CHAPTER VI
THE SOLUBLE PROBLEMS OF RIGID DYNAMICS

65. The motion of systems with one degree of freedom; motion round a fixed axis, etc.

We now proceed to apply the principles which have been developed in the foregoing chapters in order to determine the motion of holonomic systems of rigid bodies in those cases which admit of solution by quadratures.

It is natural to consider first those systems which have only one degree of freedom. We have seen (§42) that such a system is immediately soluble by quadratures when it possesses an integral of energy: and this principle is sufficient for the integration in most cases. Sometimes, however (e.g. when we are dealing with systems in which one of the surfaces or curves of constraint is forced to move in a given manner), the problem as originally formulated does not possess an integral of energy, but can be reduced (e.g. by the theorem of §29) to another problem for which the integral of energy holds; when this reduction has been performed, the problem can be integrated as before.

The following examples will illustrate the application of these principles.

(i) Motion of a rigid body round a fixed axis.

Consider the motion of a single rigid body which is free to turn about an axis, fixed in the body and in space. Let I be the moment of inertia of the body about the axis, so that its kinetic energy is $(1/2)I\dot{\theta}^2$, where θ is the angle made by a moveable plane, passing through the axis and fixed in the body, with a plane passing through the axis and fixed in space. Let Θ be the moment round the axis of all the external forces acting on the body, so that $\Theta\delta\theta$ is the work done by these forces in the infinitesimal displacement which changes θ to $\theta+\delta\theta$. The Lagrangian equation of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right)-\frac{\partial T}{\partial\theta}=\Theta$$

then gives

$$I\ddot{\theta}=\Theta\,,$$

which is a differential equation of the second order for the determination of θ.

If the forces are conservative, and $V(\theta)$ denotes the potential energy, this equation becomes

$$I\ddot{\theta}=\Theta\,,$$

which on integration gives the equation of energy

$$\frac{1}{2}I\dot{\theta}^2+V(\theta)=c\,,\quad c:const.$$

Integrating again, we have

$$t = I^{1/2}\int \{2(c-V)\}^{-1/2}\, d\theta + const.$$

and this relation between θ and t determines the motion, the two constants of integration being determined by the initial conditions.

The most important case is that in which gravity is the only external force, and the axis is horizontal. In this case let G be the centre of gravity of the body, C the foot of the perpendicular drawn from G to the axis, and let $CG = h$. The potential energy is $-Mgh\cos\theta$, where M is the mass of the body and θ is the angle made by CG with the downward vertical: and the equation of motion is

$$\ddot{\theta} + \frac{Mgh}{I}\sin\theta = 0\,.$$

This is the same as the equation of motion of a *simple pendulum* of length I/Mh, and the motion can therefore be expressed in terms of elliptic functions as in §44, the solution being of the form

$$\sin\frac{\theta}{2} = k\,\mathrm{sn}\left\{\left(\frac{Mgh}{I}\right)^{1/2}(t-t_0), k\right\}$$

in the *oscillatory case*, and of the form

$$\sin\frac{\theta}{2} = \mathrm{sn}\left\{\frac{1}{k}\left(\frac{Mgh}{I}\right)^{1/2}(t-t_0), k\right\}$$

in the *circulating case*. The quantity I/Mh is called the *length of the equivalent simple pendulum*.

If O be a point on the line CG such that $OC=I/Mh$, the points O and C are called respectively the *centre of oscillation* and the *centre of suspension*. A curious result in this connexion is that the centre of oscillation and the centre of suspension are ***convertible***, i.e., if O is the centre of oscillation when C is the centre of suspension, then C will be the centre of oscillation when O is the centre of suspension. To prove this result, we have by §59

$$\text{Moment\_of\_inertia\_of\_the\_body\_about\_}O = \text{Moment\_of\_inertia\_about\_}G + M\cdot GO^2$$
$$= I - M\cdot CG^2 + M\cdot GO^2$$

and therefore we have

$$\frac{\text{Moment\_of\_inertia\_of\_body\_about\_}O}{\text{Distance\_of\_centre\_of\_gravity\_from\_}O} = \frac{I - Mh^2 + M(I/Mh-h)^2}{I/Mh-h}$$
$$= Mh + M(I/Mh-h)$$
$$= I/h$$

If therefore the body were suspended from O, the equation of motion would still be

$$\ddot{\theta} + \frac{Mgh}{I}\sin\theta = 0\,,$$

which establishes the result. It is evident that the period of oscillation would be the same about either of the points C and O.

(ii) Motion of a rod on which an insect is crawling.

We shall next study the motion of a straight uniform rod, of mass m .and length $2a$, whose extremities can slide on the circumference of a smooth fixed horizontal circle of radius c; an insect of mass equal to that of the rod is supposed to crawl along the rod at a constant rate v relative to the rod.

Let θ be the angle made by the rod at time t with some fixed direction, and let x be the distance traversed by the insect from the middle point of the rod. The kinetic energy of the rod is $(1/2)m\{c^2-(2/3)a^2\}\dot{\theta}^2$, and the kinetic energy of the insect is due to a component of velocity $\{\dot{x}-(c^2-a^2)^{1/2}\dot{\theta}\}$ along the rod and a component of velocity $x\dot{\theta}$ perpendicular to the rod, so the total kinetic energy of the system is

$$T=\frac{1}{2}m\left(c^2-\frac{2a^2}{3}\right)\dot{\theta}^2+\frac{1}{2}m\{\dot{x}-(c^2-a^2)^{1/2}\dot{\theta}\}^2+\frac{1}{2}mx^2\dot{\theta}^2:$$

there is no potential energy.

Since $x=vt$, (t being measured from the epoch when x is zero), we have

$$T=\frac{1}{2}m\left(c^2-\frac{2a^2}{3}\right)\dot{\theta}^2+\frac{1}{2}m\{v-(c^2-a^2)^{1/2}\dot{\theta}\}^2+\frac{1}{2}mv^2t^2\dot{\theta}^2.$$

The coordinate θ, which is now the only coordinate, is ignorable, and we have therefore

$$\frac{\partial T}{\partial\dot{\theta}}=const.$$

or

$$m\left(c^2-\frac{2a^2}{3}\right)\dot{\theta}^2-m(c^2-a^2)^{1/2}\{v-(c^2-a^2)^{1/2}\dot{\theta}\}^2+mv^2t^2\dot{\theta}^2=const.$$

or

$$\dot{\theta}(2c^2-\frac{5}{3}a^2+v^2t^2)=const.$$

Integrating this equation, we have

$$\theta-\theta_0=k\arctan\left\{vt(2c^2-\frac{5}{3}a^2)^{-1/2}\right\},$$

where θ_0 and k are constants. This formula determines the position of the rod at any time.

(iii) Motion of a cone on a perfectly rough inclined plane.

(iv) Motion of a rod on a rotating frame.

(v) Motion of a disc, one of whose points is forced to move in a given manner.

(vi) Motion of a disc rolling on a constrained disc and linked to it.

66. The motion of systems with two degrees of freedom.

In the dynamics of rigid bodies, as in the dynamics of a particle, the possibility of solving by quadratures a problem with two degrees of freedom generally depends on the presence of an ignorable coordinate. The integral corresponding to the ignorable coordinate can often be interpreted physically as an integral of *momentum* or *angular momentum*. The formation and solution of the differential equations is effected by application of the principles developed in the preceding chapters : this will be shewn by the following illustrative examples.

(i) Rod passing through ring.

(ii) 0ne cylinder rolling on another under gravity.

(iii) Rod moving in a free circular frame.

(iv) Hoop and ring.

67. Initial motions.

We have already explained in §32 the general principles used in finding the initial character of the motion of a system which starts from rest at a given time. The following examples will serve to illustrate the procedure for systems of rigid bodies.

(i) A particle hangs by a string of length *b* from a point in the circumference of a disc of twice its mass and of radius *a*. The disc can turn about its axis, which is horizontal, and the diameter through the point of attachment of the string is initially horizontal. To find the initial path of the particle.

(ii) A ring of mass *m* can slide freely on a uniform rod of mass *M* and length 2*a*, which can turn about one end. Initially the rod is horizontal, with the ring at a distance *ra* from the fixed end. To find the initial curvature of the path of the ring in space.

68. The motion of systems with three degrees of freedom.

The possibility of solving by quadratures the motion of a system of rigid bodies which has three degrees of freedom depends generally (as in the case of systems with two degrees of freedom) either on the occurrence of ignorable coordinates, giving rise to integrals of *momentum* and *angular momentum*, or on a disjunction of the *kinetic potential* into parts which depend on the coordinates separately. The following examples illustrate the procedure.

(i) Motion of a rod in a given field of force.

(ii) Motion of a rod and cylinder on a plane.

69. Motion of a body about a fixed point under no forces.

One of the most important problems in the dynamics of systems with three degrees of freedom is that of determining the motion of a

rigid body, one of whose points is fixed, when no external forces are supposed to act. This problem is realised (§64) in the motion of a rigid body relative to its centre of gravity, under the action of any forces whose resultant passes through the centre of gravity.

In this system the *angular momentum* of the body about every line which passes through the fixed point and is fixed in space is constant (§40), and consequently the line through the fixed point for which this angular momentum has its greatest value is fixed in space. Let this line, which is called the ***invariable line***, be taken as axis *OZ*, and let *OX* and *Y* be two other axes through the fixed point which are perpendicular to *OZ* and to each other. The angular momenta about the axes *OX* and *OY* are **zero**, for if this were not the case the resultant of the angular momenta about *OX*, *OY*, *OZ* would give a line about which the angular momentum would be greater than the angular momentum about *OZ*, which is contrary to hypothesis. It follows (§39) that the angular momentum about any line through making an angle θ with *OZ* is $d\cos\theta$, where *d* denotes the angular momentum about *OZ*.

The position of the body at any time *t* is sufficiently specified by the knowledge of the positions at that time of its **three principal axes of inertia** at the fixed point : let these lines be taken as moving axes *Oxyz*; let (θ,ϕ,ψ) denote the three **Eulerian angles** which specify the position of the axes *Oxyz* with reference to the axes *OXYZ*, let (*A*, *B*, *C*) be the **principal moments of inertia** of the body at *O*, supposed arranged in descending order of magnitude, and let $(\omega_1,\omega_2,\omega_3)$ be the three components of angular velocity of the system about the axes *Ox*, *Oy*, *Oz* respectively, so that (§§10, 62)

$$\begin{cases} A\omega_1 = -d\sin\theta\cos\psi \\ B\omega_2 = d\sin\theta\sin\psi \\ C\omega_1 = d\cos\theta \end{cases},$$

or (§16)

$$\begin{cases} \dot\theta\sin\psi - \dot\phi\sin\theta\cos\psi = -\dfrac{d}{A}\sin\theta\cos\psi \\ \dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi = \dfrac{d}{B}\sin\theta\sin\psi \\ \dot\psi + \dot\phi\cos\theta = \dfrac{d}{C}\cos\theta \end{cases}.$$

These are really **three integrals** of the differential equations of motion of the system (only one arbitrary constant however occurs, namely *d*, our special set of axes being such as to make the other two constants of integration zero); we can therefore take these instead of the usual Lagrangian differential equations of motion in order to determine θ,ϕ,ψ .

Solving for $\dot\theta,\dot\phi,\dot\psi$, we have

$$\begin{cases} \dot{\theta} = \dfrac{(A-B)d}{AB}\sin\theta\cos\psi\sin\psi \\ \dot{\phi} = \dfrac{d}{A}\cos^2\psi + \dfrac{d}{A}\sin^2\psi \\ \dot{\psi} = \left(\dfrac{d}{C} - \dfrac{d}{A}\cos^2\psi - \dfrac{d}{B}\sin^2\psi\right)\cos\theta \end{cases}.$$

The integral of energy (which is a consequence of these three equations) may be written down at once by use of §63; it is

$$A\omega_1{}^2 + B\omega_2{}^2 + C\omega_3{}^2 = c,$$

where c is a constant; replacing $\omega_1, \omega_2, \omega_3$ by their values in terms of θ and ψ, this equation can be written in either of the forms

$$\frac{A-B}{AB}\sin^2\theta\cos^2\psi = -\frac{Bc-d^2}{Bd^2} + \frac{B-C}{BC}\cos^2\theta,$$

or

$$\frac{A-B}{AB}\sin^2\theta\cos^2\psi = \frac{Ac-d^2}{Ad^2} - \frac{A-C}{AC}\cos^2\theta.$$

Since $A>B>C$, the quantity $(cA-d^2)$ or $B(A-B)\omega_2{}^2 + C(A-C)\omega_3{}^2$ is positive, and $(cC-d^2)$ is negative: the quantity $(Bc-d^2)$ may be either positive or negative: for definiteness we shall suppose it to be positive.

The first of the three differential equations may, by use of the last equations, be written

$$\frac{d}{dt}\cos\theta = -d\left\{-\frac{Bc-d^2}{Bd^2} + \frac{B-C}{BC}\cos^2\theta\right\}^{1/2}\left\{\frac{Ac-d^2}{Ad^2} - \frac{A-C}{AC}\cos^2\theta\right\}^{1/2}$$

This equation shews that $\cos\theta$ is a ***Jacobian elliptic function*** of a linear function of t; and the two preceding equations shew that $\sin\theta\cos\psi$ and $\sin\theta\sin\psi$ are the other two Jacobian functions.

We therefore write

$$\sin\theta\cos\psi = P\,\mathrm{cn}\,u,\quad \sin\theta\sin\psi = Q\,\mathrm{sn}\,u,\quad \cos\theta = R\,\mathrm{dn}\,u$$

where P, Q, R are constants and u is a linear function of t, say $\lambda t + \varepsilon$; the quantities P, Q, R, λ, and the modulus k of the elliptic functions, are then to be chosen so as to make the above equations coincide with the equations

$$\begin{cases} k^2\,\mathrm{cn}^2 u = -k'^2 + \mathrm{dn}^2 u \\ k^2\,\mathrm{sn}^2 u = 1 - \mathrm{dn}^2 u \\ \dfrac{d}{du}\mathrm{dn}\,u = -k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u \end{cases}.$$

The comparison gives

$$P^2 = \frac{A(d^2 - cC)}{d^2(A-C)}, \quad Q^2 = \frac{B(d^2 - cC)}{d^2(B-C)}, \quad R^2 = \frac{C(cA - d^2)}{d^2(A-C)},$$

$$k^2 = \frac{(A-B)(d^2 - cC)}{(B-C)(Ac - d^2)}, \quad \lambda^2 = \frac{(B-C)(cA - d^2)}{ABC}.$$

The equation for k^2 shews that k is real, and the equation

$$1 - k^2 = \frac{(A-C)(Bc - d^2)}{(B-C)(Ac - d^2)}$$

shews that $(1-k^2)$ is positive, i.e., that k<1. The quantities P, Q, R, λ are also evidently real.

Now a real quantity a may be defined by the mutually consistent equations

$$\operatorname{sn} ia = i\left\{\frac{C(Ac - d^2)}{A(d^2 - cC)}\right\}^{1/2}, \quad \operatorname{cn} ia = \left\{\frac{d^2(A-C)}{A(d^2 - cC)}\right\}^{1/2},$$

$$\operatorname{dn} ia = \left\{\frac{B(A-C)}{A(B-C)}\right\}^{1/2}.$$

Since

$$(k')^{-1/2} \operatorname{dn} ia = \frac{\vartheta_{00}(ia/2K)}{\vartheta_{01}(ia/2K)},$$

where the ***theta-functions*** are defined by the expansions

$$\vartheta_{00}(\nu) = 1 + 2q\cos 2\pi\nu + 2q^4 \cos 4\pi\nu + 2q^9 \cos 6\pi\nu + \cdots,$$

$$\vartheta_{01}(\nu) = 1 - 2q\cos 2\pi\nu + 2q^4 \cos 4\pi\nu - 2q^9 \cos 6\pi\nu + \cdots,$$

$$\vartheta_{10}(\nu) = 2q^{1/4} \cos \pi\nu + 2q^{9/4} \cos 3\pi\nu + 2q^{25/4} \cos 5\pi\nu + \cdots,$$

$$\vartheta_{11}(\nu) = 2q^{1/4} \sin \pi\nu - 2q^{9/4} \sin 3\pi\nu + 2q^{25/4} \sin 5\pi\nu + \cdots,$$

and $q = e^{-\pi K'/K}$, we have

$$\frac{1 + 2q\cosh 2\gamma + 2q^4 \cosh 4\gamma + \cdots}{1 - 2q\cosh 2\gamma + 2q^4 \cosh 4\gamma - \cdots} = (k')^{-1/2}\left\{\frac{B(A-C)}{A(B-C)}\right\}^{1/2},$$

where γ stands for $\pi a/2K$; from this equation γ (and consequently a) may readily be determined by successive approximation.
(p.147)

(p.152)
70. Poinsot's kinematical representation of the motion; the polhode and herpolhode.

An elegant method of representing kinematically the motion of a

body about a fixed point under no forces is the following, which is due to Poinsot.

The equation of the ***momental ellipsoid*** of the body at the fixed point, referred to the moving axes *Oxyz*, is

$$Ax^2 + By^2 + Cz^2 = 1 .$$

Consider the tangent-plane to the ellipsoid which is perpendicular to the **invariable line**. If p denotes the perpendicular on this tangent-plane from the origin, we have (since the direction-cosines of p are $A\omega_1/d, B\omega_2/d, C\omega_3/d$)

$$\begin{aligned} p^2 &= \frac{A\omega_1^2 + B\omega_2^2 + C\omega_3^2}{A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2}, \quad (c/d^2 : const.) \\ &= \frac{c}{d^2} \end{aligned}$$

Since the perpendicular on the plane is constant in magnitude and direction, the plane is fixed in space: so the momental ellipsoid always touches a fixed plane.

Moreover, if (x', y', z') are the coordinates of the point of contact of the ellipsoid and the plane, we have on identifying the equations $Axx' + Byy' + Czz' = 1$ and $A\omega_1 x + B\omega_2 y + C\omega_3 z = pd$ the values

$$x' = \frac{\omega_1}{pd} = \frac{\omega_1}{\sqrt{c}}, \quad y' = \frac{\omega_2}{pd} = \frac{\omega_2}{\sqrt{c}}, \quad z' = \frac{\omega_3}{pd} = \frac{\omega_3}{\sqrt{c}},$$

and hence the radius vector to the point (x', y', z') is the instantaneous axis of rotation of the body. It follows that the **body moves as if it were rigidly connected to its momental ellipsoid**, and the latter body were to roll about the fixed point on a fixed plane perpendicular to the invariable line, without sliding; the angular velocity being proportional to the radius to the point of contact, so that the component of angular velocity about the invariable line is constant.

Example 1. If a body which is moveable about a fixed point is initially at rest and then is acted on continually by a couple of constant magnitude and orientation, shew that Poinsot's construction still holds good, but that the component angular velocity about the invariable line is no longer constant but varies directly as the time. (Coll. Exam.)

For in any interval of time *dt* the addition of angular momentum to the body is *Ndt* about the fixed axis *OZ* of the couple; so that the resultant angular momentum of the system at time *t* is *Nt* about *OZ*. Now the components of angular momentum about the principal axes of inertia *Oxyz* are $A\omega_1, B\omega_2, C\omega_3$, where *A*, *B*, *C* are the principal

moments of inertia and $(\omega_1, \omega_2, \omega_3)$ are the components of angular velocity; hence we have

$$A\omega_1 = -Nt\sin\theta\cos\psi\ , \quad B\omega_2 = Nt\sin\theta\sin\psi\ , \quad C\omega_3 = Nt\cos\theta\ ,$$

where θ, ϕ, ψ are the Eulerian angles which fix the position of the axes *Oxyz* with reference to fixed axes *OXYZ*. But these equations differ from those which occur in the motion of a body under no forces only in the substitution of *tdt* for *dt*; so the motion will be the same as in the problem of motion under no forces, except that the velocities are multiplied by *t;* whence the result follows.

Example 2. In the motion of a body, one of whose points is fixed, under no forces, let a hyperboloid be rigidly connected with the body, so as to have the principal axes of inertia of the body at the point as axes, and to have the squares of its axes respectively proportional to $d^2 - Ac$, $d^2 - Bc$, $d^2 - Cc$, where *A*, *B*, *C* are the moments of inertia of the body at the fixed point, *c* is twice its kinetic energy, and *d* is the resultant angular momentum. Shew that the motion of this hyperboloid can be represented by causing it to roll without sliding on a circular cylinder, whose axis passes through the fixed point and is parallel to the axis of resultant angular momentum.

The curve which in Poinsot's construction is traced on the momental ellipsoid by the point of contact with the fixed plane is called the ***polhode***. Its equations, referred to the principal moments of inertia, are clearly the equation of the ellipsoid together with the equation p = constant, i.e., they are

$$Ax^2 + By^2 + Cz^2 = 1\ ,$$

$$A^2x^2 + B^2y^2 + C^2z^2 = d^2/c\ .$$

Example 1. Shew that when $A = B$, the polhode is a circle.
Example 2. Taking $A \geq B \geq C$, shew that there are two kinds of polhodes, one kind consisting of curves which surround the axis *Oz* of the momental ellipsoid, and correspond to $cB > d^2 > cC$, while the other kind consists of curves which surround the axis *Ox*, and correspond to $cA > d^2 > cB$; and that the limiting case between these two kinds of polhodes is a singular polhode which corresponds to $cB - d^2 = 0$, and consists of two ellipses which pass through the extremities of the mean axis.

The curve which is traced on the fixed plane by the point of contact with the moving ellipsoid is called the ***herpolhode***.

To find the **equation of the herpolhode**, let ρ, χ be the polar coordinates of the point of contact, when the foot of the perpendicular from the fixed point on the fixed plane is taken as pole. If (x', y', z') denote the coordinates of the same point referred to the moving axes *Oxyz*, we have

$$x'^2 + y'^2 + x'^2 = SQ$$
$$= \rho^2 + \frac{c}{d^2},$$

(where SQ stands for 'square of radius from point of suspension to point contact.')

Substituting for *x'*, *y'*, *z'* their values as given by the equations

$$\begin{cases} x' = \omega_1 / \sqrt{c} = -d \sin\theta \cos\psi / A\sqrt{c} \\ y' = \omega_2 / \sqrt{c} = d \sin\theta \sin\psi / B\sqrt{c} \\ z' = \omega_3 / \sqrt{c} = d \cos\theta / C\sqrt{c} \end{cases}$$

we have

$$\rho^2 = -\frac{c}{d^2} + \frac{d^2}{A^2 c}\sin^2\theta\cos^2\psi + \frac{d^2}{B^2 c}\sin^2\theta\sin^2\psi + \frac{d^2}{C^2 c}\cos^2\theta .$$

Replacing θ and ψ by their values in terms of *t*, this becomes

$$\rho^2 = \frac{(cA - d^2)(d^2 - cC)}{cd^2 A^2 B^2 C^2}\left\{ ACB^2 - \frac{(B-C)(A-B)d^2}{\tilde{p}(t) - e_3} \right\}$$
$$= \frac{(cA - d^2)(d^2 - cC)}{cd^2 AC} \frac{\tilde{p}(t) - \tilde{p}(l + \omega)}{\tilde{p}(t) - e_3},$$

where ω denotes the half-period corresponding to the root e_1, this equation expresses the *radius vector of the herpolhode* in terms of the time.

We have next to find the **vectorial angle** χ in terms of *t*. For this we observe that $\sqrt{c}\rho^2\dot{\chi}/d$ is six times the volume of the tetrahedron whose vertices are the fixed point, the foot of the perpendicular from the fixed point on the fixed plane, and two consecutive positions of the point of contact, divided by the interval of time elapsed between these positions, and that this quantity can also be expressed in the form

$$\begin{vmatrix} x' & y' & z' \\ Acx'/d^2 & Bcy'/d^2 & Ccz'/d^2 \\ \dot{x}' & \dot{y}' & \dot{z}' \end{vmatrix}$$

or

$$\frac{c}{d^2} x'y'z' \begin{vmatrix} 1 & 1 & 1 \\ A & B & C \\ \dot{x}'/x' & \dot{y}'/y' & \dot{z}'/z' \end{vmatrix}.$$

All the quantities involved, except $\dot{\chi}$ are known functions of *t*;

on substituting their values in terms of t, and reducing, we have

$$\dot{\chi} = \frac{d}{B\{\tilde{p}(t) - \tilde{p}(l+\omega)\}}\left\{\tilde{p}(t) - \frac{(B-C)e_2 + (A-B)e_1}{A-C}\right\},$$

which can be written in the form

$$\dot{\chi} = \frac{d}{B} + \frac{i}{2}\frac{\tilde{p}'(l+\omega)}{\tilde{p}(t) - \tilde{p}(l+\omega)}.$$

This equation can be integrated in the same way as the equation for the Eulerian angle ϕ, and gives

$$e^{2i(\chi-\chi_0)} = e^{\{2id(B-2\zeta(l+\omega)\}t}\frac{\sigma(t+l+\omega}{\sigma(t-l-\omega)},$$

where χ_0 is a constant of integration. The current coordinates (ρ_0, χ) of the herpolhode are thus expressed as functions of *t*.

Example 1. A particle moves in such a way that its angular momentum round the origin is a linear function of the square of the radius vector, while the square of its velocity is a quadratic function of the square of the radius vector, the coefficient of the highest power being negative; shew that the path is the herpolhode of a Poinsot motion, in which however *A*, *B*, *C* are not restricted to be positive.

Example 2. Discuss the cases in which the polhode consists of (a) two ellipses intersecting on the mean axis of the momenta! ellipsoid, (b) two parallel circles, (c) two points; shewing that in these cases the herpolhode becomes respectively a spiral curve (whose equation can be expressed in terms of elementary functions), a circle, or a point.

71. Motion of a top on a perfectly rough plane; determination of the Eulerian angle θ.

A ***top*** is defined to be a material body which is symmetrical about an axis and terminates in a sharp point (called the *apex* or *vertex*) at one end of the axis.

We shall now study the motion of a top when spinning with its apex placed on a perfectly rough plane, so that is practically a fixed point. The problem is essentially that of determining the motion of a solid of revolution under the influence of gravity, when a point on its axis is fixed in space. Let (*A*, *A*, *C*) denote the moments of inertia of the top about rectangular axes *Oxyz*, **fixed relative to the top and moving with it**, the **origin being the apex** and the axis *Oz* being the axis of symmetry of the top; let (θ, ϕ, ψ) be the Eulerian angles defining the position of these axes with reference to fixed rectangular axes *OXYZ*, of which *OZ* is directed vertically upwards.

The kinetic energy is (§63)

$$T = \frac{1}{2}(A\omega_1^2 + A\omega_2^3 + C\omega_3^2),$$

where $\omega_1, \omega_2, \omega_3$ denote the components relative to the moving axes

of the angular velocity of the top, so that (§16) we have

$$\begin{cases} \omega_1 = \dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi \\ \omega_2 = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi \\ \omega_3 = \dot{\psi} + \dot{\phi}\cos\theta \end{cases},$$

the kinetic energy is therefore

$$T = \frac{1}{2}A\dot{\theta}^2 + \frac{1}{2}A\dot{\phi}^2\sin^2\theta + \frac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2,$$

and the potential energy is

$$V = Mgh\cos\theta\ ,$$

where M is the mass of the top and h is the distance of its centre of gravity from the apex.

The kinetic potential is therefore

$$L = T - V = \frac{1}{2}A\dot{\theta}^2 + \frac{1}{2}A\dot{\phi}^2\sin^2\theta + \frac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgh\cos\theta\ .$$

The coordinates ϕ and ψ are evidently ignorable; the corresponding integrals are

$$\frac{\partial T}{\partial\dot{\phi}} = const. \quad \text{and} \quad \frac{\partial T}{\partial\dot{\psi}} = const.$$

or

$$\begin{cases} A\phi\sin^2\theta + C(\dot{\psi} + \phi\cos\theta)\cos\theta = a \\ C(\dot{\psi} + \dot{\phi}\cos\theta) = b \end{cases},$$

where a and b are constants; these may be interpreted as ***integrals of angular momentum*** about the axes OZ and Oz, and so are obvious *à priori* from general dynamical principles.

The **modified kinetic potential** (§38) is

$$\begin{aligned} R &= L - a\dot{\phi} - b\dot{\psi} \\ &= \frac{1}{2}A\dot{\theta}^2 - \frac{(a - b\cos\theta)^2}{2A\sin^2\theta} - \frac{b^2}{2C} - Mgh\cos\theta \end{aligned}.$$

The term $-b^2/2C$ can be neglected, as it is merely a constant; the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial R}{\partial\dot{\theta}}\right) - \frac{\partial R}{\partial\theta} = 0\ ,$$

so the variation of θ is the same as in a dynamical system with one degree of freedom for which the kinetic energy is $(1/2)A\dot{\theta}^2$ and the potential energy is

$$\frac{(a - b\cos\theta)^2}{2A\sin^2\theta} + Mgh\cos\theta\ .$$

The connexion between θ and t is therefore given by the *integral of energy* of this reduced system, namely

$$\frac{1}{2}A\dot{\theta}^2 = -\frac{(a-b\cos\theta)^2}{2A\sin^2\theta} - Mgh\cos\theta + c,$$

where c is a constant.

Writing $\cos\theta = x$, this equation becomes

$$A^2\dot{x}^2 = -(a-bx)^2 - 2AMgh(x-x^3) + 2Ac(1-x^2).$$

The right-hand side of this equation is a cubic polynomial in x; now when x = 1, the cubic is negative; for some real values of θ, i.e., for some values of x between -1 and 1, the cubic must be positive, since the left-hand side of the equation is positive; when x = 1, the cubic is again negative; and when $x = +\infty$, the cubic is positive. The cubic has therefore two real roots which lie between -1 and 1, and the remaining root is also real and is greater than unity. Let these roots be denoted by

$$\cos\alpha, \quad \cos\beta, \quad \cosh\gamma,$$

where $\cos\beta > \cos\alpha$, so that $\alpha > \beta$.

The differential equation now becomes

$$(Mgh/2A)^{1/2}dt = \{4(x-\cos\alpha)(x-\cos\beta)(x-\cosh\gamma)\}^{-1/2}dx.$$

If we write

$$\begin{aligned} x &= \frac{2A}{Mgh}z + \frac{1}{3}(\cos ga + \cos\beta + \cosh\gamma) \\ &= \frac{2A}{Mgh}z + \frac{2Ac+b^2}{6AMgh} \end{aligned},$$

we have therefore

$$t + const. = \int\{4(z-e_1)(z-e_2)(z-e_3)\}^{-1/2}dz,$$

where the constants e_1, e_2, e_3 are given by the equations

$$\begin{cases} e_1 = \dfrac{Mgh}{2A}\cosh\gamma - \dfrac{2Ac+b^2}{12A^2} \\ e_2 = \dfrac{Mgh}{2A}\cos\beta - \dfrac{2Ac+b^2}{12A^2} \\ e_3 = \dfrac{Mgh}{2A}\cos\alpha - \dfrac{2Ac+b^2}{12A^2} \end{cases},$$

so that e_1, e_2, e_3 are all real and satisfy the relations

$$e_1 + e_2 + e_3 = 0, \quad e_1 > e_2 > e_3.$$

The connexion between z and t is therefore

$$z = \widetilde{p}(t+\varepsilon),$$

where ε is a constant of integration, and the function $\widetilde{p}$ is formed

with the roots e_1, e_2, e_3; and hence we have

$Weierstrass's\_elliptic\_function:$

$$u = \int_y^\infty \frac{ds}{\sqrt{4s^2 - g_2 s - g_3}}$$

$$y = \widetilde{p}(u)$$

$$4s^2 - g_{2s} - g_3 = (s-e_1)(s-e_2)(s-e_3)$$

$$e_1 + e_2 + e_3 = 0$$

$$g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3) = 2(e_1^2 + e_2^2 + e_3^2)$$

$$g_3 = 4e_1e_2e_3$$

$$\widetilde{p}(\omega_1/2) = e_1,\_\widetilde{p}(\omega_2/2) = e_2,\_\widetilde{p}(\omega_3/2) = e_3$$

$$x = \frac{2A}{Mgh}\widetilde{p}(t+\varepsilon) + \frac{2Ac+b^2}{6AMgh}.$$

Now in order that $\dot{x}$ may be real for real values of t, it is evident that x must lie between $\cos\alpha$ and $\cos\beta$, i.e., $\widetilde{p}(t+\varepsilon)$ must lie between e_2 and e_3 for real values of t; and therefore the imaginary part of the constant ε must be the half-period ω_3 corresponding to the root e_3. The real part of ε depends on the epoch from which the time is measured, and so can be taken to be zero by suitably choosing this epoch. We have therefore finally

$$\cos\theta = \frac{2A}{Mgh}\widetilde{p}(t+\omega_3) + \frac{2Ac+b^2}{6AMgh},$$

and **this is the equation which expresses the Eulerian angle θ in terms of the time**.

Example 1. If the circumstances of projection of the top are such that initially

$\theta = 60°$, $\dot{\theta} = 0$, $\dot{\phi} = 2(Mgh/3A)^{1/2}$, $\dot{\psi} = (3A-C)(Mgh/3AC^2)^{1/2}$,

shew that the value of θ at any time t is given by the equation

$$\sec\theta = 1 + \sec h\left(\sqrt{\frac{Mgh}{A}}t\right),$$

so that the axis of the top continually approaches the vertical.

For in this case we readily find for the constants a, b, c the values

$$a = b = (3MghA)^{1/2}, \quad c = Mgh,$$

so the differential equation to determine x is

$$\left(\frac{dx}{dt}\right)^2 = \frac{Mgh}{A}(1-x)^2(2x-1),$$

whence the result follows.

Example 2. A solid of revolution can turn freely about a fixed point in its axis of symmetry, and is acted on by forces derived from a potential energy function $\mu\cot^2\theta$, where θ is the angle between this axis and a fixed line; shew that the equations of motion can be integrated in terms of elementary functions.

For proceeding as in the problem of the top on the perfectly rough plane, we find for the integral of energy of the reduced problem the equation

$$\frac{1}{2}A\dot{\theta}^2 = -\frac{(a-b\cos\theta)^2}{2A\sin^2\theta} - \mu\frac{\cos^2\theta}{\sin^2\theta} + c.$$

Writing $\cos\theta = x$, this becomes

$$A^2\dot{x}^2 = -(a-bx)^2 - 2A\mu x^2 + Ac(1-x^2) .$$

The quadratic on the right-hand side is negative when $x = 1$ and $x = -1$, but is positive for some values of x between -1 and +1, since the left-hand side is positive for some real values of θ; the quadratic has therefore two real roots between -1 and +1. Calling these $\cos\alpha$ and $\cos\beta$, the equation is of the form

$$\lambda^2\dot{x}^2 = (\cos\alpha - x)(x - \cos\beta) ,$$

the solution of which is

$$x = \cos\alpha\sin^2(t/2\lambda) + \cos\beta\cos^2(t/2\lambda) .$$

72. Determination of the remaining Eulerian angles, and of the Cayley-Klein parameters; the spherical top.

When the Eulerian angle θ has been obtained in terms of the time, as in the last article, it remains to determine the other Eulerian angles ϕ and ψ. For this purpose we use the two integrals corresponding to the ignorable coordinates; these, when solved for $\dot{\phi}$ and $\dot{\psi}$, give

$$\begin{cases} \dot{\phi} = \dfrac{a - b\cos\theta}{A\sin^2\theta} \\ \dot{\psi} = \dfrac{b}{C} - \dfrac{(a - b\cos\theta)\cos\theta}{A\sin^2\theta} \end{cases} .$$

If we regard the motion as specified by the constants of the body (M, A, C, h) and the constants of integration (a, b, c), it is evident from these equations and the equation for $\dot{\theta}$ that C does not occur except in the constant term of the expression for $\dot{\psi}$; and therefore an **auxiliary top** whose moments of inertia are (A, A, A) can be projected in such a way that its axis of symmetry always occupies the same position as the axis of symmetry of the top considered, the only difference in the motion of the two tops being that the auxiliary top has throughout the motion a constant extra spin $b(C - A)/AC$ about its axis of symmetry. A top such as this auxiliary top, whose moments of inertia are all equal, is called a ***spherical top***. It follows therefore that the motion of any top can be simply expressed in terms of the motion of a spherical top, and that there is no real loss of generality in supposing any top under consideration to be spherical.

If then we take $C = A$, the equations to determine ϕ and ψ become

$$\begin{cases} \dot{\phi} = \dfrac{a - b\cos\theta}{A\sin^2\theta} = \dfrac{a+b}{2A(\cos\theta + 1)} - \dfrac{a-b}{2A(\cos\theta - 1)} \\ \dot{\psi} = \dfrac{b - a\cos\theta}{A\sin^2\theta} = \dfrac{a+b}{2A(\cos\theta + 1)} + \dfrac{a-b}{2A(\cos\theta - 1)} \end{cases} .$$

Substituting for $\cos\theta$ its value from the equation

$$\cos\theta = \frac{2A}{Mgh}\widetilde{p}(t+\omega_3) + \frac{2Ac+b^2}{6AMgh},$$

and writing

$$\widetilde{p}(l) = \frac{Mgh}{2A} - \frac{2Ac+b^2}{12A^2},$$

$$\widetilde{p}(k) = -\frac{Mgh}{2A} - \frac{2Ac+b^2}{12A^2},$$

so that l and k are known imaginary constants (being in fact the values of $t+\omega_3$ corresponding to the values 0 and π of θ), the differential equations become

$$\begin{cases} \dot{\phi} = \dfrac{Mgh(a+b)}{4A^2}\dfrac{1}{\widetilde{p}(t+\omega_3)-\widetilde{p}(k)} - \dfrac{Mgh(a-b)}{4A^2}\dfrac{1}{\widetilde{p}(t+\omega_3)-\widetilde{p}(l)} \\ \dot{\psi} = \dfrac{Mgh(a+b)}{4A^2}\dfrac{1}{\widetilde{p}(t+\omega_3)-\widetilde{p}(k)} + \dfrac{Mgh(a-b)}{4A^2}\dfrac{1}{\widetilde{p}(t+\omega_3)-\widetilde{p}(l)} \end{cases}.$$

Now the connexion between the function $\widetilde{p}$ and its derivate $\widetilde{p}'$ can be at once written down by substituting for x from the equation

$$x = \frac{2A}{Mgh}\widetilde{p}(t+\omega_3) + \frac{2Ac+b^2}{6AMgh}$$

in the equation

$$A^2\left(\frac{dx}{dt}\right)^2 = -(a-bx)^2 - 2AMgh(x-x^3) + 2Ac(1-x^2);$$

if the argument of the $\widetilde{p}$ -function is k, it follows from the definition of k that the corresponding value of x is -1; and so the last equation gives

$$A^2\{2A\widetilde{p}'(k)/Mgh\}^2 = -(a+b)^2,$$

or

$$\widetilde{p}'(k) = iMgh(a+b)/2A^2.$$

Similarly we have

$$\widetilde{p}'(l) = iMgh(a-b)/2A^2,$$

and therefore the equations for ϕ and ψ can be written in the form

$$\begin{cases} 2i\dot{\phi} = \dfrac{\widetilde{p}'(k)}{\widetilde{p}(t+\omega_3)-\widetilde{p}(k)} - \dfrac{\widetilde{p}'(l)}{\widetilde{p}(t+\omega_3)-\widetilde{p}(l)} \\ 2i\dot{\psi} = \dfrac{\widetilde{p}'(k)}{\widetilde{p}(t+\omega_3)-\widetilde{p}(k)} + \dfrac{\widetilde{p}'(l)}{\widetilde{p}(t+\omega_3)-\widetilde{p}(l)} \end{cases}.$$

Now the function

$$\frac{\tilde{p}'(k)}{\tilde{p}(t+\omega_3)-\tilde{p}(k)}$$

is an elliptic function, whose poles in any period-parallelogram are congruent with $t+\omega_3=k$ and $t+\omega_3=-k$, the corresponding residues being 1 and -1; and the function is zero when $t+\omega_3=0$. Hence we have

$$\frac{\tilde{p}'(k)}{\tilde{p}(t+\omega_3)-\tilde{p}(k)}=\zeta(t+\omega_3-k)-\zeta(t+\omega_3+k)+2\zeta(k),$$

and therefore

$$\int\frac{\tilde{p}'(k)dt}{\tilde{p}(t+\omega_3)-\tilde{p}(k)}=\log\frac{\sigma(t+\omega_3-k)}{\sigma(t+\omega_3+k)}+2\zeta(k)t+const.$$

The integrals of the equations for ϕ and ψ can therefore be written in the form

$$\begin{cases} e^{2i(\phi-\phi_0)}=e^{2\{\zeta(k)-\zeta(l)\}t}\dfrac{\sigma(t+\omega_3-k)\sigma(t+\omega_3+l)}{\sigma(t+\omega_3+k)\sigma(t+\omega_3-l)} \\ e^{2i(\psi-\psi_0)}=e^{2\{\xi(k)+\xi(l)\}t}\dfrac{\sigma(t+\omega_3-k)\sigma(t+\omega_3-l)}{\sigma(t+\omega_3+k)\sigma(t+\omega_3+l)} \end{cases},$$

where ϕ_0 and ψ_0 are constants of integration.

These equations lead to simple expressions for the Cayley-Klein parameters $\alpha,\beta,\gamma,\delta$ (§12), which define the position of the moving axes *Oxyz* with reference to the fixed axes *OXYZ*; for by definition we have

$$\alpha=\cos(\theta/2)e^{i(\phi+\psi)/2},\quad \beta=i\sin(\theta/2)e^{i(\phi-\psi)/2},$$

$$\gamma=i\sin(\theta/2)e^{i(\psi-\phi)/2},\quad \delta=\cos(\theta/2)e^{-i(\phi+\psi)/2}.$$

But we have

$$\begin{aligned} 2\cos^2(\theta/2) &= 1+\cos\theta \\ &= 1+\frac{2A}{Mgh}\tilde{p}(t+\omega_3)+\frac{2Ac+b^2}{6AMgh} \\ &= \frac{2A}{Mgh}\{\tilde{p}(t+\omega_3)-\tilde{p}(k)\} \\ &= -\frac{2A}{Mgh}\frac{\sigma(t+\omega_3+k)\sigma(t+\omega_3-k)}{\sigma^2(k)\sigma^2(t+\omega_3)} \end{aligned},$$

or

$$\cos(\theta/2)=\left(\frac{-A}{Mgh}\right)^{1/2}\frac{\{(\sigma(t+\omega_3+k)\sigma(t+\omega_3-k)\}^{1/2}}{\sigma(k)\sigma(t+\omega_3)}.$$

Similarly we find

$$\sin(\theta/2) = \left(\frac{A}{Mgh}\right)^{1/2} \frac{\{(\sigma(t+\omega_3+l)\sigma(t+\omega_3-l)\}^{1/2}}{\sigma(l)\sigma(t+\omega_3)},$$

and on combining these with the expressions for $e^{2i\phi}$ and $e^{2i\psi}$ already found, we have

$$\begin{cases} \alpha = \left(\dfrac{-A}{Mgh}\right)^{1/2} \dfrac{e^{i(\phi_0+\psi_0)/2}}{\sigma(k)} \dfrac{\sigma(t+\omega_3-k)}{\sigma(t+\omega_3)} e^{t\zeta(k)} \\ \beta = \left(\dfrac{-A}{Mgh}\right)^{1/2} \dfrac{e^{i(\phi_0-\psi_0)/2}}{\sigma(l)} \dfrac{\sigma(t+\omega_3+l)}{\sigma(t+\omega_3)} e^{-t\zeta(l)} \\ \gamma = \left(\dfrac{-A}{Mgh}\right)^{1/2} \dfrac{e^{i(\psi_0-\phi_0)/2}}{\sigma(k)} \dfrac{\sigma(t+\omega_3-l)}{\sigma(t+\omega_3)} e^{t\zeta(l)} \\ \delta = \left(\dfrac{-A}{Mgh}\right)^{1/2} \dfrac{e^{-i(\phi_0+\psi_0)/2}}{\sigma(k)} \dfrac{\sigma(t+\omega_3+k)}{\sigma(t+\omega_3)} e^{-t\zeta(k)} \end{cases}.$$

These equations express the parameters $\alpha, \beta, \gamma, \delta$ as functions of the time.

Example 1. A gyrostat of mass M moves about a faced point in its axis of symmetry; the moments of inertia about the axis of figure and a perpendicular to it through the fixed point are C and A respectively, and the centre of gravity is at a distance h from the fixed point. The gyrostat is held so that its axis makes an angle $\arccos(1/\sqrt{3})$ with the downward vertical, and is given an angular velocity $\sqrt{AMgh\sqrt{3}}/C$ about its axis. If the axis be now left free to move about the fixed point, shew that it will describe the cone

$$\sin^2\theta \sin 2\phi = (-\cos\theta - 1/\sqrt{3})^{3/2}(-\cos\theta + \sqrt{3})^{1/2}$$

or

$$\sin^2\theta \sin 2\phi = \frac{2\sqrt{2}}{\sqrt{3}\sqrt[4]{3}}(\sqrt{3}/2 + \cos\theta)^{1/2}$$

where ϕ is the azimuthal angle and θ the inclination of the axis to the upward vertical. (Camb. Math. Tripos, Part I, 1894.)

For in this problem we have initially

$$\cos\theta = 1 - /\sqrt{3}\,,\ \ \phi = 0\,,\ \ \dot{\theta} = 0\,,\ \ \dot{\phi} = 0\,,\ \ \dot{\psi} = \sqrt{AMgh\sqrt{3}}/C\,,$$

and these initial values give

$$a = -\sqrt{MAgh}/\sqrt[4]{3}\,,\ \ b = \sqrt[4]{3}\sqrt{MAgh}\,,\ \ c = -Mgh/\sqrt{3}\,.$$

Substituting in the general differential equation for θ, namely

$$\frac{1}{2}A\dot{\theta}^2 = -\frac{(a-b\cos\theta)^2}{2A\sin^2\theta} - Mgh\cos\theta + c\,,$$

we have

$$A\dot{\theta}^2\sin^2\theta = -Mgh(\cos\theta + 1/\sqrt{3})(\sqrt{3} + 2\cos\theta)(-\cos\theta + \sqrt{3})\,,$$

while the equation

$$\dot{\phi} = \frac{a - b\cos\theta}{A\sin^2\theta}$$

gives

$$\dot{\phi} = -\sqrt{\frac{Mgh\sqrt{3}}{A}}\,\frac{\cos\theta + 1/\sqrt{3}}{\sin^2\theta}.$$

Dividing this equation by the square root of the preceding equation, we have

$$\phi = 3^{1/4}\int(-\cos\theta - 1/\sqrt{3})^{1/2}(\sqrt{3} + 2\cos\theta)^{-1/2}(-\cos\theta + \sqrt{3})^{-1/2}\operatorname{cosec}\theta d\theta$$

$$= 3^{1/4}\int(x - 1/\sqrt{3})^{1/2}(\sqrt{3} - 2x)^{-1/2}(x + \sqrt{3})^{-1/2}(1 - x^2)^{-1}dx$$

where $x = -\cos\theta$.

Now if we write

$$u = (x - 1/\sqrt{3})^{3/2}(x + \sqrt{3})^{1/2}(\sqrt{3}/2 - x)^{-1/2},$$

we have by differentiation

$$\frac{du}{dx} = \frac{3}{2}(1 - x^2)(x - 1/\sqrt{3})^{1/2}(x + \sqrt{3})^{-1/2}(\sqrt{3}/2 - x)^{3/2}$$

and

$$1 + \frac{3^{5/2}}{8}u^2 = \frac{3^{3/2}(1 - x^2)^2}{8(\sqrt{3}/2 - x)}.$$

We have therefore

$$\phi = \frac{3^{3/4}}{4\sqrt{2}}\int\frac{du}{1 + 3^{3/2}u^2/8},$$

or

$$\phi = \arctan(3^{3/4}2^{-3/2}u),$$

or

$$\tan 2\phi = 3^{3/4}2^{-3/2}(-\cos\theta - 1/\sqrt{3})^{3/2}(-\cos\theta + \sqrt{3})(\sqrt{3}/2 + \cos\theta)^{-1/2}$$

which is equivalent to the result given above.

Example 2. Shew that the logarithms of the Cayley-Klein parameters, considered as functions of $\cos\theta$, are *elliptic integrals of the third kind*.

Example 3. Obtain the expressions found above for the Cayley-Klein parameters as functions of the time t by shewing that they satisfy differential equations typified by

$$\frac{d^2y}{dt^2} + Yy = 0,$$

where Y denotes a *doubly-periodic function* of t, these equations being of the *Hermite-Lame type* which is soluble by doubly-periodic

functions of the second kind.

A simple type of motion of the top is that in which the axis of symmetry maintains a constant inclination to the vertical; in this case, which is generally known as the **steady motion** of the top, $\dot{\theta}$ and $\ddot{\theta}$ are permanently zero; since we have

$$\frac{1}{2}A\dot{\theta}^2 = -\frac{(a-b\cos\theta)^2}{2A\sin^2\theta} - Mgh\cos\theta + c$$

it follows that

$$0 = \frac{d}{d\theta}\left\{\frac{(a-b\cos\theta)^2}{2A\sin^2\theta} + Mgh\cos\theta\right\}.$$

Performing the differentiation, and substituting for $(a-b\cos\theta)$ its value $A\dot{\phi}\sin^2\theta$, we have

$$0 = -b\dot{\phi} + A\dot{\phi}^2\cos\theta + Mgh.$$

This equation gives the relation between the constants $\dot{\phi},\theta$ and b (which depends on the rate of spinning of the top on its axis) in steady motion.

73. Motion of a top on a perfectly smooth plane.

We shall now consider the motion of a top which is spinning with its apex in contact with a smooth horizontal plane. The reaction of the plane is now vertical, so the horizontal component of the velocity of the centre of gravity, *G*, of the top is constant; we can therefore without loss of generality suppose that this component is zero, so that the point *G* moves vertically in a fixed line, which we shall take as axis of Z; two horizontal lines fixed in space and perpendicular to each other will be taken as axes of *X* and *Y*.

Let *Gxyz* be the principal axes of inertia of the top at *G*, and (*A*, *A*, *C*) the moments of inertia about them, *Gz* being the axis of symmetry; and let (θ,ϕ,ψ) be the Eulerian angles defining their position with reference to the axes of *X*, *Y*, Z.

The height of *G* above the plane is $h\cos\theta$, where *h* denotes the distance of *G* from the apex of the top; the part of the kinetic energy due to the motion of *G* is therefore $(1/2)Mh^2\sin^2\theta\cdot\dot{\theta}^2$, where *M* is the mass of the top; and so, as in §71, the total kinetic energy is

$$T = \frac{1}{2}Mh^2\sin^2\theta\cdot\dot{\theta}^2 + \frac{1}{2}A\dot{\theta}^2 + \frac{1}{2}A\dot{\phi}^2\sin^2\theta + \frac{1}{2}C(\dot{\psi}+\dot{\phi}\cos\theta)^2,$$

and the potential energy is

$$\mathrm{V} = \mathrm{Mgh}\cos\theta.$$

Proceeding now exactly as in §71, we have two integrals corresponding to the ignorable coordinates ϕ and ψ, namely

$$\begin{cases} A\dot{\phi}\sin^2\theta + C(\dot{\psi}+\dot{\phi}\cos\theta)\cos\theta = a \\ C(\dot{\psi}+\dot{\phi}\cos\theta) = b \end{cases},$$

where a and b are constants; and on performing the process of ignoration of coordinates we obtain for the modified kinetic potential the expression

$$\frac{1}{2}(A+Mh^2\sin^2\theta)\dot{\theta}^2 - \frac{(a-b\cos\theta)^2}{2A\sin^2\theta} - Mgh\cos\theta,$$

so the variation of θ is the same as in the system with one degree of freedom for which the kinetic energy is

$$\frac{1}{2}(A+Mh^2\sin^2\theta)\dot{\theta}^2,$$

and the potential energy is

$$\frac{(a-b\cos\theta)^2}{2A\sin^2\theta} + Mgh\cos\theta.$$

The connexion between θ and t is given by the integral of energy of this latter system, namely

$$\frac{1}{2}(A+Mh^2\sin^2\theta)\dot{\theta}^2 = -\frac{(a-b\cos\theta)^2}{2A\sin^2\theta} - Mgh\cos\theta + c,$$

where c is a constant. Writing $\cos\theta = x$, this becomes

$$A(A+Mh^2-Mh^2x^2)\dot{x}^2 = -(a-bx)^2 - 2AMgh(x-x^3) + 2Ac(1-x^2).$$

The variables x and t are separated in this equation, so the solution can be expressed as a quadrature; but the evaluation of the integral involved will require in general ***hyperelliptic functions***, or ***automorphic functions of genus two***.

74. Kowalevski's top.

The problem of the motion under gravity of a body one of whose points is fixed is not in general soluble by quadratures; and the cases considered in §69 (in which the fixed point is the centre of gravity of the body, so that gravity does not influence the motion), and in §71 (in which the fixed point and the centre of gravity lie on an axis of symmetry of the body), were for long the only ones known to be integrable. In 1888 however Mme. S. Kowalevski shewed that the problem is also soluble when two of the principal moments of inertia at the fixed point are equal and double the third, so that $A = B = 2C$, and when further the centre of gravity is situated in the plane of the equal moments of inertia.

Let the line through the fixed point O and the centre of gravity be taken as the axis Ox, and let the centre of gravity be at a distance a from the fixed point; let (θ, ϕ, ψ) be the Eulerian angles which define the position of the principal axes of inertia $Oxyz$ with reference

to fixed rectangular axes *OXYZ*, of which the axis *OZ* is vertical; let $(\omega_1, \omega_2, \omega_3)$ be the components along the axes *Oxyz* of the angular velocity of the body, and let *M* be its mass. The kinetic and potential energies are given by the equations

$$T = \frac{1}{2}(A\omega_1^2 + A\omega_2^2 + C\omega_3^2)$$
$$= C\left\{\dot\theta^2 + \dot\phi^2 \sin^2\theta + \frac{1}{2}(\dot\psi + \dot\phi\cos\theta)^2\right\},$$

$$V = -Mga\sin\theta\cos\psi .$$

The coordinate ϕ is evidently ignorable, giving an integral

$$\frac{\partial T}{\partial\dot\phi} = const.$$

or

$$2\phi\sin^2\theta + (\dot\psi + \dot\phi\cos\theta)\cos\theta = k ,$$

where *k* is a constant; and the integral of energy is

$$T + V = const.$$

or

$$\dot\theta^2 + \dot\phi^2\sin^2\theta + \frac{1}{2}(\dot\psi + \dot\phi\cos\theta)^2 - \frac{Mga}{C}\sin\theta\cos\psi = h .$$

Mme. Kowalevski shewed that another algebraic integral exists, which can be found in the following way.

The kinetic potential is

$$L = C\dot\theta^2 + C\dot\phi^2\sin^2\theta + \frac{1}{2}C(\dot\psi + \dot\phi\cos\theta)^2 + Mga\sin\theta\cos\psi$$

and the equations of motion are

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot\theta}\right) - \dfrac{\partial L}{\partial\theta} = 0 \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot\psi}\right) - \dfrac{\partial L}{\partial\psi} = 0 \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot\phi}\right) = 0 \end{cases};$$

the first of these is

$$2\ddot\theta = (\dot\phi\cos\theta - \dot\psi)\dot\phi\sin\theta + \frac{Mga}{C}\cos\theta\cos\psi ,$$

and on eliminating $\ddot\psi$ between the second and third, we obtain

$$2\frac{d}{dt}(\dot\phi\sin\theta) = -(\dot\phi\cos\theta - \dot\psi)\dot\theta + \frac{Mga}{C}\cos\theta\sin\psi .$$

Adding the first of these equations multiplied by *i* to the second, we have

$$2\frac{d}{dt}(\dot\phi\sin\theta + i\dot\theta) = i(\dot\phi\cos\theta - \dot\psi)(\dot\phi\sin\theta + i\dot\theta) + i\frac{Mga}{C}\cos\theta e^{-i\psi} ,$$

an equation which can be written in the form

First :

$$\frac{d}{dt}\left(2C\dot\theta\right) - 2C\dot\phi^2\sin\theta\cos\theta - C(\dot\psi + \dot\phi\cos\theta)\dot\phi(-\sin\theta)$$
$$- Mga\cos\theta\cos\psi = 0$$
$$2\ddot\theta - 2\dot\phi^2\sin\theta\cos\theta + \dot\psi\dot\phi\sin\theta + \dot\phi\cos\theta\dot\phi\sin\theta$$
$$- (Mga / C)\cos\theta\cos\psi = 0$$
$$2\ddot\theta + (\dot\psi - \dot\phi\cos\theta)\dot\phi\sin\theta - (Mga / C)\cos\theta\cos\psi = 0$$

Second :

$$\frac{d}{dt}\left(C(\dot\psi + \dot\phi\cos\theta)\right) - Mga\sin\theta(-\sin\psi) = 0$$
$$\ddot\psi + \ddot\phi\cos\theta + \dot\phi(-\sin\theta)\dot\theta + (Mga / C)\sin\theta\sin\psi = 0$$

Third :

$$\frac{d}{dt}\left(2C\dot\phi\sin^2\theta + C(\dot\psi + \dot\phi\cos\theta)\cos\theta\right) = 0$$
$$2\ddot\phi\sin^2\theta + 2\dot\phi\cdot 2\sin\theta\cos\theta\cdot\dot\theta + \ddot\psi\cos\theta + \dot\psi(-\sin\theta)\dot\theta$$
$$+ \ddot\phi\cos^2\theta + \dot\phi\cdot 2\cos\theta(-\sin\theta)\cdot\dot\theta = 0$$
$$\ddot\phi(2\sin^2\theta + \cos^2\theta) + \ddot\psi\cos\theta - \dot\theta\dot\psi\sin\theta$$
$$+ 2\dot\theta\dot\phi\sin\theta\cos\theta = 0$$

$$2\frac{d}{dt}\left\{(\dot\phi\sin\theta+i\dot\theta)^2+\frac{Mga}{C}\sin\theta e^{-i\psi}\right\}$$
$$=i(\dot\phi\cos\theta-\dot\psi)\left\{(\dot\phi\sin\theta+i\dot\theta)^2+\frac{Mga}{C}\sin\theta e^{-i\psi}\right\},$$

or

$$\frac{1}{U}\frac{dU}{dt}=i(\dot\phi\cos\theta-\dot\psi),$$

where

$$U=(\dot\phi\sin\theta+i\dot\theta)^2+\frac{Mga}{C}\sin\theta e^{-i\psi}.$$

Similarly, if

$$V=(\dot\phi\sin\theta-i\dot\theta)^2+\frac{Mga}{C}\sin\theta e^{i\psi}$$

we have

$$\frac{1}{V}\frac{dV}{dt}=-i(\dot\phi\cos\theta-\dot\psi).$$

It follows that

$$\frac{1}{U}\frac{dU}{dt}+\frac{1}{V}\frac{dV}{dt}=0,$$

or

$$UV=const.$$

We have therefore the equation

$$\left\{(\dot\phi\sin\theta+i\dot\theta)^2+\frac{Mga}{C}\sin\theta e^{-i\psi}\right\}\left\{(\dot\phi\sin\theta-i\dot\theta)^2+\frac{Mga}{C}\sin\theta e^{i\psi}\right\}=const.$$

or

$$(\dot\theta^2+\dot\phi^2\sin^2\theta)^2+\frac{Mga}{C}\sin\theta+\frac{Mga}{C}\sin\theta\left\{e^{i\psi}(\dot\phi\sin\theta+i\dot\theta)^2+e^{-i\psi}(\dot\phi\sin\theta-i\dot\theta)^2\right\}=const.$$

and this is the required ***third algebraic integral*** of the system.

The first integrals which have been found constitute a system of three differential equations, each of the first order, for the determination of $\dot\theta,\dot\phi,\dot\psi$, and they can be regarded as replacing the original differential equations of motion. The variable ϕ does not occur explicitly in them and we can therefore use one of the three equations in order to eliminate $\dot\phi$ from the other two; we shall then have a system of two differential equations, each of the first order, to determine θ and ψ. It has been shewn by Mme. Kowalevski that these equations can be solved by means of *hyperelliptic functions*; for this solution reference may be made to the memoir already referred to.

Example. Let $\gamma_1,\gamma_2,\gamma_3$ denote the direction-cosines of *Ox*, *Oy*, *Qz* referred to *OZ*, and let variables *x*, *y*, τ be denned by the equations

$$\omega_2^2 x = \left(\omega_1^2 - \omega_2^2 + \frac{Mga\gamma_1}{C}\right)\left\{\left(\omega_3\omega_1 + \frac{Mga\gamma_3}{C}\right)^2 - \omega_3^2\omega_2^2\right\}$$
$$+ 2\omega_3\omega_2\left(2\omega_1\omega_2 + \frac{Mga\gamma_2}{C}\right)\left(\omega_3\omega_1 + \frac{Mga\gamma_3}{C}\right),$$

$$\omega_2^2 y = \left(2\omega_1\omega_2 + \frac{Mga\gamma_3}{C}\right)\left\{\left(\omega_3\omega_1 + \frac{Mga\gamma_3}{C}\right)^2 - \omega_3^2\omega_2^2\right\}$$
$$- 2\omega_3\omega_2\left(\omega_3\omega_1 + \frac{Mga\gamma_3}{C}\right)\left(\omega_1^2 - \omega_2^2 + \frac{Mga\gamma_1}{C}\right),$$

$$\omega_2^2 d\tau = \left\{\left(\omega_3\omega_1 + \frac{Mga\gamma_3}{C}\right)^2 + \omega_3^2\omega_2^2\right\}dt.$$

Shew by use of Kowalevski's integral (without using the integrals of energy or angular momentum) that the equations of motion can be written in the form

$$\frac{d^2x}{d\tau^2} = -\frac{\partial V}{\partial x}, \quad \frac{d^2y}{d\tau^2} = -\frac{\partial V}{\partial y},$$

where V is a function of x and y only, so that the problem is transformed into that of the motion of a particle in a plane conservative field of force. (Kolosoff.)

R. Liouville has stated that the only other general case in which the motion under gravity of a rigid body with one point fixed has a third algebraic integral is that in which

1°. The momental ellipsoid of the point of suspension is an ellipsoid of revolution.

2°. The centre of gravity of the body is in the equatorial plane of the momental ellipsoid.

3°. If (A, A, C) are the principal moments of inertia at the point of suspension, the ratio $2C/A$ is an integer; this integer can be arbitrarily chosen.

On this, cf. the memoirs cited in the footnote on the preceding page.

Example. A heavy body rotates about a fixed point O, the principal moments of inertia at which satisfy the relation $A = B = 4C$; and the centre of gravity of the body lies in the equatorial plane of the momental ellipsoid, at a distance h from O. Shew that if the constant of angular momentum about the vertical through O vanishes, there exists an integral

$$\omega_3(\omega_1^2 + \omega_2^2) + gh\omega_1\cos\theta = const.,$$

where $\omega_1, \omega_2, \omega_3$ are the components of angular velocity about the principal axes $Oxyz$, Ox being the line from O to the centre of gravity; and hence that the problem can be solved by quadratures, leading to hyperelliptic integrals. (Tchapligine.)

[Reader's addition]

From M. Toda, *Thirty Lectures on Waves and Nonlinear Problems*, Asakura Shoten, 1995.

Tops of Kowalevskaja (p.202)

The equation of motion for Kawalevskaya under the conditions $A = B = 2$, $C = 1$ and $\eta_0 = \zeta_0$ will be

$$\begin{cases} 2\dot{\omega}_1 = \omega_2\omega_3 \\ 2\dot{\omega} = -\omega_1\omega_3 + c_0\gamma_3 \, , \\ \dot{\omega}_3 = -c_0\gamma_2 \end{cases} \tag{20}$$

where

$$c_0 = Mg\xi_0 \, . \tag{20'}$$

Since

$$\begin{cases} \dot{\gamma}_1 = \omega_3\gamma_2 - \omega_2\gamma_3 \\ \dot{\gamma}_2 = \omega_1\gamma_3 - \omega_3\gamma_1 \\ \dot{\gamma}_3 = \omega_2\gamma_1 - \omega_1\gamma_2 \end{cases} \tag{21}$$

we have the integral called the integral of Kowalevskaja:

$$\xi\bar{\xi} = k^2 \, , \tag{22}$$

where

$$\xi = (\omega_1 + i\omega_2)^2 - c_0(\gamma_1 + i\gamma_2) \, , \tag{23}$$

and $\bar{\xi}$ is the complex conjugate of ξ.