G. B. Whitham, Linear and Nonlinear Waves
を読む
－第 5 章　双曲型方程式－

H. N.

2013 年 8 月 26 日

目次

We consider a general quasi-linear first order system

$$A_{ij}\frac{\partial u_j}{\partial t}+a_{ij}\frac{\partial u_j}{\partial x}+b_j=0, \qquad (i=1,\dots,n) \tag{5.1}$$

where the matrices A, a and the vector $\boldsymbol{b}$ may be functions of $u_1,\dots,u_n$ as well as x and t.

5.1 **Characteristics and Classification**

We consider the linear combination with a vector $\boldsymbol{l}=\boldsymbol{l}(x,t,\boldsymbol{u})$.

$$l_i\left(A_{ij}\frac{\partial u_j}{\partial t}+a_{ij}\frac{\partial u_j}{\partial x}\right)+l_ib_j=0 \tag{5.2}$$

We will investigate whether can be chosen so that equation (5.2) takes the form

$$m_i\left(\beta\frac{\partial u_j}{\partial t}+\beta\frac{\partial u_j}{\partial x}\right)+l_ib_j=0. \tag{5.3}$$

If this is possible, (5.3) provides a relation between the directional derivatives of all the u_j in the single direction (α,β). Putting $x=X(\eta)$ and $t=T(\eta)$, the total derivative of on the curve is

$$\frac{du_j}{d\eta}=T'\frac{\partial u_j}{\partial t}+X'\frac{\partial u_j}{\partial x}.$$

We may take

$$\alpha=X'(\eta),\quad \beta=T'(\eta)$$

and write (5.3) as

$$m_j\frac{du_j}{d\eta}+l_jb_j=0. \tag{5.4}$$

The condition for (5.2) to be in the form (5.4) are

$$l_iA_{ij}=m_jT',\quad l_ja_{ij}=m_jX'$$

and we may eliminate the m_j to give

$$l_j(A_{ij}X'-a_{ij}T')=0. \tag{5.5}$$

The necessary and sufficient condition for a nontrivial solution to this equation is

$$\left|A_{ij}X'-a_{ij}T'\right|=0. \tag{5.6}$$

This is a condition on the direction of the curve. Such a curve is said to be a *characteristic* and the corresponding equation (5.4) is said to be in *characteristic form*.

5.2 Examples of Classification

Example 1. First consider the wave equation

$$u_{tt}-\gamma u_{xx}=0$$

This can be written as a system by introducing $u_{x=v}$, $u_t=w$ and writing

$$\begin{cases} v_t-w_x=0 \\ w_t-\gamma v_x=0 \end{cases}.$$

The linear combination

$$l_1(v_t-w_x)+l_2(w_t-\gamma v_x)=0$$

takes the characteristic form

$$l_1(v_t + cv_x) + l_2(w_t + cw_x) = 0$$

provided

$$\begin{cases} -\gamma l_2 = cl_1 \\ -l_1 = cl_2 \end{cases}.$$

There are nontrivial solution when $c^2 = \gamma$. If $\gamma > 0$, we may take

$$c = +\sqrt{\gamma}\,,\ \ l_1 = -\sqrt{\gamma}\,,\ \ l_2 = 1\,;$$

$$c = -\sqrt{\gamma}\,,\ \ l_1 = +\sqrt{\gamma}\,,\ \ l_2 = 1\,.$$

The two vectors ***l*** are linearly independent, hence the system is *hyperbolic*.
If $\gamma < 0$, there are no real characteristic forms; in fact, the equation is the prototype of *elliptic* equations.

Example 2. The heat equation

$$u_t - u_{xx} = 0$$

is equivalent to the system

$$\begin{cases} u_t - v_x = 0 \\ u_x - v = 0 \end{cases},$$

or in a vector form

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 0 \\ -v \end{pmatrix} = 0\,.$$

Then

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \ a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \ \boldsymbol{u} = \begin{pmatrix} u \\ v \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 0 \\ -v \end{pmatrix}.$$

It is clear that the combination

$$l_1(u_t - v_x) + l_2(u_t - v) = 0$$

can be in characteristic form only if $l_1 = 0$. Thus the only solution is $\boldsymbol{l} = (0,1)$ or a scalar multiple of this. Since there is only one vector ***l*** for the second order system, it is not hyperbolic. If we check the general formalism, (5.6) reduces this case to

$$|AX' - aT'| = \left|\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X' - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T'\right| = \begin{vmatrix} X' & T' \\ -T' & 0 \end{vmatrix},$$

that is, $T'^2 = 0$. Thus the x axis is a double characteristics, but there is only one characteristic form

$$u_t - v = 0\,.$$

Example 3. The simplest second order hyperbolic equation is

$$u_{xt} = 0\,;$$

an equivalent system is

$$\begin{cases} u_t - v = 0 \\ v_x = 0 \end{cases},$$

or

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}_t+\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}_x+\begin{pmatrix}-v\\ 0\end{pmatrix}=0\,.$$

In this case both matrices ***A*** and ***a*** are singular but (5.7) is satisfied and there is no trouble. Equation (5.6) is

$$|AX'-aT'|=\left|\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}X'-\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}T'\right|=\begin{vmatrix}X' & 0\\ 0 & -T'\end{vmatrix}$$

that is, $X'T'=0$.

Both the *t* axis and *x* axis are characteristics, and the original equations are already in characteristic form.

Example 4. Consider now

$$u_{tt}-\gamma u_{xx}+u=0\,.$$

If we introduce $u_x=v$, $u_t=w$, as in Example 1, the extra undifferentiated term in u prevents the completely obvious elimination of *u* and might suggest keeping three equations. If we choose

$$\begin{cases} u_x-v=0\\ u_t-w=0\\ w_t-\gamma v_x+u=0\end{cases}$$

or

$$\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}_t+\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & -\gamma & 0\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}_x+\begin{pmatrix}-v\\ -w\\ u\end{pmatrix}=0$$

as an equivalent system, we have the trouble that both matrices A and a are singular, and so are all linear combination of them. Equation (5.6) is

$$|AX'-aT'|=\left|\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}X'-\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & -\gamma & 0\end{pmatrix}T'\right|=\begin{vmatrix}-T' & 0 & 0\\ X' & 0 & 0\\ 0 & \gamma T' & X'\end{vmatrix}$$

and is clearly satisfied for all values of (X',T'). However, this system is excluded by (5.7).

At least in the case $\gamma>0$, we ca implement the suggestion noted earlier that the system is probably too big and ca be reduced. We can spot the reduction by writing the equation as

$$\left(\frac{\partial}{\partial t}-\sqrt{\gamma}\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+\sqrt{\gamma}\frac{\partial}{\partial x}\right)u+u=0\,.$$

Then the introduction of

$$\varphi=u_t+\sqrt{\gamma}u_x$$

leads to the second order system

$$\begin{cases}\varphi_t-\sqrt{\gamma}\varphi_x+u=0\\ u_t+\sqrt{\gamma}u_x-\varphi=0\end{cases}.$$

or

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\varphi\\ u\end{pmatrix}_t+\begin{pmatrix}-\sqrt{\gamma} & 0\\ 0 & \sqrt{\gamma}\end{pmatrix}\begin{pmatrix}\varphi\\ u\end{pmatrix}_x+\begin{pmatrix}u\\ -\varphi\end{pmatrix}=0\,.$$

This has nonsingular coefficients. In fact it is already in characteristic form, and there are just two characteristics.

Example 5. An alternative system that might be proposed for the equation

$$u_{tt} - \gamma u_{xx} + u = 0$$

is

$$\begin{cases} u_t - w = 0 \\ v_t - w_x = 0 \\ w_t - \gamma v_x + u = 0 \end{cases}$$

or

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_t + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -\gamma & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_x + \begin{pmatrix} -w \\ 0 \\ u \end{pmatrix} = 0 .$$

This differ from Example 4 in that the equation $v_t - w_x = 0$, obtained by eliminating u, has been substituted for $u_x - v = 0$. Now A is the unit matrix and we should have no trouble. The condition (5.6) is found to be

$$|AX' - aT'| = \left| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} X' - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -\gamma & 0 \end{pmatrix} T' \right| = \begin{vmatrix} X' & 0 & 0 \\ 0 & X' & T' \\ 0 & \gamma T' & X' \end{vmatrix} = 0 ,$$

that is,

$$X'(X'^2 - \gamma T'^2) = 0 .$$

Two of the roots $X' = \pm\sqrt{\gamma}T'$ are clearly the characteristics of the original equation, but why has an extra characteristic $X' = 0$ arisen? The system is not large as a system, but it is no longer equivalent to the original equation. It is in fact equivalent to

$$\frac{\partial}{\partial t}(u_{tt} - \gamma u_{xx} + u) = 0 .$$

The extra characteristic corresponds to the extra t derivative.

Example 6. The system

$$\begin{cases} u_t + C(u,v) u_x = 0 \\ v_t + C(u,v) v_x = 0 \end{cases}$$

is clearly an example with one characteristic on which $dx/dt=C$, but with two independent characteristic forms. Hence it is hyperbolic.

Example 7 The system

$$\begin{cases} u_t + C(u) u_x = 0 \\ v_t + C(u) v_x + C'(u) v u_x = 0 \end{cases}$$

occurs in dispersive waves. The only possible characteristic form is the first equation as it stands. Hence the system is *not hyperbolic*. Yet because of the exceptional case that the first equation ca be solved independently of the second, we can integrate the first equation along the characteristics $dx/dt=C$. Then once u is known in a ***whole region***, u_x can be calculated to find v. For these purposes it is like a hyperbolic system with a double characteristic, yet formally it would be classified as *parabolic*.

In Examples 2 to 7 the classification is not completely straightforward. We now add a few nonlinear examples

where there is no problem in the classification but they are typical and rather well known. We list pertinent information with a minimum of explanation.

Example 8: Gas Dynamics. In the compressible inviscid flow of gas with velocity u, pressure p, density ρ, and entropy S, the equations (see Chapter 6) are

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0 \\ u_t + uu_x + \dfrac{1}{\rho} p_x = 0 \\ S_t + uS_x = 0 \end{cases},$$

where $p = p(\rho, S)$. The characteristic equations are

$$\frac{dp}{dt} \pm \rho a \frac{du}{dt} = 0 \qquad \text{on} \quad \frac{dx}{dt} = u \pm a ,$$

$$\frac{dS}{dt} = 0 \qquad \text{on} \quad \frac{dx}{dt} = u ,$$

where $a^2 = \left(\dfrac{\partial p}{\partial \rho}\right)_{S=const}$. For a gas with constant specific heats $p = \kappa \rho^{\gamma} e^{S/c_v}$ and $a^2 = \gamma p / \rho$.

Example 9: River Waves and Shallow Water Theory. The equations were given in (3.37); the characteristic forms are

$$\frac{d}{dt}(v \pm 2\sqrt{g'h}) = g'S - C_f \frac{v^2}{h} \qquad \text{on} \quad \frac{dx}{dt} = v \pm \sqrt{g'h} .$$

Example 9'. The reduced kinematic approximation (3.38) has a single characteristic form

$$\frac{dh}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = \frac{3}{2}\left(\frac{g'S}{C_f}\right)^{1/2} h^{1/2} .$$

Example 10: Magnetogasdynamics. For a conducting gas in a magnetic field the equations (using standard notation) are sometimes taken to be

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0 \\ \rho(u_t + uu_x) + p_x = jB \\ \dfrac{1}{\gamma - 1}(p_t + up_x) - \dfrac{\gamma}{\gamma - 1}\dfrac{p}{\rho}(\rho_t + u\rho_x) = \dfrac{j^2}{\sigma} \\ B_t + E_x = 0 \\ \varepsilon_0 E_t + \dfrac{1}{\mu} B_x + j = 0 \end{cases},$$

where $j = \sigma(E - uB)$. The characteristic velocities are $\pm(\varepsilon_0 \mu)^{-1/2}$, $u \pm a$ and u.

Example 10'. When the conductivity σ is very high the following reduced set may be derived and is often used as an adequate approximation:

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0 \\ B_t + uB_x + Bu_x = 0 \\ \rho(u_t + uu_x) + p_x + \dfrac{1}{\mu} BB_x = 0 \\ \dfrac{1}{\gamma - 1}(p_t + up_x) - \dfrac{\gamma}{\gamma - 1}\dfrac{p}{\rho}(\rho_t + u\rho_x) = 0 \end{cases}.$$

The characteristic velocities are now $u \pm (a^2 + B^2 / \mu\rho)^{1/2}$, u, u.

Example 11. Nonlinear Effects for Electromagnetic Waves. In a simple but probably unrealistic formulation of the effects in nonlinear optics, one may take

$$\begin{cases} \dfrac{\partial B}{\partial t} + \dfrac{\partial E}{\partial x} = 0 \\ \dfrac{\partial D}{\partial t} + \dfrac{1}{\mu}\dfrac{\partial B}{\partial x} = 0 \end{cases},$$

with $D = D(E)$. The characteristic equations are

$$\frac{dB}{dt} \pm \frac{1}{c(E)}\frac{dE}{dt} = 0 \qquad \text{on} \quad \frac{dx}{dt} = \pm c(E)$$

where $c(E) = [\mu D'(E)]^{-1/2}$. Dispersive effects usually make the relation $D = D(E)$ inadequate.

Example 12: Nonlinear Elastic Waves in a Bar. The one dimensional equation for waves in a bar may be formulated in terms of the displacement $\xi(x,t)$ of a section initially at position x and the stress $\sigma(x,t)$, as

$$\rho_0 \xi_{tt} = \sigma_x,$$

where ρ_0 is the initial density in the unstrained state. If the strain $\varepsilon = \xi_x$ and velocity $u = \xi_t$ are introduced, the equivalent pair

$$\begin{cases} \rho_0 u_t - \sigma_x = 0 \\ \varepsilon_t - u_x = 0 \end{cases}$$

may be used. The linear theory takes $\sigma \propto \varepsilon$, but nonlinear effects may be included by taking σ as a more general function $\sigma = \sigma(\varepsilon)$. The characteristic velocities are $\pm[\sigma'(\varepsilon)/\rho_0]^{1/2}$. For the appropriate choices of $\sigma(\varepsilon)$, there are interesting effects in the wave propagation; in particular, it is a little surprise perhaps to find that chocks are produced in the *unloading* phase of a disturbance. An account is includes¥d in Courant and Friedrichs (1948, p.235).

5.3 Riemann Invariants

Each equation I characteristic form introduces a particular linear combination of the derivatives. For simplicity we consider the reduced form (5.10), where the linear combination concerned is $l_i du_i / dt$. In a linear problem, the vector $\boldsymbol{l}$ is independent of $\boldsymbol{u}$ so that a new variable $r = l_i u_i$ simplifies the form of the equation to

$$\frac{dr}{dt} + f(x,t,\boldsymbol{u}) = 0.$$

In nonlinear problems, however, $\boldsymbol{l}$ may depend on $\boldsymbol{u}$ and it is not always possible to achieve this form. It would be necessary to find λ and r such that

$$l_i du_i = \lambda dr,$$

or, equivalently,

$$l_i = \lambda \frac{\partial r}{\partial u_i} . \tag{5.13}$$

(Here x and t are held fixed; the differential dr refers only to change in $\boldsymbol{u}$.) This is a special case of Pfaff's problem for the integrability of differential forms. For n=2 we may eliminate r and find an equation for λ which clearly has a solution. For n>2, however, elimination of both λ and r from (5.13) gives condition on the l_i which must satisfied for this to be possible.

For a hyperbolic system, the n characteristic equations take a particularly simple form if it should turn out that a variable r_k can be introduced corresponding to each differential form $l_i^{(k)} du_i$. Then the functions r_k can be used as new variables in place of the u_i and the characteristic equations can be written as

$$\frac{dr_k}{dt} + f_k(x,t,\boldsymbol{r}) = 0 \quad \text{on} \quad \frac{dx}{dt} = c_k(x,t,\boldsymbol{r}) \tag{5.14}$$

This can always be done for linear problems, and in that case the f_k are linear in r. For nonlinear problems it can be done when n=2, but it may not be possible for n>2.

Such variables were introduced by Riemann in his work on plane waves in gas dynamics, a case with n=2. In that particular case (see Section 6.7), the f_k are zero so that r_1 and r_2 are constant on their respective characteristics; the functions r_1 and r_2 are the called *Riemann invariants*. In general, we might call the r_k *Riemann variables*.

5.8 Shock Waves

The situation as regards the breaking of waves and the introduction of shock waves is very much the same as in the case of a single quasi-linear equation. Some solutions which are initially single valued, and even continuous, will develop multivalued regions: waves will break. This is again interpreted as an inadequacy of the assumptions leading to (5.1), but the appropriate saving features can be well approximated by allowing discontinuities in $\boldsymbol{u}$.

We again take the view that in formulating the differential equations, there will have been an earlier stage where the equations were in integrated form

$$\frac{d}{dt}\int_{x_2}^{x_1} f_i dx + [g_i]_{x_2}^{x_1} + \int_{x_2}^{x_1} h_i dx = 0 , \tag{5.54}$$

where f_i, g_i, h_i are various quantities of physical interest in the problem. For example, in problems of mechanics f_i and g_i could be the density and flux of mass, or the density and flux of momentum, or the density and flux of energy. The quantity h_i allows for a distributed source term, such as a body force in the momentum equation. Equation (5.54) is a *conservation equation* for the physical quantity concerned (mass, momentum, energy, etc.).

The densities f_i will be functions of (x, t) and of n basic variables $\boldsymbol{u} = (u_1, \ldots, u_n)$; in general, there will be n equations (5.54) in the statement of the appropriate physical laws. Various simplifying assumptions will then be made to relate the g_i, h_i to x, t, $\boldsymbol{u}$. At the first level of approximation, the g_i and h_i will just be functions of x, t, $\boldsymbol{u}$. If $\boldsymbol{u}$ has continuous first derivatives, (5.54) may then be written in the differentiated form

$$\frac{\partial f_i(x,t,\boldsymbol{u})}{\partial t} + \frac{\partial g_i(x,t,\boldsymbol{u})}{\partial x} + h_i(x,t,\boldsymbol{u}) = 0 . \tag{5.55}$$

This is a differential equation in *conservation form*.

If discontinuities in $\boldsymbol{u}$ are to be included, the integrated form (5.54) must be used and the dependence of g_i and h_i on $\boldsymbol{u}$ left open at first. If a discontinuous *shock* occurs at $x = s(t)$, exactly the same argument given in Section 2.3 gives the shock conditions

$$-U[f_i] + [g_i] = 0\,, \quad i=1,\ldots,n, \tag{5.56}$$

where U is the shock velocity $\dot{s}(t)$. We then argue that in the continuous parts of the solution on the two sides of the shock, it is still a good approximation to take

$$g_i = g_i(x,t,\boldsymbol{u})\,,\quad h_i = h_i(x,t,\boldsymbol{u})$$

Therefore (5.56) is applied with the same functional dependence of g_i on $\boldsymbol{u}$. As in Chapter 2, a more accurate choice of the g_i will involve derivatives of $\boldsymbol{u}$, and the shocks will be smoothed out into thin regions of rapid change. However, the discontinuity treatment is simpler and usually is adequate.

The formal mathematical definition of *weak solutions* of (5.55), leading to the jump conditions (5.56), follows closely the discussion in Section 2.7. Evaluating the derivatives in (5.55), we see that it is a case of the system (5.1) in which

$$A_{ij} = \frac{\partial f_i}{\partial u_j}\,,\quad a_{ij} = \frac{\partial g_i}{\partial u_j}\,, \tag{5.57}$$

The discussion of weak solutions is applicable only to these special cases. Moreover the important warning about nonuniqueness must be emphasized. In typical cases, starting from the relevant system (5.1), it will be possible to find more than n different equations in the conservation form (5.55). Shock conditions (5.56) from a choice of any n of them will be satisfactory mathematically, but only those n equations that correspond to the original physical statements in (5.54) will give the correct solutions for the problem. A good example of this nonuniqueness occurs in gas dynamics (see Chapter 6). In view of the nonuniqueness, the connection with the physical laws is stressed here.