j \quad (i \neq j).$$

If T is alternating, then for any $v_1,\ldots,v_k$, we have

$$\begin{aligned}0 &= T(v_1,\ldots,v_i+v_j,\ldots,v_i+v_j,\ldots,v_k)\\ &= T(v_1,\ldots,v_i,\ldots,v_i,\ldots,v_k)+T(v_1,\ldots,v_i,\ldots,v_j,\ldots,v_k)\\ &\quad +T(v_1,\ldots,v_j,\ldots,v_i,\ldots,v_k)+T(v_1,\ldots,v_j,\ldots,v_j,\ldots,v_k)\\ &= 0+T(v_1,\ldots,v_i,\ldots,v_j,\ldots,v_k)+T(v_1,\ldots,v_j,\ldots,v_i,\ldots,v_k)+0\end{aligned}.$$

Therefore, T is ***skew-symmetric***:

$$T(v_1,\ldots,v_i,\ldots,v_j,\ldots,v_k) = -T(v_1,\ldots,v_j,\ldots,v_i,\ldots,v_k).$$

Of course, if T is skew-symmetric, then T is also alternating. [This is not true in the special case of a vector space over a field where 1 + 1 = 0; in this case, skew-symmetry is the same as symmetry, and the condition of being alternating is the stronger one.]

We will denote by $\Omega^k(V)$ the set of all alternating $T \in \widetilde{T}^k(V)$. It is clear that $\Omega^k(V) \subset \widetilde{T}^k(V)$ is a **subspace** of $\widetilde{T}^k(V)$. Moreover, if $f: V \to W$ is a linear transformation, then $f^*: \widetilde{T}^k(W) \to \widetilde{T}^k(V)$ preserves these subspaces - $f^*: \Omega^k(W) \to \Omega^k(V)$. Notice that $\Omega^1(V) = \widetilde{T}^1(V) = V^*$, so $\Omega^1(V)$ has dimension n. It is also convenient to set $\Omega^0(V) = \widetilde{T}^0(V) = R$. At the moment it is not clear what the **dimension** of $\Omega^k(V)$ equals for k > 1, but one case is well-known. The most familiar example of an alternating T is the determinant function $\det \in \widetilde{T}^n(R^n)$, considered as a function of the n rows of a matrix – we shall soon see that this function is, in a certain sense, the most general alternating function. Most discussions of the determinant begin by showing that of any two alternating n-linear functions on R^n, one is a multiple of the other; in other words, $\dim \Omega^n(R^n) \le 1$. Then one proves $\dim \Omega^n(R^n) = 1$ by actually constructing the non-zero function *det* (it follows, of course, that $\dim \Omega^n(V) = 1$ if V is any n-dimensional vector space). The construction of *det* is usually by a messy explicit formula, which is a special case of the definition to follow.

Let S_k denote the set of all **permutations** of {1 , . . . , k}; an element $\sigma \in S_k$ is a function $i \mapsto \sigma(i)$. If $(v_1,\ldots,v_k)$ is a k-tuple (of any objects) we set

$$\sigma \cdot (v_1,\ldots,v_k) = (v_{\sigma(1)},\ldots,v_{\sigma(k)}).$$

This definition has a built-in *confusion*. On the right side, the first element, for example, is the $\sigma(1)^{st}$ of the *v*'s on the left side; if these *v*'s have indices running in some order *other* than *1*, . . . , *k*, then the first element on the right is *not* necessarily that *v* whose index is $\sigma(1)$. The simplest way to figure out something like $\sigma\cdot(v_3,v_2,v_1,\ldots)$ is to rename things: $v_3 = w_1$, $v_2 = w_2$, $v_1 = w_3$, ….Thus warned, we compute

$$\sigma\cdot(\rho\cdot(v_1,\ldots,v_k)) = \sigma\cdot(v_{\sigma(1)},\ldots,v_{\sigma(k)})$$

by setting

$$v_{\rho(1)} = w_1\,,\ \ldots,\ \ v_{\rho(k)} = w_k\,,$$

so that

$$\begin{aligned}\sigma\cdot(\rho\cdot(v_1,\ldots,v_k)) &= \sigma\cdot(w_1,\ldots,w_k)\\ &= (w_{\sigma(1)},\ldots,w_{\sigma(k)})\\ &= (v_{\rho(\sigma(1))},\ldots,v_{\rho(\sigma(k))})\end{aligned}$$

since $w_\alpha = v_{\rho(\alpha)}$. Thus

(*) $$\sigma\cdot(\rho\cdot(v_1,\ldots,v_k)) = (\rho\sigma)\cdot(v_1,\ldots,v_k)$$

Now for any $T \in T^k(V)$ we define the "***alternation*** of *T*"

$$AltT = \frac{1}{k!}\sum_{\sigma\in S_k}\operatorname{sgn}\sigma\cdot T\circ\sigma\,,$$

i.e.,

$$AltT(v_1,\ldots,v_k = \frac{1}{k!}\sum \operatorname{sgn}\sigma\cdot T(v_{\sigma(1)},\ldots,v_{\sigma(k)})\,,$$

where $\operatorname{sgn}\sigma$ is +1 if σ is an even permutation and -1 if σ is odd.

1. PROPOSITION.

(1) *If* $T\in\widetilde{T}^k(V)$*, then* $Alt(T)\in\Omega^k(V)$.

(2) *If* $\omega\in\Omega^k(V)$*, then* $Alt\omega = \omega$.

(3) *If* $T\in\widetilde{T}^k(V)$*, then* $Alt(Alt(T)) = Alt(T)$.

PROOF. Left to the reader (or see pp. 78-79 of *Calculus on Manifolds*). QED.

We now define, for $\omega\in\Omega^k(V)$ and $\eta\in\Omega^l(V)$, an element $\omega\wedge\eta\in\Omega^{k+l}(V)$, the ***wedge product*** of ω and η, by

$$\omega\wedge\eta = \frac{(k+l)!}{k!\,l!}Alt(\omega\otimes\eta)\,.$$

The funny coefficient is not essential, but it makes some things work out more nicely, as we shall soon see. It is clear that

(1) $\wedge$ is ***bilinear***:

$$(\omega_1+\omega_2)\wedge\eta=\omega_1\wedge\eta+\omega_2\wedge\eta$$
$$\omega\wedge(\eta_1+\eta_2)=\omega\wedge\eta_1+\omega\wedge\eta_2$$
$$a\omega\wedge\eta=\omega\wedge a\eta=a(\omega\wedge\eta)$$

(2) $f^*(\omega\wedge\eta)=f^*\omega\wedge f^*\eta$

Moreover, it is easy to see that

(3) $\wedge$ is "***anti-commutative***": $\omega\wedge\eta=(-1)^{kl}\eta\wedge\omega$.

In particular, if *k* is odd then

$$\omega\wedge=0.$$

Finally, ***associativity*** of $\wedge$ is proved in the following way.

2. THEOREM.

(1) *If* $S\in\widetilde{T}^k(V)$ *and* $T\in\widetilde{T}^l(V)$ *and* $Alt(S)=0$, *then*

$$Alt(S\otimes T)=Alt(T\otimes S)=0.$$

(2) $Alt(Alt(\omega\otimes\eta)\otimes\theta)=Alt(\omega\otimes\eta\otimes\theta)=Alt(\omega\otimes Alt(\eta\otimes\theta))$

(3) *If* $\omega\in\Omega^k(V)$, $\eta\in\Omega^l(V)$, $\theta\in\Omega^m(V)$, *then*

$$(\omega\wedge\eta)\wedge\theta=\omega\wedge(\eta\wedge\theta)=\frac{(k+l+m)!}{k!l!m!}Alt(\omega\otimes\eta\otimes\theta).$$

PROOF.

(1) We have

$$\begin{aligned}&(k+l)!\,Alt(S\otimes T)(v_1,\dots,v_{k+l})\\ &\quad=\sum_{\sigma\in S_{k+l}}\operatorname{sgn}\sigma\cdot(S\otimes T)\cdot(\sigma\cdot(v_1,\dots,v_{k+l}))\\ &\quad=\sum_{\sigma\in S_{k+l}}\operatorname{sgn}\sigma\cdot S(v_{\sigma(1)},\dots,v_{\sigma(k)})\cdot T(v_{\sigma(k+1)},\dots,v_{\sigma(k+l)}))\end{aligned}.$$

Now let $G\subset S_{k+l}$ consist of all σ which leave *k+1*,…,*k+l* fixed. Then

$$\begin{aligned}&\sum_{\sigma\in S_{k+l}}\operatorname{sgn}\sigma\cdot S(v_{\sigma(1)},\dots,v_{\sigma(k)})\cdot T(v_{\sigma(k+1)},\dots,v_{\sigma(k+l)}))\\ &=\left[\sum_{\sigma'\in S_k}\operatorname{sgn}\sigma'\cdot S(v_{\sigma'(1)},\dots,v_{\sigma'(k)})\right]\cdot T(v_{k+1},\dots,v_{k+l})\\ &=0\end{aligned}.$$

Suppose now that $\sigma_0\notin G$. Let $\sigma_0G=\{\sigma_0\sigma':\sigma'\in G\}$. Then

$$\begin{aligned}&\sum_{\sigma\in\sigma_0G}\operatorname{sgn}\sigma\cdot(S\otimes T)(\sigma\cdot(v_1,\dots,v_{k+l}))\\ &=\operatorname{sgn}\sigma_0\cdot\sum_{\sigma'\in G}\operatorname{sgn}\sigma'\cdot(S\otimes T)(\sigma'\cdot(\sigma_0\cdot(v_1,\dots,v_{k+l})))\end{aligned}$$

by (*). We have just shown that this is 0 (since $\sigma_0\cdot(v_1,\dots,v_{k+l})$ is just some other

(k + l)-tuple of vectors). Notice that $G \cap \sigma_0 G = \phi$, for if $\sigma \in G \cap \sigma_0 G$, then $\sigma = \sigma_0 \sigma'$ for some $\sigma' \in G$, so $\sigma_0 = \sigma(\sigma')^{-1} \in G$, a contradiction. We can then continue in this way, breaking S_{k+l} up into disjoint subsets, the sum over each being 0. The relation $Alt(T \otimes S) = 0$ is proved similarly.

(2) Clearly

$$Alt(Alt(\eta \otimes \theta) - \eta \otimes \theta) = Alt(\eta \otimes \theta) - Alt(\eta \otimes \theta = 0,$$

so (1) implies that

$$\begin{aligned} 0 &= Alt(\omega \otimes [Alt(\eta \otimes \theta) - \eta \otimes \eta]) \\ &= Alt(\omega \otimes Alt(\eta \otimes \theta)) - Alt(\omega \otimes \eta \otimes \theta) \end{aligned};$$

the other equality is proved similarly.

(3) We have

$$\begin{aligned} (\omega \wedge \eta) \wedge \theta &= \frac{(k+l+m)!}{(k+l)!m!} Alt((\omega \wedge \eta) \otimes \theta) \\ &= \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} Alt((\omega \otimes \eta \otimes \theta) \end{aligned}.$$

The other equality is proved similarly. QED.

Notice that (2) just states that $\wedge$ is ***associative*** even if we had omitted the factor (k+l)!/ k!l! in the definition. On the other hand, the factor 1/ k! in the definition of Alt is essential - without it, we would not have *Alt*(*Alt T*)=*Alt T*, and the first equation in the proof of (2) would fail.

[If we had defined $\overline{\text{Alt}}$ just like *Alt*, but without the factor 1/ k!, then $\wedge$ could be defined by

$$\omega \wedge \eta = \frac{1}{k!l!} \overline{Alt}(\omega \otimes \eta).$$

This makes sense, *even over a field of finite characteristic*, because each term in the sum $Alt(\omega \otimes \eta)(v_1, \ldots, v_{k+1})$ occurs k!l! times (since ω and η are alternating), and 1/k!l! can be interpreted as meaning that these k!l! terms are replaced by just one.]

The factor (k+l)!/ k!l! has been inserted into the definition of $\wedge$ for the following reason. If $v_1, \ldots, v_n$ is a basis of V, and $\phi_1, \ldots, \phi_n$ is the dual basis, then

$$\begin{aligned} \phi_1 \wedge \cdots \wedge \phi_n &= \frac{(1+\cdots+1)!}{1!\cdots 1!} Alt(\phi_1 \otimes \cdots \otimes \phi_n) \\ &= \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \cdot (\phi_1 \otimes \cdots \otimes \phi_n) \circ \sigma \end{aligned}.$$

In particular,

$$(\phi_1 \wedge \cdots \wedge \phi_n)(v_1, \ldots, v_n) = 1.$$

(So if $v_1, \ldots, v_n$ is the standard basis for R^n, then $\phi_1 \wedge \cdots \wedge \phi_n = \det$.) A basis for $\Omega^k(V)$ can now be described.

3. THEOREM. *The set of all*

$$\phi_{i1} \wedge \cdots \wedge \phi_{ik} \qquad 1 \le i1 < \cdots < ik \le n$$

is a basis for $\Omega^k(V)$*, which therefore has dimension*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

(In particular, $\Omega^k(V) = \{0\}$ *for k >n.)*

PROOF. If $\omega \in \Omega^k(V) \subset T^k(V)$, we can write

$$\omega = \sum a_{i1,\ldots,ik} \phi_{i1} \otimes \cdots \otimes \phi_{ik}.$$

So

$$\omega = Alt(\omega) = \sum_{i1,\ldots,ik} a_{i1,\ldots,ik} Alt(\phi_1 \otimes \cdots \otimes \phi_{ik})$$

Each $Alt(\phi_1 \otimes \cdots \otimes \phi_{ik})$ is either 0 or $\pm(1/k!)\phi_{j1} \wedge \cdots \wedge \phi_{jk}$ for some $j1 < \cdots < jk$, so the elements $\phi_{j1} \wedge \cdots \wedge \phi_{jk}$ for $j1 < \cdots < jk$ span $\Omega^k(V)$. If

$$0 = \sum_{i1 < \cdots < ik} a_{i1,\ldots,ik} \phi_{i1} \wedge \cdots \wedge \phi_{ik}$$

then applying both sides to $(v_{i1}, \ldots, v_{ik})$ gives all $a_{i1,\ldots,ik} = 0$. QED

4. COROLLARY. *If* $\omega_1, \ldots, \omega_k \in \Omega^1(V)$, *then* $\omega_1, \ldots, \omega_k$ *are linearly independent if and only if*

$$\omega_1 \wedge \cdots \wedge \omega_k \ne 0.$$

PROOF. If $\omega_1, \ldots, \omega_k$ are linearly independent, there is a basis $v_1, \ldots, v_k, \ldots, v_n$ of *V* such that the dual basis vectors $\phi_1, \ldots, \phi_k, \ldots, \phi_n$ satisfy $\phi_i = \omega_i$ for $1 \le i \le k$. Then $\omega_1 \wedge \cdots \wedge \omega_k$ is a basis element of $\Omega^k(V)$, so it is not 0. On the other hand, if

$$\omega_1 = a_2\omega_2 + \cdots + a_k\omega_k,$$

then

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k = (a_2\omega_2 + \cdots + a_k\omega_k) \wedge \omega_2 \wedge \cdots \wedge \omega_k = 0. \text{ (QED)}$$

To abbreviate formulas, it is convenient to let *I* denote a typical "***multi-index***" $(i_1, \ldots, i_k)$, and let ϕ_I denote $\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}$. Then every element of $\Omega^k(V)$ is

uniquely expressible as

$$\sum_I a_I \phi_I \ .$$

Notice that Theorem 3 implies that every $\omega \in \Omega^k(R^n)$ is a linear combination of the functions

$$(v_1, \dots, v_k) \mapsto \text{determinant\_of\_a\_} k \times k \text{\_minor\_of\_} \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}.$$

One more simple theorem is in order; before we proceed to apply our construction to manifolds.

5. THEOREM. *Let* $v_1, \dots, v_n$ *be a basis for V, let* $\omega \in \Omega^n(V)$*, and let*

$$w_i = \sum_{j=1}^n \alpha_{ji} v_j \ , \quad i = 1, \dots, n \, .$$

Then

$$\omega(w_1, \dots, w_n) = \det(\alpha_{ij}) \cdot \omega(v_1, \dots, v_n) \, .$$

PROOF. Define $\eta \in \widetilde{T}(R^n)$ by

$$\eta((a_{11}, \dots, a_{n1}), \dots, (a_{n1}, \dots, a_{nn})) = \omega\left(\sum_{j=1}^n a_{j1} v_j, \dots, \sum_{j=1}^n a_{jn} v_j \right).$$

Then clearly $\eta \in \Omega^n(R^n)$, so $\eta = c \cdot \det$ for some $c \in R$, and

$$c = \eta(e_1, \dots, e_n) = \omega(v_1, \dots, v_n) \, . \text{ (QED)}$$

6. COROLLARY. *If V is n-dimensional and* $0 \neq \omega \in \Omega^n(V)$*, then there is a unique orientation* μ *for V such that*

$$[v_1, \dots, v_n] = \mu \text{, if and only if } \omega(v_1, \dots, v_n) > 0 \, .$$

With our new algebraic construction at hand, we are ready to apply it to ***vector bundles***. If $\xi = \pi : E \to B$ is a vector bundle, we obtain a new bundle $\Omega^k(\xi)$ by replacing each fibre $\pi^{-1}(p)$ with $\Omega^k(\pi^{-1}(p))$. A section ω of $\Omega^k(\xi)$ is a function with $\omega(p) \in \Omega^k(\pi^{-1}(p))$ for each $p \in B$. If η is a section of $\Omega^l(\xi)$, then we can define a section $\omega \wedge \eta$ of $\Omega^{k+l}(\xi)$ by $(\omega \wedge \eta)(p) = \omega(p) \wedge \eta(p) \in \Omega^{k+l}(\pi^{-1}(p))$.

In particular, sections of $\Omega^k(TM)$, which are just alternating covariant tensor fields of order *k*, are called ***k-forms*** on *M*. A ***1-form*** is just a covariant vector field. Since $\Omega^k(TM)$ can obviously be made into a C^∞ vector bundle, we can speak

of C^∞ forms; all forms will be understood to be C^∞ forms unless the contrary is explicitly stated. Remember that covariant tensors actually map contravariantly: If $f: M \to N$ is C^∞, and ω is a *k*-form on *N*, then $f^*\omega$ is a k-form on *M*. We can also define $\omega_1 + \omega_2$ and $\omega \wedge \eta$. The following properties of *k*-forms are obvious from the corresponding properties for $\Omega^k(V)$:

$$(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta \,,$$
$$\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2 \,,$$
$$f\omega \wedge \eta = \omega \wedge f\eta = f(\omega \wedge \eta) \,,$$
$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega \,,$$
$$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta \,.$$

If (x,U) is coordinate system, then the $dx^i(p)$ are a basis for ${M_p}^*$, so the $dx^{i_1}(p) \wedge \cdots \wedge d^{i_k}(p)$, $(i_1 < \cdots < i_k)$ are a basis for $\Omega^k(p)$. Thus every *k*-form ω can be written uniquely as

$$\omega = \sum_{i_1 < \cdots i_k} \omega_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \,,$$

or, if we denote $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ by dx^I for the multi-index $I = (i_1, \ldots, i_k)$,

$$\omega = \sum_I \omega_I dx^I \,.$$

The problem of finding the relationship between the ω_I and the functions ω'_I when

$$\omega = \sum_I \omega_I dx^I = \sum_I \omega'_I \, dy^I$$

is left to the reader (Problem 16), but we will do one special case here.

7. THEOREM. *If $f: M \to N$ is a C^∞ function between n-manifolds, (x, U) is a coordinate system around $p \in M$, and (y, V) a coordinate system around $q = f(p) \in N$, then*

$$f^*(g dy^1 \wedge \cdots \wedge dy^n) = (g \circ f) \cdot \det\left(\frac{\partial (y^i \circ f)}{\partial x^j}\right) dx^1 \wedge \cdots \wedge dx^n \,.$$

PROOF. It suffices to show that

$$f^*(dy^1 \wedge \cdots \wedge dy^n) = \det\left(\frac{\partial (y^i \circ f)}{\partial x^j}\right) dx^1 \wedge \cdots \wedge dx^n \,.$$

Now, by Problem 4-1,

$$
\begin{aligned}
&f^*(dy^1 \wedge \cdots \wedge dy^n)(p)\left(\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p\right) \\
&= dy^1(q) \wedge \cdots \wedge dy^n(q)\left(f_*\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, f_*\left.\frac{\partial}{\partial x^n}\right|_p\right) \\
&= dy^1(q) \wedge \cdots \wedge dy^n(q)\left(\sum_{i=1}^n \frac{\partial(y^i \circ f)}{dx^1}(p)\left.\frac{\partial}{\partial y^i}\right|_q, \ldots, \sum_{i=1}^n \frac{\partial(y^i \circ f)}{dx^n}(p)\left.\frac{\partial}{\partial y^i}\right|_q\right) \\
&= \det\left(\frac{\partial(y^i \circ f)}{\partial x^j}(p)\right)
\end{aligned}
$$

by Theorem 5. (QED)

8. COROLLARY. *If (x, U) and (y, V) are two coordinate systems on M and*

$$gdy^1 \wedge \cdots \wedge dy^n = hdx^1 \wedge \cdots \wedge dx^n,$$

then

$$h = g \cdot \det\left(\frac{\partial y^i}{\partial x^j}\right).$$

PROOF. Apply the theorem with *f* = identity map. (QED)

[This corollary shows that *n*-forms are the geometric objects corresponding to the "even scalar densities" defined in Problem 4-10.]

If $\xi = \pi : E \to B$ is an *n*-plane bundle, then a nowhere zero section ω of $\Omega^n(\xi)$ has a special significance: For each $p \in B$, the non-zero $\omega(p) \in \Omega^n(\pi^{-1}(p))$ determines an orientation μ_p of $\pi^{-1}(p)$ by Corollary 6. It is easy to see that the collection of orientations $\{\mu_p\}$ satisfy the "compatibility condition" set forth in Chapter 3, so that $\mu = \{\mu_p\}$ is an orientation of ξ. In particular, if there is a nowhere zero *n*-form ω on an *n*-manifold *M*, then *M* is orientable (i.e., the bundle *TM* is orientable). The converse also holds:

9. THEOREM. *If a C^∞ manifold M is orientable, then there is an n-form ω on M which is nowhere 0.*

PROOF. By Theorem 2-13 and 2-15, we can choose a cover *O* of *M* by a collection of coordinate systems $\{(x,U)\}$, and a partition of unity $\{\phi_U\}$ subordinate to *O*. Let μ be an orientation of *M*. For each (*x*, *U*) choose an *n*-form ω_U on *U* such that for $v_1, \ldots, v_n \in M_p$, $p \in U$ we have

$$\omega_U(v_1, \ldots, v_n) > 0 \quad \text{if and only if} \quad [v_1, \ldots, v_n] = \mu_p.$$

Now let

$$\omega = \sum_{U \in O} \phi_U \omega_U \ .$$

Then ω is a C^∞ n-form. Moreover, for every p, if $v_1, \ldots, v_n \in M_p$ satisfy $[v_1, \ldots, v_n] = \mu_p$, then each

$$(\phi_U \omega_U)(p)(v_1, \ldots, v_n) \geq 0 ,$$

and strict inequality holds for at least one U. Thus $\omega(p) \neq 0$. (QED)

Notice that the bundle $\Omega^n(TM)$ is 1-dimensional. We have shown that if M is orientable, then $\Omega^n(TM)$ has a nowhere 0 section, which implies that it is trivial. Conversely, of course, if the bundle $\Omega^n(TM)$ is trivial, then it certainly has a nowhere 0 section, so M is orientable. [Generally, if ξ is a k-plane bundle, then $\Omega^k(\xi)$ is trivial if and only if ξ is orientable, provided that the base space B is "paracompact" (every open cover has a locally-finite refinement).]

Just as $\Omega^0(V)$ has been introduced as another name for R, a 0-form on M will just mean a function f on M (and $f \wedge \omega$ will just mean $f \cdot \omega$). For every 0-form f we have the 1-form df (recall that $df(X) = X(f)$, which in a coordinate system (x, U) is given by

$$df = \sum_{j=1}^{n} \frac{\partial f}{dx^j} dx^j \ .$$

If ω is a k-form

$$\omega = \sum_I \omega_I dx^I \ ,$$

then each $d\omega_I$ is a 1-form, and we can define a $(k+1)$-form $d\omega$, the differential of ω, by

$$\begin{aligned} d\omega &= \sum_I d\omega_I dx^I \\ &= \sum_I \sum_{\alpha=1}^{n} \frac{\partial \omega_I}{\partial x^\alpha} dx^\alpha \wedge dx^I \end{aligned} \ .$$

It turns out that this definition does not depend on the coordinate system. This can be proved in several ways. The **first way** is to use a brute-force computation comparing the coefficients ω'_I in the expression

$$\omega = \sum_I \omega'_I \, dx^I$$

with the ω_I.

The **second method** is a lot sneakier. We begin by finding some properties of $d\omega$ (still defined with respect to this particular coordinate system).

10. PROPOSITION.

(1) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.

(2) *If* ω_1 *is a k-form, then* $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$.

(3) $d(d\omega) = 0$. *Briefly,* $d^2 = 0$.

PROOF. (1) is clear. To prove (2) we first note that because of (1) it suffices to consider only

$$\omega_1 = fdx^I, \quad \omega_2 = gdx^I .$$

Then $\omega_1 \wedge \omega_2 = fgdx^I \wedge dx^J$ and

$$\begin{aligned} d(\omega_1 \wedge \omega_2) &= d(fg) \wedge dx^I \wedge dx^J \\ &= gdf \wedge dx^I \wedge dx^J + fdg \wedge dx^I \wedge dx^J \\ &= d\omega_1 \wedge \omega_2 + (-1)^k fdx^I \wedge dg \wedge dx^I \\ &= d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2 \end{aligned} .$$

(3) It clearly suffices to consider only *k*-forms of the form

$$\omega = fdx^I .$$

Then

$$d\omega = \sum_{\alpha=1}^{n} \frac{\partial f}{\partial x^\alpha} dx^\alpha \wedge dx^I ,$$

so

$$d(d\omega) = \sum_{\alpha=1}^{n} \left(\sum_{\beta=1}^{n} \frac{\partial^2 f}{\partial x^\beta \partial x^\alpha} dx^\beta \wedge dx^\alpha \wedge dx^I \right) .$$

In this sum, the terms

$$\frac{\partial^2 f}{\partial x^\beta \partial x^\alpha} dx^\beta \wedge dx^\alpha \wedge dx^I$$

and

$$\frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} dx^\alpha \wedge dx^\beta \wedge dx^I$$

cancel in pairs. (QED)

We next note that these properties characterize d on U.

11. PROPOSITION. *Suppose d' takes k-forms on U to (k + 1)-forms on U, for all k, and satisfies*

(1) $d'(\omega_1 + \omega_2) = d'\omega_1 + d'\omega_2$.

(2) $d'(\omega_1 \wedge \omega_2) = d'\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d'\omega_2$.

(3) $d'(d'f) = 0$.

(4) $d'f = (the\_old)df$.

Then $d' = d$ *on U.*

PROOF. It is clearly enough to show that $d'\omega = d\omega$ when $\omega = fdx^I$. Now by (2),

$$\begin{aligned} d'(fdx^I) &= d'f \wedge dx^I + f \wedge d'(dx^I) \\ &= df \wedge dx^I + f \wedge d'(dx^I) \end{aligned}$$

by (4). So it suffices to show that $d'(dx^I) = 0$, where

$$\begin{aligned} dx^I &= dx^{i1} \wedge \cdots \wedge dx^{ik} \\ &= d'x^{i1} \wedge \cdots \wedge d'x^{ik} \end{aligned}$$

by (4). We will use induction on *k*. Assuming it for *k* - 1 we have

$$\begin{aligned} & d'(dx^I) = d'(d'x^{i1} \wedge \cdots \wedge d'x^{ik}) \\ & = d'(d'x^{i1}) \wedge d'x^{i2} \wedge \cdots \wedge d'x^{ik} - d'x^{i1} \wedge d'(d'x^{i1} \wedge \cdots \wedge d'x^{ik}) . \\ & = 0 - 0 \end{aligned}$$

(We used (2) and (3) and the inductive hypothesis.) (QED)

12. COROLLARY. *There is a unique operator d from the k-forms on M to the (k+1)-forms on M, for all k, satisfying*

$$d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2 ,$$

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2 ,$$

$$d^2 = 0 ,$$

and agreeing with the old d on functions.

PROOF. For each coordinate system (x, *U*) we have a unique d_U defined. Given the form ω, and $p \in M$, pick any *U* with $p \in M$ and define

$$d\omega(p) = d_U(\omega \mid U)(p) . \text{ (QED)}$$

The **third way** of proving that the definition of *d* does not depend on the coordinate system is to give an invariant definition.

13. THEOREM. *If* ω *is a k-form on M, then there is a unique (k + 1)-form* $d\omega$ *on M such that for every set of vector fields* $X_1, \ldots, X_{k+1}$ *we have*

$$(*) \quad \begin{aligned} & d\omega(X_1, \ldots, X_{k+1}) \\ & = \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) + \\ & \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}) \\ & (\equiv \Sigma_1 + \Sigma_2) \end{aligned} ,$$

where $\hat{X}_i$ *indicates that it is omitted. This (k + 1)-form agrees with* $d\omega$ *as defined previously.*

PROOF. The operator which takes $(X_1,\ldots,X_{k+1})$ to $\Sigma_1+\Sigma_2$ is clearly linear over R. Moreover, it is actually linear over the C^∞ functions *F*. In fact, if X_{i0} is replaced by fX_{i0}, then Σ_1 becomes

$$f\Sigma_1+\sum_{i\neq i0}(-1)^{i+1}(X_i f)\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{k+1}),$$

and using the formulas

$$[fX,Y]=f[X,Y]-Yf\cdot X,$$
$$[X,fY]=f[X,Y]+Xf\cdot Y,$$

it is easily seen that Σ_2 becomes

$$\begin{aligned}f\Sigma_2&+\sum(-1)^{i+i0}(X_i f)\omega(X_{i0},X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_{i0},\ldots,X_{k+1})\\&-\sum(-1)^{i0+j}(X_j f)\omega(X_{i0},X_1,\ldots,\hat{X}_{i0},\ldots,\hat{X}_j,\ldots,X_{k+1})\end{aligned};$$

a brief inspection then shows that $\Sigma_1+\Sigma_2$ becomes $f\Sigma_1+f\Sigma_2$.

Theorem 4-2 shows that there is a unique covariant tensor field $d\omega$ satisfying (*). It is easy to check that $d\omega$ is alternating, so that it is a $(k + 1)$-form.

To compute $d\omega$ in a coordinate system (*x*, *U*) it clearly suffices to compute $d(fdx^I)$. Moreover, by renumbering, we might as well assume

$$\omega=fdx^1\wedge\cdots\wedge dx^k.$$

For $d\omega$, as for any form, we have

$$d\omega=\sum_{\alpha_1<\cdots<\alpha_{k+1}}d\omega\left(\frac{\partial}{\partial x^{\alpha_1}},\ldots,\frac{\partial}{\partial x^{\alpha_{k+1}}}\right)dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_{k+1}}.$$

It is clear from (*) that

$$d\omega\left(\frac{\partial}{\partial x^{\alpha_1}},\ldots,\frac{\partial}{\partial x^{\alpha_{k+1}}}\right)=0$$

unless some $(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_{k+1})$ is a permutation of $(1,\ldots,k)$.

Since the α 's are increasing, this happens only if

$$(\alpha_1,\ldots,\alpha_{k+1})=(1,\ldots,k,j)\quad j>k,$$

in which case

$$d\omega\left(\frac{\partial}{\partial x^{\alpha_1}},\ldots,\frac{\partial}{\partial x^{\alpha_k}},\frac{\partial}{\partial x^{\alpha_j}}\right)=(-1)^k\frac{\partial f}{\partial x^j},$$

so

$$\begin{aligned} d\omega &= \sum_{j>k} (-1)^k \frac{\partial f}{\partial x^k} dx^1 \wedge \cdots \wedge dx^k \wedge dx^j \\ &= \sum_{j>k} \frac{\partial f}{\partial x^j} dx^j \wedge dx^1 \wedge \cdots \wedge dx^k \\ &= \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} dx^j \wedge dx^1 \wedge \cdots \wedge dx^k \end{aligned},$$

which is just the old definition. (QED)

This is our first real example of an ***invariant definition*** of an important tensor, and our first use of Theorem 4-2. We do not find $d\omega(p)(v_1,\ldots,v_{k+1})$ directly, but first find $d\omega(X_1,\ldots,X_{k+1})$, where X_i are vector fields extending v_i, and then evaluate this function at *p*. By some sort of ***magic***, this turns out to be independent of the extensions $X_1,\ldots,X_{k+1}$. This may not seem to be much of an improvement over using a coordinate system and checking that the definition is independent of the coordinate system. But we can hardly hope for anything better. After all, although $d\omega(X_1,\ldots,X_{k+1})(p)$ does not depend on the values of X_i except at *p*, it does depend on the values of ω at points other than *p* – this must enter into our formula somehow. One other feature of our definition is common to most invariant definitions of tensors - the presence of a term involving brackets of various vector fields. This term is what makes the operator linear over the C^∞ functions, but it disappears in computations in a coordinate system.

In the particular case where ω is a ***1-form***, Theorem 13 gives the following formula.

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

This enables us to state a second version of Theorem 6-5 (***The Frobenius Integrability Theorem***) in terms of differential forms. Define the ring $\Omega(M)$ to be the direct sum of the ***rings*** of *l*-forms on *M*, for all *l*. If Δ is a *k*-dimensional distribution on *M*, then $\tilde{I}(\Delta) \subset \Omega(M)$ will denote the ***subring*** generated by the set of all forms ω with the property that (if ω has degree *l*)

$$\omega(X_1,\ldots,X_l) = 0 \text{, whenever } X_1,\ldots,X_l \text{ belong to } \Delta.$$

It is clear that $\omega_1 + \omega_2 \in \tilde{I}(\Delta)$ if $\omega_1, \omega_2 \in \tilde{I}(\Delta)$, and that $\eta \wedge \omega \in \tilde{I}(\Delta)$ if $\omega \in \tilde{I}(\Delta)$ [thus, $\tilde{I}(\Delta)$ is an ***ideal*** in the ring $\Omega(M)$]. Locally, the ideal $\tilde{I}(\Delta)$ is generated by *n-k* independent 1-forms $\omega^{k+1},\ldots,\omega^n$. In fact, around any point $p \in M$ we can choose a coordinate system (*x*,*U*) so that

$$\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^k}\right|_p \quad \text{span } \Delta_p.$$

Then

$$dx^1(p) \wedge \cdots \wedge dx^k(p) \quad \text{is non-zero on} \quad \Delta_p\,.$$

By continuity, the same is true for q sufficiently close to p, which by Corollary 4 implies that $dx^1(q),\ldots,dx^k(q)$ are linearly independent in Δ_q. Therefore, there are C^∞ functions f_β^α such that

$$dx^\alpha(q) = \sum_{\beta=1}^{k} f_\beta^\alpha(q)dx^\beta(q) \quad \text{restricted to} \quad \Delta_q\,. \qquad (\alpha = k+1,\ldots,n)$$

We can therefore let

$$\omega^\alpha = dx^\alpha - \sum_{\beta=1}^{k} f_\beta^\alpha dx^\beta\,.$$

14. PROPOSITION (THE FROBENIUS INTEGRABILITY THEOREM; *SECOND VERSION*). *A distribution Δ on M is integrable if and only if $d(\widetilde{I}(\Delta)) \subset \widetilde{I}(\Delta)$.*

PROOF. Locally we can choose 1-forms $\omega^1,\ldots,\omega^n$ which span $M_q{}^*$ for each q such that $\omega^{k+1},\ldots,\omega^n$ generate $\widetilde{I}(\Delta)$. Let $X_1,\ldots,X_n$ be the vector fields with

$$\omega^i(X_j) = \delta_j^i\,.$$

Then $X_1,\ldots,X_k$ span Δ. So Δ is integrable if and only if there are functions C_{ij}^β with

$$[X_i, X_j] = \sum_{\beta=1}^{k} C_{ij}^\beta X_\beta \ , (i,j\text{=}1,\ldots,k)$$

Now

$$d\omega^\alpha(X_i,X_j) = X_i(\omega^\alpha(X_j)) - X_j(\omega^\alpha(X_i)) - \omega^\alpha([X_i,X_j])\,.$$

For $1 \le i,j \le k$ and $\alpha > k$, the first two terms on the right vanish. So $d\omega^\alpha(X_i,X_j) = 0$ if and only if $\omega^\alpha([X_i,X_j]) = 0$. But each $\omega^\alpha([X_i,X_j]) = 0$ if and only if each $[X_i,X_j]$ belongs to Δ (i.e., if Δ is integrable), while each $d\omega^\alpha(X_i,X_j) = 0$ if and only if $d\omega^\alpha \in \widetilde{I}(\Delta)$. (QED)

Notice that since the $\omega^i \wedge \omega^j$ ($i < j$) span $\Omega^2(M_q)$ for each q, we can

always write

$$d\omega^{\alpha} = \sum_{i<j} c_{ij}^{\alpha} \omega^{i} \wedge \omega^{j} = \sum_{j} \theta_{j}^{\alpha} \wedge \omega^{j} \quad \text{for certain forms } \theta_{j}^{\alpha} .$$

If $\alpha > k$, and $i_0, j_0 \le k$ are distinct, we have

$$0 = d\omega^{\alpha}(X_{i_0}, X_{j_0}) = \sum_{j} (\theta_{j}^{\alpha} \wedge \omega^{j})(X_{i_0}, X_{j_0}) = \theta_{j_0}^{\alpha}(X_{i_0}),$$

so we can write the condition $d(\widetilde{I}(\varDelta)) \subset \widetilde{I}(\varDelta)$ as

$$d\omega^{\alpha} = \sum_{\beta > k} \theta_{\beta}^{\alpha} \wedge \omega^{\beta} .$$

Once we have introduced a coordinate system (x, U) such that the slices

$$\{q \in U : x^{k+1}(q) = a^{k+1}, \ldots, x^{n}(q) = a^{n}\}$$

are integral submanifolds of $\varDelta$, the forms $dx^{k+1}, \ldots, dx^{n}$ are a basis for $\widetilde{I}(\varDelta)$, so $\omega^{k+1}, \ldots, \omega^{n}$ must be linear combinations of them. We therefore have the following.

15. COROLLARY. *If $\omega^{k+1}, \ldots, \omega^{n}$ are linearly independent I-forms in a neighborhood of $p \in M$, then there are I-forms θ_{β}^{α} ($\alpha, \beta > k$) with*

$$d\omega^{\alpha} = \sum_{\beta} \theta_{\beta}^{\alpha} \wedge \omega^{\beta}$$

if and only if there are functions f_{β}^{α}, g^{β} ($\alpha, \beta > k$) with

$$\omega^{\alpha} = \sum_{\beta} f_{\beta}^{\alpha} dg^{\beta} .$$

Although Theorem 13 warms the heart of many an invariant lover, the cases $k > 1$ will hardly ever be used (a very significant exception occurs in the last chapter of Volume V). Problem 18 gives another invariant definition of $d\omega$, using induction on the degree of ω, which is much simpler. The reader may reflect on the difficulties which would be involved in using the definition of Theorem 13 to prove the following important property of *d*:

16. PROPOSITION. *If $f : M \to N$ is C^{∞} and ω is a k-form on N, then*

$$f^{*}(d\omega) = d(f^{*}\omega) .$$

PROOF. For $p \in M$, let (x, U) be a coordinate system around $f(p)$. We can assume

$$\omega = g dx^{i_1} \wedge \cdots \wedge dx^{i_k} .$$

We will use induction on k. For $k = 0$ we have, tracing through some definitions,

$$f^{*}(dg)(X) = dg(f_{*}X) = [f_{*}X]g = X(g \circ f) = d(g \circ f)(X)$$

(and, of course, $f^{*}g$ is to be interpreted as $g \circ f$). Assuming the formula for $k-1$, we have

$$
\begin{aligned}
I &\equiv d(f^*\omega) \\
&= d((f^* g dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}) \wedge f^* dx^{i_k}) \\
&= d(f^*(g dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}) \wedge f^* dx^{i_k}) + 0
\end{aligned}
$$

since $df^* dx^{i_k} = dd(x^{i_k} \circ f) = 0$

$$I = f^*(d(g dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}})) \wedge f^* dx^{i_k}$$

by the inductive hypothesis

$$
\begin{aligned}
I &= f^*(dg \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}) \wedge f^* dx^{i_k} \\
&= f^*(dg \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}} \wedge dx^{i_k} \\
&= f^*(d\omega)
\end{aligned} .
$$

(QED)

One property of d qualifies, by the criterion of the previous chapter, as a basic theorem of differential geometry. The relation $d^2 = 0$ is just an elegant way of stating that mixed partial derivatives are equal. There is another set of terminology for stating the same thing. A form ω is called ***closed*** if $d\omega = 0$ and ***exact*** if $\omega = d\eta$ for some form η. (The terminology "exact" is classical – differential forms used to be called simply "differentials"; a differential was then called "exact" if it actually was the differential of something. The term "closed" is based on an analogy with chains, which will be discussed in the next chapter.) Since $d^2 = 0$, every exact form is closed. In other words, $d\omega = 0$ is a necessary condition for solving $\omega = d\eta$. If ω is a 1-form

$$\omega = \sum_{i=1}^{n} \omega_i dx^i ,$$

then the condition $d\omega = 0$, i.e.,

$$\frac{\partial \omega_i}{\partial c^j} = \frac{\partial \omega_j}{\partial x^i}$$

is necessary for solving $\omega = df$, i.e.,

$$\frac{\partial f}{\partial x^i} = \omega_i .$$

Now we know from Theorem 6-1 that these conditions are also sufficient. For 2-forms the situation is more complicated, however. If ω is a 2-form on R^3,

$$\omega = A dy \wedge dz - B dx \wedge dz + C dx \wedge dy ,$$

then

$$\omega = d(P dx + Q dy + R dz)$$

if and only if

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = A ,$$

$$\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = B \,,$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = C \,.$$

The necessary condition, $d\omega = 0$, is

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0 \,.$$

In general, we are dealing with a rather strange collection o f partial differential equations (carefully selected so that we can get integrability conditions). It turns out that these necessary conditions are also sufficient: if ω is closed, then it is exact. Like our results about solutions to differential equations, this result is true only locally. The reasons for restricting ourselves to local results are now somewhat different, however. Consider the case of a closed 1 -form ω on R^2:

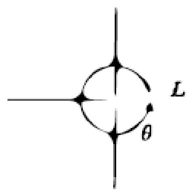
$$\omega = f dx + g dy \,, \text{ with } \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \,.$$

We know how to find a function α on ***all*** of R^2 with $\omega = d\alpha$, namely

$$\alpha(x, y) = \int_{x_0}^{x} f(t, y_0)\,dt + \int_{y_0}^{y} g(x, t)\,dt \,.$$

On the other hand, the situation is very different if ω is defined only on $R^2 - \{0\}$. Recall that if $L \subset R^2$ is $[0, \infty) \times \{0\}$, then

$$\theta : R^2 - L \to R \,,$$



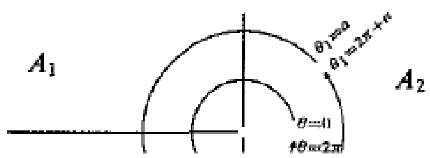
defined in Chapter 2, is C^∞; in fact,

$$(r, \theta) : R^2 - L \to \{r : r > 0\} \times (0, 2\pi)$$

is the inverse of the map

$$(a, b) \mapsto (a \cos b, a \sin b) \,,$$

whose derivative at (a, b) has determinant equal to $a \neq 0$. By deleting a different ray L_1 we can define a different function θ_1. Then $\theta_1 = \theta$ in the region A_1 and $\theta_1 = \theta + 2\pi$ in the region A_2. Consequently $d\theta$ and $d\theta_1$ agree on their common domain, so that together they define a 1-form ω on $R^2 - \{0\}$. A computation (Problem 20) shows that



$$\omega = \frac{-y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy \,.$$

The 1-form ω is usually denoted by $d\theta$, but this is an abuse of notation, since $\omega = d\theta$ only on $R^2 - L$. In fact, ω is not df for any C^1 function

$f: R^2 - \{0\} \to R$. Indeed, if $\omega = df$, then

$$df = d\theta \quad \text{on} \quad R^2 - L,$$

so $d(f-\theta)=0$ on $R^2 - L$, which implies that $\frac{\partial f}{\partial x} = \frac{\partial \theta}{\partial x}$ and $\frac{\partial f}{\partial y} = \frac{\partial \theta}{\partial y}$ and hence $f = \theta + \text{constant}$ on $R^2 - L$, which is impossible. Nevertheless, $d\omega = 0$ [the two relations

$$d(d\theta) = 0 \quad \text{on} \quad R^2 - L,$$
$$d(d\theta_1) = 0 \quad \text{on} \quad R^2 - L_1$$

clearly imply that this is so]. So ω is closed, but not exact. (It is still exact in a neighborhood of any point of $R^2 - \{0\}$.)

Clearly ω is also not exact in any small region containing 0. This example shows that it is the shape of the region, rather than its size, that determines whether or not a closed form is necessarily exact.

A manifold M is called (smoothly) ***contractible to a point*** $p_0 \in M$ if there is a C^∞ function

$$H: M \times [0,1] \to M$$

such that

$$H(p,1) = p, \quad H(p,0) = p_0 \quad \text{for} \quad p \in M.$$

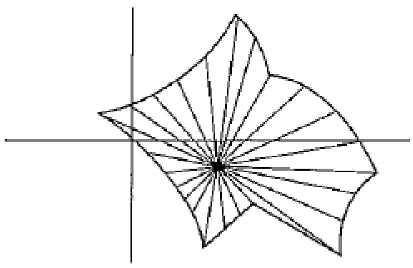
For example, R^n is smoothly contractible to $0 \in R^n$; we can define

$$H: R^n \times [0,1] \to R^n$$

by

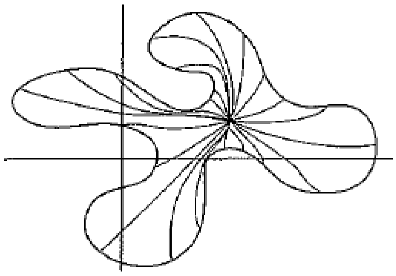
$$H(p,t) = tp.$$

More generally, $U \subset R^n$ is contractible to $p_0 \in U$ if U has the property that $p \in U$ implies $p_0 + t(p - p_0) \in U$ for $0 \le t \le 1$ (such a region U is called ***star-shaped*** with respect to p_0).



Of course, many other regions are also contractible to a point. If we think of [0,

1] as representing time, then for each time t we have a map $p \mapsto H(p,t)$ of M into itself; at time 1 this is just the identity map, and at time 0 it is the constant map.



We will show that if M is smoothly contractible to a point, then every closed form on M is exact. (By the way, this result and our investigation of the form $d\theta$ prove the intuitively obvious fact that $R^2 - \{0\}$ is ***not*** contractible to a point; the same result holds for $R^n - \{0\}$, but we will not be in a position to prove this until the next chapter.) The trick in proving our result is to analyze $M \times [0,1]$ (for any manifold M), and pay hardly any attention at all to H.

For $t \in [0,1]$ we define

$$i_t : M \to M \times [0,1]$$

by

$$i_t(p) = (p,t).$$

We claim that if ω is a form on $M \times [0,1]$ with $d\omega = 0$, then

$$i_t{}^*\omega - i_0{}^*\omega \quad \text{is exact;}$$

we will see later (and you may try to convince yourself right now) that the theorem follows trivially from this.

Consider first a 1-form ω on $M \times [0,1]$. We will begin by working in a coordinate system on $M \times [0,1]$. There is an obvious function t on $M \times [0,1]$ namely, the projection π on the second coordinate), and if (x, U) is a coordinate system on M, while π_M is the projection on M, then

$$(x^1 \circ \pi_M, \ldots, x^n \circ \pi_M, t)$$

is a coordinate system on $U \times [0,1]$. We will denote $x^i \circ \pi_M$ by $\bar{x}^i$, for convenience. It is easy to check (or should be) that

$$i_\alpha{}^*\left(\sum_{i=1}^{n} \omega_i d\bar{x}^i + f dt\right) = \sum_{i=1}^{n} \omega_i(\cdot,\alpha) dx^i,$$

where

$$\omega_i(\cdot,\alpha) \text{ denotes the function } p \mapsto \omega_i(p,\alpha).$$

Now for $\omega = \sum_{i=1}^{n} \omega_i d\bar{x}^i + f dt$ we have

$$d\omega = [\text{terms not involving } dt] - \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial t} d\bar{x}^i \wedge dt + \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{x}^i} d\bar{x}^i \wedge dt .$$

So $d\omega = 0$ implies that

$$\frac{\partial \omega_i}{\partial t} = \frac{\partial f}{\partial \bar{x}^i} .$$

Consequently,

$$\begin{aligned} \omega_i(p,1) - \omega_i(p,0) &= \int_0^1 \frac{\partial \omega_i}{\partial t}(p,t)dt \\ &= \int_0^1 \frac{\partial f}{\partial \bar{x}^i}(p,t)dt \end{aligned} ,$$

so

(1) $$\sum_{i=1}^{n} \omega_i(p,1)dx^i - \sum_{i=1}^{n} \omega_i(p,0)dx^i = \sum_{i=1}^{n} \left(\int_0^1 \frac{\partial f}{\partial \bar{x}^i}(p,t)dt \right) dx^i .$$

If we define $g : M \to R$ by

$$g(p) = \int_0^1 f(p,t)dt ,$$

then

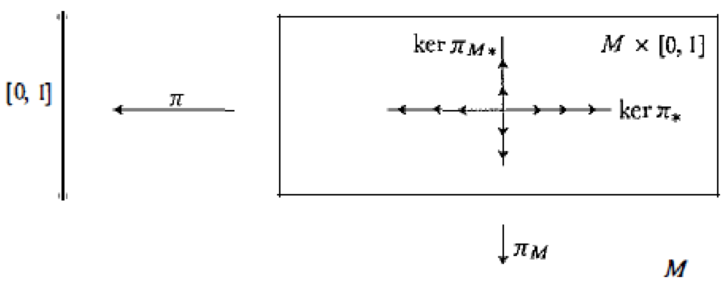
(2) $$\frac{\partial g}{\partial x^i}(p) = \int_0^1 \frac{\partial f}{\partial \bar{x}^i}(p,t)dt .$$

Equations (1) and (2) show that

$$i_1{}^* \omega - i_0{}^* \omega = dg .$$

Now although we seem to be using a coordinate system, the function *f*, and hence *g* also, is really independent of the coordinate system. Notice that for the tangent space of $M \times [0,1]$ we have

($*$) $$(M \times [0,1])_{(p,t)} = \mathrm{Ker}\pi_* \oplus \mathrm{Ker}\pi_{M*} .$$



If a vector space V is a direct sum $V = V_1 \oplus V_2$ of two subspaces, then any

$\omega \in \Omega^1(V)$ can be written

$$\omega = \omega_1 + \omega_2$$

where

$$\omega_1(v_1 + v_2) = \omega(v_1)$$
$$\omega_2(v_1 + v_2) = \omega(v_2)\ .$$

Applying this to the decomposition (*), we write the 1-form ω on $M \times [0,1]$ as $\omega_1 + \omega_2$; there is then a unique *f* with $\omega_2 = fdt$.

In general, for a *k*-form ω, it is easy to see (Problem 22) that we can write ω uniquely as

$$\omega = \omega_1 + (dt \wedge \eta)$$

where $\omega_1(v_1, \ldots, v_k) = 0$ if some $v_i \in \mathrm{Ker}\pi_{M*}$, and η is a (*k* - 1)-form with the analogous property. Define a (*k* - 1)-form $I\omega$ on *M* as follows:

$$I\omega(p)(v_1, \ldots, v_{k-1}) = \int_0^1 \eta(p,t)(i_{l*}v_1, \ldots, i_{l*}v_{k-1})dt\ .$$

We claim that $d\omega = 0$ implies that $i_1{}^*\omega - i_0{}^*\omega = d(I\omega)$. Actually, it is easier to find a formula for $i_1{}^*\omega - i_0{}^*\omega$ that holds even when $d\omega \neq 0$.

17. THEOREM. *For any k-form* ω *on* $M \times [0,1]$ *we have*

$$i_1{}^*\omega - i_0{}^*\omega = d(I\omega) + I(d\omega)\ .$$

(*Consequently*, $i_1{}^*\omega - i_0{}^*\omega = d(I\omega)$ *if* $d\omega = 0$.)

PROOF. Since $I\omega$ is already invariantly defined, we can just as well work in a coordinate system $(\bar{x}^1, \ldots, \bar{x}^n, t)$. The operator I is clearly linear, so we just have to consider two cases.

(1) $\omega = fd\bar{x}^{i_1} \wedge \cdots \wedge d\bar{x}^{i_k} = fd\bar{x}^I$. Then

$$d\omega = \_\_ + \frac{\partial f}{\partial t} dt \wedge d\bar{x}^I\ ;$$

it is easy to see that

$$\begin{aligned} I(d\omega)(p) &= \left(\int_0^1 \frac{\partial f}{\partial t}(p,t)dt \right) dx^I(p) \\ &= [f(p,1) - f(p,0)]dx^I(p)\ . \\ &= i_i{}^*\omega(p) - i_0{}^*\omega(p) \end{aligned}$$

Since $I\omega = 0$, this proves the result in this case.

(2) $\omega = fdt \wedge d\bar{x}^{i_1} \wedge \cdots \wedge d\bar{x}^{i_{k-1}} = fdt \wedge d\bar{x}^I$. Then $i_1{}^*\omega = i_0{}^*\omega = 0$. Now

$$I(d\omega)(p) = I\left(-\sum_{\alpha=1}^{n}\frac{\partial f}{\partial \bar{x}^{\alpha}}dt \wedge d\bar{x}^{\alpha} \wedge d\bar{x}^{I}\right)(p)$$
$$= -\sum_{\alpha=1}^{n}\left(\int_0^1 \frac{\partial f}{\partial \bar{x}^{\alpha}}(p,t)dt\right)dx^{\alpha} \wedge dx^{I}$$

and

$$d(I\omega) = d\left(\int_0^1 f(p,t)dt\right)dx^{I}$$
$$= \sum_{\alpha=1}^{n}\frac{\partial}{\partial x^{\alpha}}\left(\int_0^1 f(p,t)dt\right)dx^{\alpha} \wedge dx^{I}.$$

Clearly $I(d\omega) + d(I\omega) = 0$. (QED)

18. COROLLARY. *If M is smoothly contractible to a point* $p_0 \in M$ *, then every closed form* ω *on M is exact.*

PROOF. We are given $H : M \times [0,1] \to M$ with

$$\begin{cases} H(p,1) = p \\ H(p,0) = p_0 \end{cases} \quad \text{for all } p \in M.$$

Thus

$$H \circ i_1 : M \to M \quad \text{is the identity,}$$
$$H \circ i_0 : M \to M \quad \text{is the constant map } p_0$$

So

$$\omega = (H \circ i_1)^*(\omega) = i_1{}^*(H^*\omega)$$
$$0 = (H \circ i_0)^*(\omega) = i_0{}^*(H^*\omega).$$

But

$$d(H^*\omega) = H^*(d\omega) = 0,$$

so

$$\begin{aligned} \omega - 0 &= i_1{}^*(H^*\omega) - i_0{}^*(H^*\omega) \\ &= d(I(H^*\omega)) \end{aligned} \quad \text{by the Theorem. (QED)}$$

Corollary 18 is called the ***Poincaré Lemma*** by most geometers, while $d^2 = 0$ is called the ***Poincaré Lemma*** by some (I don't even know whether Poincaré had anything to do with it.) In the case of a star-shaped open subset *U* of R^n, where we have an explicit formula for *H*, we can find (Problem 23) an explicit formula for $I(H^*\omega)$, for every form ω on *U*. Since the new form is given by an integral, we can solve the system of partial differential equations $\omega = d\eta$ explicitly in terms of integrals. There are classical theorems about vector fields in R^3 which can be derived from the Poincaré Lemma and its converse (Problem 27), and originally *d*

was introduced in order to obtain a uniform generalization of all these results. Even though the Poincaré Lemma and its converse fit very nicely into our pattern for basic theorems about differential geometry, it has always been something of a mystery to me just why d turns out to be so important. An answer to this question is provided by a theorem of Palais, *Natural Operations on Differential Forms*, Trans. Amer. Math. Soc. 92 (1959), 125- 141. Suppose we have any operator D from k-forms to l-forms, such that the following diagram commutes for every C^{∞} map $f : M \to N$ [it actually suffices to assume that the diagram commutes only for diffeomorphisms f].

$$\begin{tikzcd}
k\text{-forms on } M \arrow[d, "D"'] & k\text{-forms on } N \arrow[l, "f^*"'] \arrow[d, "D"] \\
l\text{-forms on } M & l\text{-forms on } M \arrow[l, "f^*"']
\end{tikzcd}$$

Palais' theorem says that, with few exceptions, $D = 0$. Roughly, these exceptional cases are the following. If $k = l$, then D can be a multiple of the identity map, but nothing else. If $l = k + 1$, then D can only be some multiple of d. (As a corollary, $d^2 = 0$, since d^2 makes the above diagram commute!) There is only one other case where a non-zero D exists - when k is the dimension of M and $l = 0$. In this case, D can be a multiple of "integration", which we discuss in the next chapter.