CHAPTER 9

The Propagation of Weak Shocks

9.1 The Nonlinearization Technique

A second extension of the techniques is required because the nonlinear propagation speed is often a function of the derivatives, φ_t and φ_s, rather than of φ itself. However, the procedure goes through. The expressions for φ_t and φ_s are each written in the form corresponding to (9.17) and the revised characteristics are determined from the appropriate expansion

$$\frac{dt}{ds} = \frac{1}{c} = \frac{1}{c_0} - \alpha_1 \varphi_t - \alpha_2 \varphi_s \quad \text{on} \quad \tau = const.$$

From (9.17) the corresponding first terms for φ_t, φ_s are

$$\varphi_t = \Phi(s) f'(\tau), \quad \varphi_s = -\frac{1}{c_0} \Phi(s) f'(\tau),$$

and the characteristic relation becomes

$$\frac{\partial t}{\partial s} = \frac{1}{c_0} - k f'(\tau) \Phi(s)$$

where

$$k = \alpha_1 - \alpha_2 / c_0$$

The characteristics are given by

$$t = \frac{s}{c_0} - k f'(\tau) \int_0^s \Phi(s') ds' + T(\tau). \qquad (9.19)$$

A specific example of this is provided by *spherical waves in gas dynamics*. The linear theory is the acoustic theory and φ_t, φ_s are related to the pressure and velocity perturbations. From (7.3)-(7.4) we have

$$\frac{p - p_0}{p_0} = -\frac{\gamma}{a_0^2} \varphi_t = \frac{\gamma F(t - r/a_0)}{r},$$

$$\frac{u}{a_0} = \frac{1}{a_0} \varphi_r = \frac{F(t - r/a_0)}{r} - \frac{f(t - r/a_0)}{a_0 r^2},$$

where

$$F(\tau) = -\frac{f'(\tau)}{a_0^2}.$$

The perturbation in the sound speed a will also be needed, and it is given by

$$\frac{a-a_0}{a_0}=\frac{\gamma-1}{2\gamma}\frac{p-p_0}{p_0}=\frac{\gamma-1}{2}\frac{F(t-r/a_0)}{r}$$

The nonlinearized solution is

$$\frac{p-p_0}{p_0}=\frac{\gamma F(\tau)}{r}, \tag{9.20}$$

$$\frac{a-a_0}{a_0}=\frac{\gamma-1}{2}\frac{F(\tau)}{r}, \tag{9.20}$$

$$\frac{u}{a_0}=\frac{F(\tau)}{r}-\frac{f(\tau)}{a_0 r^2}, \tag{9.21}$$

where $\tau(t,r)$ is to be determined from the improved characteristics. The *exact* characteristic equations were given in (6.135). The outgoing characteristics have velocity $a + u$. Therefore the first order correction to the characteristics requires

$$\frac{dp}{dt}\pm\rho a\frac{du}{dt}+2\frac{\rho a^2 u}{r}=0$$

$$on\ \_\ \frac{dr}{dt}=u\pm a\ \_(6.135)$$

$$\frac{dt}{dr}=\frac{1}{a+u}\approx\frac{1}{a_0}-\frac{a+u-a_0}{a_0^2}. \tag{9.22}$$

From (9.20) and (9.21), this means that

$$\frac{dt}{dr}=\frac{1}{a_0}-\frac{\gamma+1}{2a_0}\frac{F(\tau)}{r}+\frac{1}{a_0^2}\frac{f(\tau)}{r^2},$$

and integrating

$$t=\frac{r}{a_0}-\frac{\gamma+1}{2a_0}F(\tau)\log r-\frac{1}{a_0^2}\frac{f(\tau)}{r}+T(\tau). \tag{9.23}$$

[The relation between $F(\tau)$ and $f(\tau)$ is modified to $f'(\tau)=-a_0^2F(\tau)T'\tau)$ if $T'(\tau)\neq 1$.] Since our interest is in the region $a_0\tau/r<<1$, and the term $f(\tau)/r$ is always relatively small in this region, it is sufficient to use (the expression for u is corrected)

$$\frac{p-p_0}{p_0}=\frac{\gamma F(\tau)}{r},$$

$$\frac{a-a_0}{a_0}=\frac{\gamma-1}{2}\frac{F(\tau)}{r},$$

$$\frac{u}{a_0}=\frac{F(\tau)}{r},$$

$$t=\frac{r}{a_0}-\frac{\gamma+1}{2a_0}F(\tau)\log r+T(\tau). \tag{9.24}$$

This is a rather trivial example of retaining only the geometrical acoustics approximation to (9.21) and (9.23). Cylindrical and other waves in gas dynamics are handled similarly and the geometrical acoustics approximation provides a greater simplification similar to the step from (9.11) to (9.12).

When derivatives appear in the expression for c it is more convenient to take them as new dependent variables. Then, in all cases, the geometrical optics approximation leads to expressions for the dependent variables which are proportional to

$$\Phi(s)F(\tau)\,,$$

where $\Phi(s)$ is an amplitude function and $F(\tau)$ describes the wave profile. The corrected propagation speed, *using this approximation*, takes the form

$$c \approx c_0 + c_0^2 k\Phi(s)F(\tau)\,, \tag{9.25}$$

where the coefficient k is a constant determined by the particular relation of c to the dependent variables. The improved characteristics satisfy

$$\frac{\partial t}{\partial s} = \frac{1}{c_0} - k\Phi(s)F(\tau) \tag{9.26}$$

and are given by

$$t = \frac{s}{c_0} - kF(\tau)\int_0^s \Phi(s')ds' + T(\tau)\,. \tag{9.27}$$

Shock Determination.

Shocks, when required, are fitted in using the weak shock condition

$$U = \frac{1}{2}(c_1 + c_2)\,,$$

where U is the shock velocity and c_1 , c_2 now denote the propagation speeds on the two sides. In the present context, it is convenient to describe curves in the (s, t) plane by giving t as a function of s, so the shock condition is used in the form

$$\left(\frac{dt}{ds}\right)_{shock} = \frac{1}{2}\left\{\left(\frac{dt}{ds}\right)_{C_1} + \left(\frac{dt}{ds}\right)_{C_2}\right\}, \tag{9.28}$$

which is equivalent to first order in the deviation of velocities from c_0 . If the shock is specified by

$$t = \frac{s}{c_0} - G(s)\,,$$

we have

$$G'(s) = \frac{1}{2}k\{F(\tau_1) + F(\tau_2)\}\Phi(s)\,,$$

$$G(s) = kF(\tau_1)\int_0^s \Phi(s')ds' - T(\tau_1)\,,$$

$$G(s) = kF(\tau_2)\int_0^s \Phi(s')ds' - T(\tau_2)$$

We then deduce the typical "*equal area*" relation

$$\frac{1}{2}\{F(\tau_1) + F(\tau_2)\}\{T(\tau_2) - T(\tau_1)\} = \int_0^{\tau} F(\tau')dT(\tau') \quad (9.29)$$

For a head shock moving into the undisturbed region, the shock is determined by (9.27) with τ related to s by

$$\frac{1}{2}kF^2(\tau)\int_0^s \Phi(s')ds' = \int_0^{\tau} F(\tau')dT(\tau') . \quad (9.30)$$

As $s \to \infty$, the equation of the shock asymptotes to

$$t = \frac{s}{c_0} - K\left\{\int_0^s \Phi(s')ds'\right\}^{1/2} + T(\tau_0) \quad (9.31)$$

where

$$K = \left\{2k\int_0^{\tau_0} F(\tau)dT(\tau)\right\}^{1/2}, \quad F(\tau_0) = 0 . \quad (9.32)$$

At the shock the flow quantities are proportional to

$$K\Phi(s)\left\{\int_0^s \Phi(s')ds'\right\}^{-1/2} . \quad (9.33)$$

The typical asymptotic wave form is the *N wave*, with balanced shock waves, and between the shocks the linear decrease in time is proportional to

$$\Phi(s)\left\{\int_0^s \Phi(s')ds'\right\}^{-1} . \quad (9.34)$$

For spherical waves $\Phi(s) = 1/s$ and the shock strength (9.33) decays like $s^{-1}(\log s)^{-1/2}$, only slightly faster than the decay for linear pulses.

For cylindrical waves $\Phi = s^{-1/2}$ and the shock strength decays like $s^{-3/4}$.

Of course plane waves are also covered by these formulas; Φ is constant and the decay law is $s^{-1/2}$, in agreement with earlier results.

The asymptotic decay laws for cylindrical and spherical waves were obtained independently by various writers and the first was probably Landau (1945)[1].

For more general two and three dimensional waves in a uniform medium

$$\Phi(s) \propto A^{-1/2}(s),$$

where $A(s)$ is the ray tube area. Further details and applications may

[1] L. D. Landau, On shock waves at large distances from the place of their origin. *Soviet*

be found in an earlier paper (Whitham, 1956). For nonuniform media s/c_0 is replaced by $\int ds/c_0$ and all the dependence on s in (9.26) must be included in $\Phi(s)$.

9.3 Sonic Booms

The central problem for sonic booms is to determine the shocks produced by an axisymmetrical body in steady supersonic flight. The effects of different body shapes, acceleration, curved flight paths, and nonuniform atmosphere are all developed in various ways from this basic problem.

For the basic problem it is convenient to take a frame of reference in which the flow is steady. The linearized theory has been discussed in detail in Section 7.5 and the nonlinearization can now proceed in close analogy with the techniques developed here for unsteady waves. The corresponding problem of plane flow treated in Section 6.17 also contributes to the background of ideas.

If U is the mainstream velocity parallel to the x axis and the perturbed velocity components in the x and r directions are now denoted by $U(1+u)$ and Uv, we have

$$u = -\frac{1}{2\pi}\int_0^{x-Br}\frac{S''(\eta)d\eta}{\sqrt{(x-\eta)^2 - B^2r^2}} \tag{9.64}$$

$$v = \frac{1}{2\pi r}\int_0^{x-Br}\frac{S''(\eta)d\eta}{\sqrt{(x-\eta)^2 - B^2r^2}} \tag{9.65}$$

where $B = \sqrt{M^2 - 1}$ and $S(x)$ is the cross-sectional area at a distance x from the nose. The disturbance is confined behind the Mach cone $x - Br = 0$, which makes the Mach angle

$$\mu_0 = \sin^{-1}\frac{1}{M} \tag{9.66}$$

with the stream direction. The quantity $x - Br$ is the linear characteristic variable and corresponds to $t - r/c_0$ in the discussion of unsteady cylindrical waves. In the (x,r) diagram, the linear characteristics are a family of parallel straight lines making an angle μ_0 to the x axis as shown in Fig. 9.2. In the region

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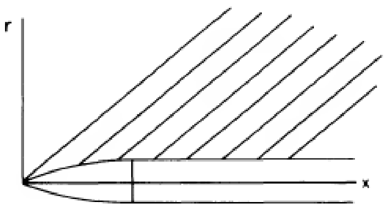

Fig. 9.2. Linear characteristic pattern in supersonic flow past a body.

$(x - Br)/Br << 1$, the approximations in (7.45)-(7.47) may be used. This region includes the front shock and the main part of the far field and it is here that the nonlinear corrections are crucial. The nonlinear effects modify the characteristics and introduce shocks as indicated in Fig. 9.3. Following the nonlinearization technique, we replace $x - Br$ by $\xi(x, r)$, where ξ is to be determined so that the curves $\xi = const.$ are an adequate approximation to the exact characteristics.

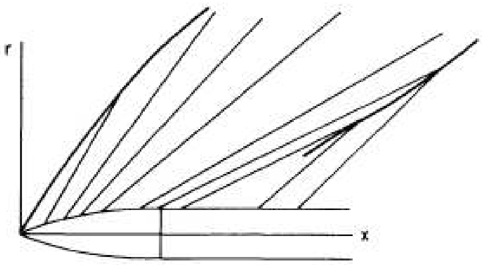

Fig. 9.3. Nonlinear characteristic pattern with shocks in supersonic flow past a body.

The modified expressions for the flow quantities are

$$u = -\frac{F(\xi)}{\sqrt{2Br}}, \qquad v = \frac{BF(\xi)}{\sqrt{2Br}}, \tag{9.67}$$

$$\frac{p - p_0}{p_0} = \gamma M^2 \frac{F(\xi)}{\sqrt{2Br}}, \quad \frac{a - a_0}{a_0} = \frac{\gamma - 1}{2} M^2 \frac{F(\xi)}{\sqrt{2Br}} \tag{9.68}$$

where

$$F(\xi) = \frac{1}{2\pi} \int_0^\xi \frac{S''(\eta)}{\sqrt{\xi - \eta}} d\eta \tag{9.69}$$

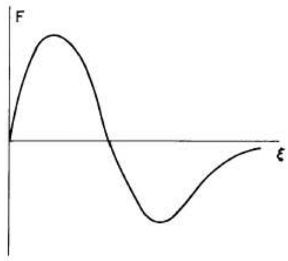

Fig. 7.3. Typical *F* curve for axisymmetric supersonic flow past a body.

A typical *F* curve was given in Fig. 7.3. As noted in (7.48), $F \to 0$ as $\xi \to 0$ and the linear theory with $\xi = x - Br$ does not predict a shock. The nonlinearization is clearly crucial here.

The exact equations for irrotational axisymmetric flow are the same as for plane flow, (6.158)-(6.159), with *y* replaced by *r* and an additional term $-a^2 v / r$ in (6.158). Since the highest derivatives are unaffected, the characteristic directions are still given by $\theta \pm \mu$, where θ is the flow direction and μ is the exact Mach angle defined by $\mu = \sin^{-1} \frac{a}{q}$. Accordingly, on $\xi = const.$,

$$\frac{dx}{dr} = \cot(\mu + \theta)\,.$$

Just as in the unsteady wave problems, the first order perturbation approximation to this is sufficient and we use

$$\frac{\partial x}{\partial r} = \cot\mu_0 - (\mu - \mu_0 + \theta)\operatorname{cosec}^2\mu_0$$

To the same order

$$\theta \approx v\,, \qquad \mu - \mu_0 \approx \frac{a_0}{U}\left(\frac{a - a_0}{a_0} - u\right)\sec\mu_0\,;.$$

hence

$$\frac{\partial x}{\partial r} = \frac{1}{B} - \frac{M^2}{B}\left(\frac{a - a_0}{a_0} - u\right) - M^2 v\,.$$

On substitution from (9.67) and (9.68), the equation becomes

$$\frac{\partial x}{\partial r} = \frac{1}{B} - \frac{(\gamma+1)M^4}{(2B)^{3/2}}\frac{F(\xi)}{r^{1/2}}\,,$$

and we have

$$x = Br - kF(\xi)r^{1/2} + \xi\,, \tag{9.70}$$

where

$$k = \frac{(\gamma+1)M^4}{2^{1/2}B^{3/2}}\,. \tag{9.71}$$

The nonlinearized solution in the region $\xi / Br << 1$ is given by (9.67), (9.68), and (9.70).

$$\frac{\partial x}{\partial r} = \frac{1}{B} - \frac{M^2}{B}\left(\frac{a - a_0}{a_0} - u\right) - M^2 v$$
$$= \frac{1}{B} - \frac{M^2}{B}\left(\frac{\gamma-1}{2}M^2\frac{F(\xi)}{\sqrt{2Br}} + \frac{F(\xi)}{\sqrt{2Br}}\right) - M^2\frac{BF(\xi)}{\sqrt{2Br}}$$
$$= \frac{1}{B} - \left(\frac{\gamma-1}{2B}M^4\frac{F(\xi)}{\sqrt{2Br}} + \right) - \frac{M^2}{B}\frac{F(\xi)}{\sqrt{2Br}} - M^2\frac{BF(\xi)}{\sqrt{2Br}}$$
$$= \frac{1}{B} - \left(\frac{\gamma-1}{2B}M^4\frac{F(\xi)}{\sqrt{2Br}} + \right) - M^2\frac{F(\xi)}{\sqrt{2Br}}\left(\frac{1}{B} + B\right)$$
$$= \frac{1}{B} - \left(\frac{\gamma-1}{2B}M^4\frac{F(\xi)}{\sqrt{2Br}} + \right) - M^2\frac{F(\xi)}{\sqrt{2Br}}\left(\frac{1+B^2}{B}\right)$$
$$= \frac{1}{B} - \left(\frac{\gamma-1}{2B}M^4\frac{F(\xi)}{\sqrt{2Br}} + \right) - M^2\frac{F(\xi)}{\sqrt{2Br}}\left(\frac{M^2}{B}\right)$$
$$= \frac{1}{B} - M^4\frac{F(\xi)}{\sqrt{2Br}}\left(\frac{\gamma-1}{2B} + \frac{1}{B}\right)$$
$$= \frac{1}{B} - M^4\frac{F(\xi)}{\sqrt{2Br}}\left(\frac{\gamma+1}{2B}\right)$$
$$= \frac{1}{B} - \frac{\gamma+1}{(2B)^{3/2}}M^4\frac{F(\xi)}{r^{1/2}}$$

The Shocks.

The characteristics overlap and a shock is required when $F'(\xi) > 0$. For a finite body,

$$\int_0^\infty F(\xi)\,d\xi = 0$$

as noted earlier, so there will be two such regions in general and two shocks. The counterpart to (9.28) is that the shock slope is the mean of the characteristic slopes on the two sides, and the shock determination is completely analogous. If a shock is described by

$$x = Br - G(r)\,,$$

we have

$$G(r) = \begin{cases} kF(\xi_1)r^{1/2} - \xi_1 \\ kF(\xi_2)r^{1/2} - \xi_2 \end{cases},$$

where

$$\frac{1}{2}\{F(\xi_1)+F(\xi_2)\}(\xi_2-\xi_1)=\int_{\xi_1}^{\xi_2}F(\xi)d\xi .$$

For the front shock which has undisturbed flow ahead of it, $F(\xi_1)=0$ and ξ_1 may be eliminated from the determination. Then, dropping the subscript on ξ_2, we have

$$\frac{1}{2}kF^2(\xi)r^{1/2}=\int_0^{\xi}F(\xi')d\xi' \tag{9.72}$$

$$x=Br-kF(\xi)r^{1/2}+\xi \tag{9.73}$$

for the determination of the shock. The flow quantities immediately behind the shock are given by (9.67) and (9.68) with $\xi(r)$ determined from (9.72).

$$\begin{aligned}F(\xi)&=\frac{1}{2\pi}\int_0^{\xi}\frac{S''(\eta)}{\sqrt{\xi-\eta}}d\eta\\&=\frac{1}{2\pi}\int_0^{\xi}\frac{2\pi\varepsilon^2}{\sqrt{\xi-\eta}}d\eta\\&=\varepsilon^2\int_0^{\xi}\frac{1}{\sqrt{\xi-\eta}}d\eta\\&=-\varepsilon^2\int_{\xi}^{0}\frac{dx}{\sqrt{x}}\\&=-\varepsilon^2\left[2\sqrt{x}\right]_{\xi}^{0}\\&=2\varepsilon^2\xi^{1/2}\end{aligned}$$

Flow Past a Slender Cone.

For a cone with semiangle ε, $S(x)=\pi\varepsilon^2x^2$ and the F function (9.69) is

$$F(\xi)=2\varepsilon^2\xi^{1/2} .$$

In that case the relation (9.72) between ξ and r for points on the shock is

$$\xi^{1/2}=\frac{3}{2}k\varepsilon^2r^{1/2}$$

$$\frac{1}{2}kF^2(\xi)r^{1/2}=\int_0^{\xi}F(\xi')d\xi'$$

$$\frac{1}{2}k\cdot 4\varepsilon^4\xi r^{1/2}=\int_0^{\xi}2\varepsilon^2\xi'^{1/2}\,d\xi'=2\varepsilon^2\cdot\frac{2}{3}\xi^{3/2}$$

$$\frac{3}{2}k\varepsilon^2r^{1/2}=\xi^{1/2}$$

The shock equation (9.73) reduces to

$$x=Br-\frac{3}{4}k^2\varepsilon^4r .$$

$$\begin{aligned}x&=Br-kF(\xi)r^{1/2}+\xi\\&=Br-k\cdot 2\varepsilon^2\xi^{1/2}r^{1/2}+\frac{9}{4}k^2\varepsilon^4r\\&=Br-k\cdot 2\varepsilon^2\cdot\frac{3}{2}k\varepsilon^2r^{1/2}\cdot r^{1/2}+\frac{9}{4}k^2\varepsilon^4r\\&=Br-\frac{3}{4}k^2\varepsilon^4r\end{aligned}$$

This corresponds to a conical shock with a semiangle of

$$\mu_0+\frac{3}{8}\frac{(\gamma+1)^2M^6}{(M^2-1)^{3/2}}\varepsilon^4 \tag{9.74}$$

The shock strength obtained from (9.68) is

$$\frac{p-p_0}{p_0}=\frac{3\gamma(\gamma+1)M^6}{2(M^2-1)}\varepsilon^4 . \tag{9.75}$$

For a cone, dimensional arguments show that the ***exact*** solution is a similarity solution with the flow quantities functions of r/x. The exact nonlinear equations may then be reduced to ordinary differential equations and integrated numerically. This is the famous *Taylor-Maccoll solution* (1933)[2], which was a **landmark** in the development of the theory of supersonic flow. The results (9.74) and (9.75) were deduced for slender cones within the similarity theory by

[2] G. I. Taylor and J. W. Maccoll, The air pressure on a cone moving at high speeds. *Proc. Roy. Soc. A* **253**, 296-312.

Lighthill (1948)[3]. It provides a valuable check on the more general approach for slender bodies. Numerical results show that (9.74)-(9.75) are very good approximations for cones up to 10° semiangle over a range of Mach numbers from about 1.1 to 3.0.

For general slender bodies, these formulas give the initial behavior of the shock. It should be noted that whereas the disturbances near the body are $O(\varepsilon^2)$, the shock strength is $O(\varepsilon^4)$. This explains in a sense why it is missed in the linear theory.

Behavior at Large Distance for Finite Bodies.

According to (9.72), for points on the shock $\xi \to \xi_0$ as $r \to \infty$, where $F(\xi_0) = 0$. Then (9.72) is asymptotically

$$F(\xi) \approx \left[\frac{2}{k}\int_0^{\xi_0} F(\xi')d\xi'\right]^{1/2} r^{-1/4}. \tag{9.76}$$

The shock is asymptotic to (sign before ξ_0 should be plus)

$$x \approx Br - \left[2k\int_0^{\xi_0} F(\xi')d\xi'\right]^{1/2} r^{1/4} + \xi_0 \tag{9.77}$$

and the shock strength is

$$\begin{aligned}\frac{p-p_0}{p_0} &\approx \frac{\gamma M^2}{(2B)^{1/2}}\left[\frac{2}{k}\int_0^{\xi_0} F(\xi)d\xi\right]^{1/2} r^{-3/4} \\ &= \frac{2^{1/4}\gamma}{(\gamma+1)^{1/2}}(M^2-1)^{1/8}\left[\int_0^{\xi_0} F(\xi)d\xi\right]^{1/2} r^{-3/4}\end{aligned} \tag{9.78}$$

This is the most important formula for sonic boom work. It shows that the boom at the ground depends very weakly on the Mach number, depends on distance like $r^{-3/4}$, and depends on the body shape through the factor

$$K = \left[\int_0^{\xi_0} F(\xi)d\xi\right]^{1/2}. \tag{9.79}$$

If the length of the body is l and the ratio of maximum diameter to length is the thickness ratio δ, the shape factor $K \propto \delta l^{3/4}$. For a body shape

$$R(x) = \begin{cases} \delta l\left\{1-\left(1-\dfrac{x}{l}\right)^3\right\} \_ 0 \le x \le l \\ \delta l \_ l \le x \end{cases},$$

we find $K = 1.04\delta l^{3/4}$.

The asymptotic wave profile is a balanced *N wave*. Between the

[3] M. J. Lighthill, The position of the shock wave in certain aerodynamic problems. *Q. J. Mech. Appl. Math.* **1**, 309-318.

shocks $\xi \approx \xi_0$, $F(\xi) \approx 0$, so that from (9.70)

$$F(\xi) \approx \frac{Br - x + \xi_0}{kr^{1/2}},$$

and from (9.68) and (9.71) the pressure ratio is

$$\frac{p - p_0}{p_0} \approx \frac{\gamma}{\gamma + 1} \frac{(M^2 - 1)^{1/2}}{M^2} \frac{Br - x + \xi_0}{r}. \tag{9.80}$$

The flow behind the rear shock is not completely undisturbed but is of smaller order than the disturbance in the N wave. For this and other details reference may be made to the original account (Whitham, 1952[4]).

Extensions of the Thoery.

Axisymmetric bodies might seem a far cry from real aircraft, but it is known that the far flow field in any direction away from a finite body can be represented as the flow due to an equivalent body of revolution. That is, in any direction the expressions (9.67)-(9.69) apply but the F function will be different for different directions. In the linear theory from which one starts, the contributions from fuselage, wings, lift distribution, and the like can be treated separately and superposed to give the final F function for each direction. The nonlinear expressions then apply with this F function. The volume contribution is related to a distribution of cross-sectional area $S(x)$, where the cuts are made by planes at an angle to the stream in accordance with the supersonic area rule. For details of the method and the nonlinear results, see Whitham (1956)[5]. When various protuberances such as wings are included $S'(x)$ becomes discontinuous and (9.69) must be modified appropriately (Whitham, 1952).

The effects of the lift distribution are of equal importance with the volume effects. In the linear theory the lift distribution $L(x)$ provides a contribution

$$\varPhi_1 = -\frac{1}{2\pi\rho_0 U} \frac{\cos\tilde{\omega}}{r} \int_0^{x-Br} \frac{(x-\eta)L(\eta)}{\sqrt{(x-\eta)^2 - B^2 r^2}} d\eta \tag{9.81}$$

to the velocity potential, where $\tilde{\omega}$ is the angle of a meridian plane through the flight path and is measured from the downward vertical. This may be approximated for $(x - Br)/Br << 1$ as before and the

[4] G. B. Whitham, The flow pattern of a supersonic projectile. *Comm. Pure Appl. Math.* **5**, 301-348.
[5] G. B. Whitham, On the propagation of weak shock waves. *J. Fluid Mech.* **1**, 290-318.

perturbations are again given by (9.67)-(9.68) with

$$F(\xi)=\frac{1}{2\pi}\frac{B\cos\tilde{\omega}}{\rho_0 U^2}\int_0^{\xi}\frac{L'(\eta)}{\sqrt{\xi-\eta}}d\eta \tag{9.82}$$

This is an interesting illustration of the "*equivalent body*" concept for asymmetric distributions. It should be noted that the approximations (9.67)-(9.68) are valid for $\xi / Br << 1$ and they are sufficient to determine the shocks. However, the pressure distribution behind the main *N wave* makes important contributions to the total lift transmitted to the ground. The full form (9.81) and, when necessary, its nonlinearization are required for a detailed accounting of the lift. This has sometimes caused confusion in the literature where it has been remarked that the pressure distribution given by (9.68), when integrated over the ground, does not give the total lift $\int_0^{\infty} L(x)dx$. The expression (9.68) applies only in the region of the main *N wave*. The full formula derived from (9.81) integrates correctly to give the total lift.

The remaining extensions to accelerating bodies and nonuniform atmospheres, the latter being always important in the real situation, can be handled analytically to some extent and the theory leans heavily on geometrical acoustics (see Friedman, Kane, and Signalla, 1963, and references given there). Further developments and comparisons with wind tunnel and observational data are reviewed in a series of papers published in the *Journal of the Acoustical Society of America* (1965). Similar comparisons have been made by various government laboratories and aircraft companies. (A popular account for laymen which contains some interesting checks of the theory with reality is presented in Boeing Document D6A10598-1). The conclusion seems to be that the theory provides good results and valuable insight in an extremely complicated practical problem.