CHAPTER 7
The Wave Equation

The equation

$$\frac{\partial^2\varphi}{\partial t^2}=c^2\nabla^2\varphi\,,\quad c=const. \qquad (7.1)$$

has become known as the *wave equation* even though **the majority of waves are not governed by it**! However, it does occur in many problems and it is the simplest equation for starting the discussion of two and three dimensional waves. There is an enormous number of possible topics and we must make some choice. Following the general theme of this book, we restrict the discussion to basic results which contribute to an understanding of waves and play a role in extensions to nonlinear theory. We make no attempt to give even an introduction to the vast amount of special and intricate analysis developed for the various boundary value problems of diffraction theory. The elementary aspects of interference and diffraction patterns are well documented in a variety of books and the more advanced theory rapidly becomes a matter of skillful use of "mathematical methods" rather than one of understanding the nature of waves more deeply.

On the other hand, the approximate *theory of geometrical optics* involves valuable general ideas which can be extended to other contexts both for linear and nonlinear problems. It is developed here for the wave equation and the extensions for nonuniform media and for anisotropic waves are noted. These extensions go beyond the strict discussion of (7.1) but the material fits in conveniently here. Other aspects of geometrical optics and the development of similar ideas in nonlinear problems are considered in later chapters.

7.1 Occurrence of the Wave Equation

The wave equation (7.1) occurs primarily in three fields: acoustics, elasticity, and electromagnetism.

Acoustics.

The equations for acoustics have been given in Section 6.6. The expressions are noted here for easy reference. The *gas dynamic equations* are linearized for small perturbations about a constant state in which

$$u=0\,,\quad \rho=\rho_0\,,\quad p=p_0=p(\rho_0)$$

The propagation speed is

$$a_0^2=\left.\frac{\partial p}{\partial \rho}\right|_{\rho=\rho_0}=p'(\rho_0)\,, \qquad (7.2)$$

and in terms of a velocity potential φ the perturbations are given by

$$u=\nabla\varphi\,, \qquad (7.3)$$

$$p-p_0=\rho_0\varphi_t\,, \qquad (7.4)$$

$$\begin{cases}\dfrac{\partial\rho}{\partial t}+\nabla(\rho\boldsymbol{u})=0\\[2mm] \dfrac{\partial\boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot grad)\boldsymbol{u}=-\dfrac{1}{\rho}gradp\end{cases}$$

$$p=p_0+p',\_\rho=\rho_0+\rho'$$

$Linearized:$

$$\begin{cases}\dfrac{\partial\rho'}{\partial t}+\rho_0\nabla(\boldsymbol{u})=0,\_(3)\\[2mm] \dfrac{\partial\boldsymbol{u}}{\partial t}=-\dfrac{1}{\rho_0}gradp',\_(4)\end{cases}$$

$$p'=\left(\frac{\partial p}{\partial\rho}\right)_S\rho'$$

$Substitution\_into\_(3):$

$$\frac{\partial p'}{\partial t}+\rho_0\left(\frac{\partial p}{\partial\rho}\right)_S\nabla(\boldsymbol{u})=0,\_(5)$$

$Velocity\_potential:$

$$\boldsymbol{u}=grad\phi$$

$Substitution\_into\_(4):$

$$\frac{\partial grad\phi}{\partial t}=-\frac{1}{\rho_0}gradp'$$

$$\frac{\partial\phi}{\partial t}=-\frac{1}{\rho_0}p',\_(7.4)$$

$Also$

$$\rho'=\frac{p'}{\left(\dfrac{\partial p}{\partial\rho}\right)_S}=\frac{p'}{a_0^2}=-\frac{\rho_0}{a_0^2}\frac{\partial\phi}{\partial t},\_(7.5)$$

$$\rho - \rho_0 = -\frac{\rho_0}{a_0^2}\varphi_t \,. \tag{7.5}$$

Substitution in the linearized continuity equation leads to the equation for φ:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot u = 0$$

$$-\frac{\rho_0}{a_0^2}\varphi_{tt} + \rho_0 \nabla^2 \varphi = 0$$

$$\varphi_{tt} = a_0^2 \nabla^2 \varphi$$

$$\varphi_{tt} = a_0^2 \nabla^2 \varphi \,. \tag{7.6}$$

Linearized Supersonic Flow.

The acoustic formulation may be used when the disturbance is caused by a moving solid body. If the disturbance is to remain small, the motion of the body must be very small (this applies to the cone of a loud speaker, for example) or the body must be very slender. The former is a typical source for sound waves and the equation must be solved subject to appropriate boundary conditions. The case of a *slender body* moving with arbitrary constant velocity relates acoustics to aerodynamics. If the body moves with constant velocity, there is an obvious advantage in transforming to a moving frame of reference fixed in the body. Let (x_1, x_2, x_3) refer to the original frame in which the motion of the gas is small and described by (7.3)-(7.6). If the body moves with speed U in the negative x_1 direction, and (x, y, z) refer to coordinates fixed with respect to the body, the transformation of coordinates is given by

$$x = x_1 + Ut \,, \quad y = x_2 \,, \quad z = x_3 \,.$$

The velocity components in the new frame are $(U + u_1, u_2, u_3)$ where $u_i = \partial\varphi / \partial x_i$. Moreover, the flow appears steady in the new frame so that

$$\begin{aligned} \varphi(x_1, x_2, x_3, t) &= \Phi(x, y, z) \\ &= \Phi(x_1 + Ut, x_2, x_3) \end{aligned} \,.$$

Therefore (7.6) becomes

$$\varphi_{tt} = a_0^2 \nabla^2 \varphi$$

$$U^2 \Phi_{tt} = a_0^2 \left\{ \Phi_{x_1 x_1} + \Phi_{x_2 x_2} + \Phi_{x_3 x_3} \right\}$$

$$\frac{U^2}{a_0^2} U^2 \Phi_{tt} = \Phi_{x_1 x_1} + \Phi_{x_2 x_2} + \Phi_{x_3 x_3}$$

$$(M^2 - 1)\Phi_{xx} = \Phi_{yy} + \Phi_{zz}$$

$$(M^2 - 1)\Phi_{xx} = \Phi_{yy} + \Phi_{zz} \,, \qquad M = \frac{U}{a_0} \,; \tag{7.7}$$

(7.4) becomes

$$p - p_0 = -\rho_0 U \Phi_x \,; \tag{7.8}$$

and the velocity components relative to the body are

$$(U + \Phi_x, \Phi_y, \Phi_z) \,. \tag{7.9}$$

For supersonic flow, $M > 1$, we recover the wave equation in a reduced number of variables with x playing the role of time. This is a linearized version of the analogy noted in Section 6.16.

7.2 Plane Waves

For one space dimension x, the wave equation is

$$\varphi_{tt} = c^2 \varphi_{xx} \,.$$

If characteristic coordinates $\alpha = x - ct$, $\beta = x + ct$ are introduced, it reduces to

$$\frac{\partial^2 \varphi}{\partial \alpha \partial \beta} = 0,$$

and the general solution is

$$\begin{aligned} \varphi &= f(\alpha) + g(\beta) \\ &= f(x - ct) + g(x + ct) \end{aligned}.$$

where f and g are arbitrary functions. The arbitrary functions are readily determined to fit prescribed initial or boundary conditions. For the signaling problem of outgoing waves with

$$\varphi_x = Q(t), \quad x = 0,$$

the solution is

$$\varphi = -cQ_1\left(t - \frac{x}{c}\right)$$

where $Q_1(t)$ is the integral of $Q(t)$. For the initial value problem,

$$\varphi = \varphi_0(x), \quad \varphi_t = \varphi_1(x), \quad t = 0, \quad -\infty < x < \infty,$$

the solution is

$$\varphi = \frac{1}{2}\{\varphi_0(x - ct) + \varphi_0(x + ct)\} + \frac{1}{2c}\int_{x-ct}^{xct} \varphi_1(\xi) d\xi. \quad (7.21)$$

7.3 Spherical Waves

For waves symmetric about the origin $\varphi = \varphi(R, t)$, where R is the distance from the origin. The wave equation reduces to

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial R^2} + \frac{2}{R}\frac{\partial^2 \varphi}{\partial R}.$$

Surprisingly enough, this may also be written

$$\frac{1}{c^2}\frac{\partial^2 (R\varphi)}{\partial t^2} = \frac{\partial^2 (R\varphi)}{\partial R^2},$$

$$LHS = \frac{R}{c^2}\frac{\partial^2 \varphi}{\partial t^2}$$

$$\begin{aligned} RHS &= \frac{\partial}{\partial R}\left\{\varphi + R\frac{\partial \varphi}{\partial R}\right\} \\ &= \frac{\partial \varphi}{\partial R} + \left\{\frac{\partial \varphi}{\partial R} + R\frac{\partial^2 \varphi}{\partial R^2}\right\} \\ &= 2\frac{\partial \varphi}{\partial R} + R\frac{\partial^2 \varphi}{\partial R^2} \end{aligned}$$

which is the one dimensional wave equation. Thus the general solution takes the simple form

$$\varphi = \frac{f(R - ct)}{R} + \frac{g(R + ct)}{R}. \quad (7.22)$$

For a source producing only outgoing waves, the solution is

$$\varphi = \frac{f(R - ct)}{R},$$

and f is determined from the properties of the source. A convenient standard form is to prescribe

$$Q(t) = \lim_{R \to 0} 4\pi R^2 \frac{\partial \varphi}{\partial R}; \quad (7.23)$$

this gives

$$Q(t) = -4\pi f(-ct)$$

$$\begin{aligned} Q(t) &= \lim_{R \to 0} 4\pi R^2 \frac{\partial \varphi}{\partial R} \\ &= \lim_{R \to 0} 4\pi R^2 \left[-\frac{f(R - ct)}{R^2} + \frac{1}{R} f'(R - ct)\right] \\ &= \lim_{R \to 0} 4\pi\left[-f(R - ct) + Rf'(R - ct)\right] \\ &= -4\pi f(-ct) \end{aligned}$$

and

$$\varphi = -\frac{1}{4\pi}\frac{Q(t - R/c)}{R}. \tag{7.24}$$

In acoustics, $\partial\varphi/\partial R$ is the *radial velocity* and $Q(t)$ is the *volume flux* of fluid.

For an initial value problem, although it is merely a matter of determining the functions f and g in (7.22), the solution is more interesting than might have been expected.

[***Balloon Problem***]

Consider the "balloon problem" in acoustics: the pressure inside a region of radius R_0 is $p_0 + P$ while the pressure outside is p_0, the gas is initially at rest, and the balloon is burst at $t = 0$. From (7.3) and (7.4), the initial conditions may be formulated as

$$\varphi = 0\,,\quad \varphi_t = \begin{cases} -P/\rho_0, & R < R_0, \\ 0, & R > R_0. \end{cases}$$

Therefore the solution

$$\varphi = \frac{f(R - a_0 t)}{R} + \frac{g(R + a_0 t)}{R} \tag{7.25}$$

must have

$$f(R) + g(R) = 0\,,\qquad 0 < R < \infty\,,$$

$$f'(R) - g'(R) = \begin{cases} \dfrac{P}{\rho_0 a_0}R, & 0 < R < R_0, \\ 0, & R_0 < R < \infty. \end{cases} \tag{7.26}$$

These conditions determine f and g for positive values of their arguments. However, in the solution (7.25), the values of f are also required for negative argument. The remaining condition comes from the behavior of the solution at the origin. Since there is no source at the origin, we require

$$\lim_{R\to 0} R^2\frac{\partial\varphi}{\partial R} = 0\,;$$

hence,

$$f(-a_0 t) + g(a_0 t) = 0\,,\quad 0 < t < \infty\,. \tag{7.27}$$

This condition determines f for negative argument in terms of the values of g for positive argument.

Solving (7.26) and (7.27), we have

$$f(\xi) = \begin{cases} \dfrac{1}{4}\dfrac{P}{\rho_0 a_0}(\xi^2 - R_0^2), & -R_0 < \xi < R_0 \\ 0, & R_0 < |\xi| \end{cases},$$

$$g(\xi) = \begin{cases} -\dfrac{1}{4}\dfrac{P}{\rho_0 a_0}(\xi^2 - R_0^2), & 0 < \xi < R_0 \\ 0, & R_0 < \xi \end{cases}.$$

Finally, the solution for the pressure disturbance is

$$p - p_0 = \frac{P}{2R}\{(R - a_0 t)F + (R + a_0 t)G\},$$

where

$$F = \begin{cases} 1, \_ -R_0 < R - a_0 t < R_0 \\ 0\_otherwise \end{cases}$$

$$G = \begin{cases} 1, \_ 0 < R + a_0 t < R_0 \\ 0\_otherwise \end{cases}.$$

The variation of pressure with time is shown in Fig. 7.1. For a

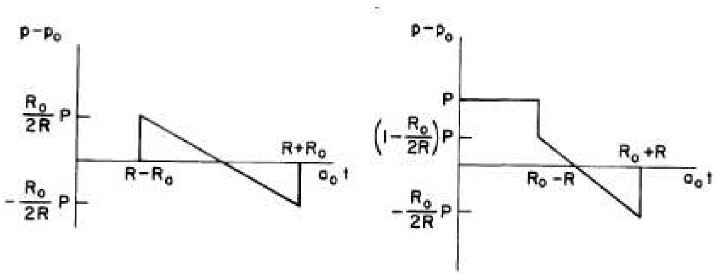

Fig. 7.1. Pressure signatures in the balloon problem.

point $R > R_0$, a discontinuous pressure increase equal to $PR_0/2R$ arrives at time $t = (R - R_0)/a_0$; the excess pressure then decreases linearly in time to a value $-PR_0/2R$ at $t = (R + R_0)/a_0$ and it then returns discontinuously to zero. Even at $R = R_0$, the discontinuity at the front of the wave is only $P/2$; the remaining $P/2$ to make up the initial discontinuity P is taken by the incoming expansion wave.

For interior points $R < R_0$, a discontinuous decrease in pressure, reducing the initial value P to $P(1 - R_0/2R)$, arrives at time $t = (R_0 - R)/a_0$; the excess pressure then decreases linearly with time to $-PR_0/2R$ at $t = (R_0 + R)/a_0$ and then returns discontinuously to zero. Notice at the center $R = 0$ these changes are infinite but the whole disturbance lasts for zero time!

It is interesting that an entirely positive initial distribution of pressure leads to an outgoing wave with equal positive and negative phases. In fact this *N wave* profile is typical in two and three dimensional waves. The reasons for it can be understood as follows. In an outgoing wave the pressure and radial velocity are given by

$$p - p_0 = \frac{\rho_0 a_0 f'(R - a_0 t)}{R},$$

$$u = \frac{f'(R - a_0 t)}{R} - \frac{f(R - a_0 t)}{R^2}.$$

The *first point* we can make is that in any wave that returns ***both*** $p - p_0$ and u to zero after the whole wave passes, ***both*** f' and f must return to zero. Hence, f' has to have both positive and negative parts if the total integral, which is f, has to return to zero. A *second point* concerns the total volume flow at large distances. For large *R*, the volume flow across a sphere of radius R is

$$4\pi R^2 u \sim 4\pi R f'(R - a_0 t).$$

This is large in R. If f', which is proportional to the pressure, were always positive, there would be an infinitely large outward flow as $R \to \infty$. An *N wave*, however, has a large outward flow immediately followed by a balancing large inward flow, so the net outflow is finite.

For *plane waves* neither of these effects arises and a positive source provides a wave with wholly positive $p - p_0$ and u.

7.4 Cylindrical Wave

7.5 Supersonic Flow Past a Body of Revolution

The most interesting use of the cylindrical wave solution is probably in supersonic aerodynamics. As noted in (7.7), the perturbation velocity potential satisfies the two dimensional wave equation with

$$x \leftrightarrow t\,,\quad M^2 - 1 \leftrightarrow \frac{1}{c^2}\,.$$

For a body of revolution, (7.7) becomes

$$B^2 \Phi_{xx} = \Phi_{rr} + \frac{1}{r}\Phi_r\,,\quad B = \sqrt{M^2 - 1}\,,$$

$\sin\mu = 1/M$

$$1 + \frac{1}{\tan^2\mu} = \frac{1}{\sin^2\mu}$$

$$\frac{1}{\tan^2\mu} = \sqrt{\frac{1}{\sin^2\mu} - 1} = \sqrt{M^2 - 1} = B$$

$r/\tan\mu = Br$

where r is distance from the flight path and x is distance from the nose of the body. The solution is zero for $x<Br$ and

$$\Phi = -\frac{1}{2\pi}\int_0^{x-Br} \frac{q(\eta)}{\sqrt{(x-\eta)^2 - B^2 r^2}}\,d\eta\,,\quad x > Br\,. \tag{7.38}$$

The *source strength* $q(\eta)$ is related to the shape of the body. The boundary condition on the body is that the velocity normal to it is zero. Hence if the body shape is given by $r = R(x)$,

$$\Phi_r = R'(x)(U + \Phi_x) \quad \text{on } r = R(x). \tag{7.39}$$

For the linearization in the equations, the body must be slender, that is, $R'(x)$ is small, and Φ_x and Φ_r are both small. Accordingly, the boundary condition is linearized to

$$q(r) = 2\pi U R(x) R'(x) = U S'(x)\,,$$

where $S(x) = \pi R^2(x)$ is the *cross-sectional area of the body* at a distance x from the nose. Intuitively one can see that $US'(x)$ is the rate at which the increasing cross-sectional area is ***pushing fluid out***, and this is the source strength. The solution for the given body therefore is,

$$\Phi = -\frac{U}{2\pi}\int_0^{x-Br} \frac{S'(\eta)\,d\eta}{\sqrt{(x-\eta)^2 - B^2 r^2}}\,,\quad x - Br > 0\,. \tag{7.40}$$

The components of the velocity perturbation are obtained by suitable modification of (7.30) as

$$\Phi_x = -\frac{U}{2\pi}\int_0^{x-Br} \frac{S''(\eta)\,d\eta}{\sqrt{(x-\eta)^2 - B^2 r^2}}\,, \tag{7.41}$$

$$\Phi_r = \frac{U}{2\pi}\int_0^{x-Br}\frac{(x-\eta)S''(\eta)d\eta}{\sqrt{(x-\eta)^2 - B^2r^2}}\,. \qquad (7.42)$$

In ***linear theory***, the pressure is given by (7.8). However, an interesting question arises here about the consistency of the linear theory, particularly with regard to the pressure. The exact expression for the pressure in potential flow is given by Bernoulli's equation [see (6.157)] as

$$a^2 = a_0^2 - \frac{\gamma-1}{2}(q^2 - U^2)\_(6.157)$$

$$\frac{p}{p_0} = \left(\frac{a}{a_0}\right)^{2\gamma/(\gamma-1)} = \left\{1 - \frac{\gamma-1}{a_0^2}\left(U\Phi_x + \frac{1}{2}\Phi_x^2 + \frac{1}{2}\Phi_r^2\right)\right\}^{\gamma/(\gamma-1)}.$$

$$\frac{a^2}{a_0^2} = 1 - \frac{\gamma-1}{2a_0^2}\left\{(U+\Phi_x)^2 + \Phi_r{}^2 - U^2\right\} = 1 - \frac{\gamma-1}{2a_0^2}\left\{2U\Phi_x + \Phi_x{}^2 + \Phi_r{}^2\right\}$$

Therefore, since $a_0^2 = \gamma p_0/\rho_0$,

$$\frac{p-p_0}{\rho_0} = -\left(U\Phi_x + \frac{1}{2}\Phi_x^2 + \frac{1}{2}\Phi_r^2\right) + \cdots.$$

$$\frac{p}{p_0} = 1 - \frac{\gamma}{a_0^2}\left(U\Phi_x + \frac{1}{2}\Phi_x^2 + \frac{1}{2}\Phi_r^2\right) = 1 - \frac{\rho_0}{p_0}\left(U\Phi_x + \frac{1}{2}\Phi_x^2 + \frac{1}{2}\Phi_r^2\right)$$

$$\frac{p}{p_0} - 1 = -\frac{\rho_0}{p_0}\left(U\Phi_x + \frac{1}{2}\Phi_x^2 + \frac{1}{2}\Phi_r^2\right)$$

When r is not small compared with the length of the body, Φ_x and Φ_r are comparable small quantities being of order δ^2 in the thickness ratio of the body δ (defined as maximum diameter divided by length). Then the linearization in which $\Phi_x{}^2$ and $\Phi_r{}^2$ are neglected is correct. However, on the body $r = R(x) = O(\delta)$, and for small r,

$$\Phi_r \sim \frac{US'(x)}{2\pi r},\quad \Phi_x \sim \frac{US''(x)}{2\pi}\log r\,.$$

Therefore on the body

$$\Phi_r = O(\delta)\,,\quad \Phi_x = O(\delta^2\log\frac{1}{\delta})\,.$$

Apart from the $\log(1/\delta)$, which in practical situations in not really large, the term $\Phi_r{}^2/2$ is as important as the term Φ_x. It appears then that one should take

$$\frac{p-p_0}{\rho_0} = -U\Phi_x - \frac{1}{2}\Phi_r{}^2\,, \qquad (7.43)$$

rather than (7.8), for a good approximation to the pressure. Lighthill (1945)[1] and Broderick (1949) showed by careful consideration of higher approximations that (7.43) is correct with an error $O(\delta^4\log^2 1/\delta)$. At the same time the consistency of the linear theory must be questioned, since the boundary condition is applied in the region where Φ_x and Φ_r are not of the same order. It is shown in the references cited that (7.41) and (7.42) are the correct lowest order terms, and the adoption of the nonlinear relation (7.43) is the only essential change.

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[1] M. J. Lighthill, Supersonic flow past bodies of revolution. *ARC R & M* 2003.

Drag.

The drag on the surface of the body due to the perturbed pressure is

$$D=\int_0^l (p-p_0)S'(x)dx\,,$$

where the integration extends over the length l of the body. The expression for Φ near the body is

$$\Phi \sim -\frac{U}{2\pi}\int_0^x S''(\eta)\log\frac{2(x-\eta)}{Br}d\eta\,,$$

[see (7.31)], and the pressure is given by (7.43). Hence the drag can be written

$$\frac{2\pi D}{\rho_0 U^2}=\int_0^l S'(x)\left\{-S''(x)\log R(x)+\frac{\partial}{\partial x}\int_0^x S''(\eta)\log\frac{2(x-\eta)}{B}d\eta-\frac{1}{4\pi}\frac{S'^2(x)}{R^2(x)}\right\}dx$$

The first and third terms combine into

$$-\int_0^l\left\{S'(x)S''(x)\log R(x)+\frac{S'^2(x)}{2R(x)}R'(x)\right\}dx\,,$$

since $S'=2\pi RR'$, and this is

$$-\int_0^l\frac{d}{dx}\left\{\frac{1}{2}S'^2(x)\log R(x)\right\}dx=0\,,$$

for a body with $S'(0)=S'(l)=0$. After integration by parts, the second term gives

$$\begin{aligned}D&=\frac{\rho_0U^2}{2\pi}\int_0^l S''(x)\int_0^x S''(\eta)\log\frac{1}{|x-\eta|}dxd\eta\\&=\frac{\rho_0U^2}{4\pi}\int_0^l\int_0^l S''(x)S''(\eta)\log\frac{1}{|x-\eta|}dxd\eta\end{aligned}\qquad(7.44)$$

[The term in $\log(2/B)$ integrates to zero.] This is the famous supersonic drag formula first obtained by von Karman and Moore in 1932[2].

Behavior Near the Mach Cone and at Large Distances.

The wavefront is $x-Br=0$; this is the Mach cone making an angle $\sin^{-1}(1/M)$ with the x axis. When $(x-Br)/Br<<1$, we have from (7.32) and (7.33), suitably transcribed to supersonic flow, that

$$\Phi\sim-\frac{U}{2\pi\sqrt{2Br}}\int_0^\xi\frac{S'(\eta)}{\sqrt{\xi-\eta}}d\eta\,,\qquad \xi=x-Br\,.$$

Hence the velocity components are

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[2] Th. von Karman and N. B. Moor, Resistance of slender bodies moving with supersonic velocities with special reference to projectiles. *Trans. Am. Soc. Mech. Engrs.*

$$\Phi_x \sim -\frac{UF(x-Br)}{\sqrt{2Br}}, \quad \Phi_r \sim UB\frac{F(x-Br)}{\sqrt{2Br}}, \quad \frac{x-Br}{Br} << 1, \tag{7.45}$$

where

$$F(\xi) = \frac{1}{2\pi}\int_0^{\xi}\frac{S''(\eta)}{\sqrt{\xi-\eta}}d\eta. \tag{7.46}$$

In this limit, Φ_x and Φ_r are the same order and the pressure is given to the first order by

$$\frac{p-p_0}{\rho_0 U^2} \sim \frac{F(x-Br)}{\sqrt{2Br}}. \tag{7.46}$$

It should again be noted that the behavior near the wavefront and at large distances can be combined in one expression.

If the body is sharp nosed with $R'(0)=\varepsilon$, then $S(x) \sim \pi\varepsilon^2 x^2$ for small x, and we have

$$F(\xi) \sim 2\varepsilon^2 \xi^{1/2}, \quad \text{as} \quad \xi \to 0. \tag{7.48}$$

Figure 7.3 is a typical $F(\xi)$ curve. The appearance of negative

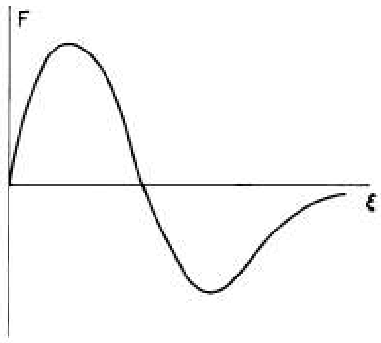

Fig. 7.3. Typical F curve for axisymmetric supersonic flow past a body.

phase is typical, even for a shell shaped body for which the source strength $US'(x)$ is always positive. In fact it is easily shown that

$$\int_0^{\infty} F(\xi)d\xi = 0,$$

and the physical explanation in terms of mass flow is similar to that given for spherical waves at the end of Section 7.3.

It may be remarked that the velocity components and pressure are continuous at the Mach cone according to this linear theory. In reality a *shock wave* is produced, and we have the important phenomenon of the *sonic boom*. This is missed because it is a nonlinear effect. The theory for it is developed in detail in Chapter 9.

7.6 Initial Value Problem in Two and Three Dimensions

54, 303-310.

One of the many "*Poisson integrals*" appearing in the theory of partial differential equations provides the solution of the wave equation for initial conditions:

$$\varphi = \varphi_0(\boldsymbol{x})\,,\quad \varphi_t = \varphi_1(\boldsymbol{x})\,,\quad t = 0\,.$$

According to Hadamard's ideas, the three dimensional problem will be easier than the two dimensional, and we start with it.

We know from the spherical wave solution discussed in Section 7.3 that

$$\psi(\boldsymbol{x},t) = \frac{f(|\boldsymbol{x}-\xi| - ct)}{|\boldsymbol{x}-\xi|}$$

is a solution for arbitrary ξ. We now argue intuitively that the initial prescribed disturbance at any point ξ gives rise to such a spherical wave and propose that the solution should be something like a superposition of all the spherical waves. That is, we consider

$$\psi(\boldsymbol{x},t) = \int_{-\infty}^{\infty} \Psi(\xi)\frac{f(|\boldsymbol{x}-\xi| - ct)}{|\boldsymbol{x}-\xi|}d\xi\,. \qquad (7.49)$$

The arbitrary function $\Psi(\xi)$ is inserted because, depending on the initial conditions, the spherical waves from different points ξ will have different strengths. The form of (7.49) suggests the introduction of spherical polars (R,θ,λ) based on the point $\boldsymbol{x}$. It then becomes

$$\psi(\boldsymbol{x},t) = \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \Psi(\boldsymbol{x}+R\boldsymbol{l}) f(R-ct) R\sin\theta dR d\theta d\lambda\,,$$

where $\boldsymbol{l}$ is the unit vector from $\boldsymbol{x}$ to ξ and its Cartesian components may be written

$$\boldsymbol{l} = (\sin\theta\cos\lambda, \sin\theta\sin\lambda, \cos\theta)\,.$$

With the idea that the initial source strength determining f acts only for an instant, we specialize this formula to the case $f(R-ct) = \delta(R-ct)$. Then

$$\psi(\boldsymbol{x},t) = ct\int_0^{\pi}\int_0^{2\pi} \Psi(\boldsymbol{x}+ct\boldsymbol{l}) f(R-ct)\sin\theta d\theta d\lambda\,. \qquad (7.50)$$

Formally, this is a solution for arbitrary Ψ. It may also be written as a surface integral

$$\psi(\boldsymbol{x},t) = \frac{1}{ct}\int_{S(t)} \Psi dS\,,$$

where $S(t)$ is the surface of the sphere with origin at x and radius ct. For a continuously differentiable function Ψ, we see from (7.50) that

$$\psi \to 0\,,\quad \psi_t \to 4\pi\Psi(\boldsymbol{x}) \quad \text{as} \quad t\to 0\,. \qquad (7.51)$$

If we choose $\Psi(\boldsymbol{x}) = \varphi_1(\boldsymbol{x})$, we shall have solved the special initial value problem

$$\psi \to 0\,,\quad \psi_t \to \varphi_1(\boldsymbol{x}) \quad \text{as} \quad t\to 0\,. \qquad (7.52)$$

The solution is

$$\psi(\boldsymbol{x},t) = \frac{1}{4\pi ct}\int_{S(t)} \varphi_1 dS\,. \qquad (7.53)$$

It represents the total contribution of the instantaneous sources which send spherical waves to the point x in time t; they are all exactly a distance ct away and their contributions traveling with speed c arrive at x just at the time t (see Fig. 7.4). Notice that all points inside $S(t)$ could still in principle have been contributing. But there is no "*tail*" for spherical waves: the sources act for an infinitesimal time and each contribution lasts only for an infinitesimal time. This will not be so in two dimensions. In any event, (7.53) is formally the solution for the initial values (7.52). It may also be written

$$\psi(\boldsymbol{x},t) = ctM[\varphi_1]\,,$$

where

$$M[\varphi_1] = \frac{1}{4\pi c^2 t^2}\int_{S(t)} \varphi_1 dS$$

stands for the mean value of φ_1 over the sphere $S(t)$.

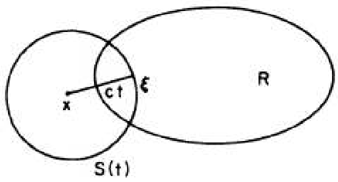

Fig. 7.4. Construction detail in Poisson's solution of the initial value problem. R is the region of the initial disturbance.

To furnish the other half of the initial condition we could specialize f in (7.49) to δ' and pursue the consequences similarly to the above. However, it is quicker to use a trick that is often useful: if ψ is a solution of a partial differential equation with constant coefficients, then so is any t or $\boldsymbol{x}$ derivative. In this case, we note that

$$\chi(\boldsymbol{x},t) = \frac{\partial \psi(\boldsymbol{x},t)}{\partial t}$$

is a solution of the wave equation, where $\psi(\boldsymbol{x},t)$ is given by (7.50). Moreover, as $t \to 0$, we see from (7.51) that

$$\chi = \psi_t \to 4\pi \Psi(\boldsymbol{x})\,,$$

$$\chi_t = \psi_{tt} = c^2\nabla^2\psi \to 0\,.$$

To give $\chi \to \varphi_0(\boldsymbol{x})$, $\chi_t \to 0$ as $t \to 0$, we must now choose $\Psi(\boldsymbol{x}) = \varphi_0(\boldsymbol{x})/4\pi$, and take

$$\chi(\boldsymbol{x},t) = \frac{\partial}{\partial t}\left\{\frac{1}{4\pi ct}\int_{S(t)} \varphi_0 dS\right\}.$$

The complete solution for general initial values therefore is

$$\begin{aligned}\varphi(\boldsymbol{x},t) &= \frac{\partial}{\partial t}\left\{\frac{1}{4\pi ct}\int_{S(t)} \varphi_0 dS\right\} + \frac{1}{4\pi ct}\int_{S(t)} \varphi_1 dS \\ &= \frac{\partial}{\partial t}\{ctM[\varphi_0]\} + ctM[\varphi_1]\end{aligned}\quad . \qquad (7.54)$$

Direct Verification of the Solution.

It remains to verify directly that (7.50) satisfies the wave equation. We have immediately that

$$\psi_{x_i x_i} = ct\int_0^{\pi}\int_0^{2\pi}\frac{\partial^2\Psi(\xi)}{\partial\xi_i^2}\sin\theta d\theta d\lambda$$
$$= \frac{1}{ct}\int_{S(t)}\frac{\partial^2\Psi}{\partial\xi_i^2}dS \quad ,$$

where $\xi = \boldsymbol{x} + ct\boldsymbol{l}$. The t derivatives require a little more manipulation. First,

$$\psi_t = \frac{\psi}{t} + c^2 t\int_0^{\pi}\int_0^{2\pi} l_i\frac{\partial\Psi}{\partial\xi_i}\sin\theta d\theta d\lambda$$
$$= \frac{\psi}{t} + \frac{1}{t}\int_{S(t)} l_i\frac{\partial\Psi}{\partial\xi_i}dS \quad ,$$
$$= \frac{\psi}{t} + \frac{1}{t}\int_{V(t)}\frac{\partial^2\Psi}{\partial\xi_i^{\,2}}dV$$

where $V(t)$ is the volume inside the sphere $S(t)$. Then

$$\psi_{tt} = -\frac{\psi}{t^2} + \frac{\psi_t}{t} - \frac{1}{t^2}\int_{V(t)}\frac{\partial^2\Psi}{\partial\xi_i^{\,2}}dV + \frac{c}{t}\int_{S(t)}\frac{\partial^2\Psi}{\partial\xi_i^{\,2}}dS \, ,$$

which reduces to

$$\psi_{tt} = \frac{c}{t}\int_{S(t)}\frac{\partial^2\Psi}{\partial\xi_i^{\,2}}dS$$

in view of the expression for ψ_t. We see that

$$\psi_{tt} = c^2\psi_{x_i x_i} \, ,$$

as required.

These arguments assume that Ψ is twice continuously differentiable. The solution (7.54) requires only that φ_0 and φ_1 are integrable in order for it to be meaningful. We might extend the meaning of solution to include all cases in which (7.54) is defined. In particular, if φ_0 and φ_1 are piecewise continuous, (7.54) is defined and $\varphi \to \varphi_0(\boldsymbol{x})$, $\varphi_t \to \varphi_1(\boldsymbol{x})$ at points of continuity.

For the balloon problem in Section 7.3, $\varphi_0 = 0$, $\varphi_1 = -P/\rho_0$ in an initial sphere. This is an example of piecewise continuous data. It is interesting to construct the solution using Poisson's integral not only for the spherically symmetric case but for an arbitrary shape of initial pressure region. This is left for the reader.

Wavefront.

If the nonzero values of $\varphi_0(\boldsymbol{x})$ and $\varphi_1(\boldsymbol{x})$ are confined to a finite region R as shown in Fig. 7.4, the solution at any point outside R is zero until the time when $S(t)$ first intersects points of R. It is clear that this occurs when ct is equal to the least distance from $\boldsymbol{x}$ to the boundary of R. This least distance is on the normal from R to $\boldsymbol{x}$. The *wavefront* at time t can be determined by turning this argument around. Construct all the normals to the boundary surface of R. Measure a

distance *ct* out along each normal. The surface formed by these points is the *wavefront*. Notice that where the surface of *R* is concave, the wavefront will be ***folded*** over itself after a while (see Fig. 7.5). This construction will be studied further in the discussion of geometrical optics.

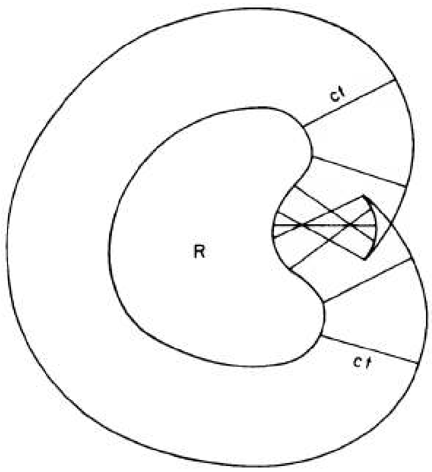

Fig. 7.5. Wavefront construction for a disturbance initially confined to the region *R*.

The disturbance at any point ***x*** outside *R* ceases when S(*t*) becomes so large that *R* is entirely within it. Thus in three dimensions an initial disturbance of finite extent produces disturbances which last only for a finite time. There is no "*tail*."