CHAPTER I

RIEMANNIAN MANIFOLDS

In this chapter, we have given a brief survey of Riemannian geometry. In §1, we give the basic definitions and formulas on **differentiable manifolds**. §2 is devoted to the study of **connections** and **covariant differentiation** on manifolds. We introduce the **torsion tensor field** and the **curvature tensor field** and derive the **structure equation of Cartan**. Moreover, we define the **Riemannian connection** on a manifold. In §3, we define the **sectional curvature** of a Riemannian manifold and give examples of space forms, that is, Riemannian manifolds of constant curvature. In §4, we discuss **transformations** on a Riemannian manifold and give some integral formulas (cf. Yano [6]). In the last §5, we prepare some results of **fibre bundles** for later use.

In §§1, 2 and 3 we followed Chapter I of Helgason [1] and Kobayashi-Nomizu [1]. In §5, we followed fairly closely Kobayashi-Nomizu [1]. We also refer to Matsushima [1].

1. MANIFOLDS AND TENSOR FIELDS

Let M be a topological space. We assume that M satisfies the ***Hausdorff separation axiom*** which states that any two different points in M can be separated by disjoint open sets. An open ***chart*** on M is a pair (U,ϕ) where U is an open subset of M and ϕ is a homeomorphism of U onto an open subset of R^n, where R^n is an n-dimensional Euclidean space.

Definition. Let M be a Hausdorff space. A differentiable structure on M of dimension n is a collection of open charts $(U_i,\phi_i)_{i\in\Lambda}$ on M where $\phi_i(U_i)$ is an open subset of R^n such that the following conditions are satisfied:

(a) $M = \bigcup_{i\in\Lambda} U_i$;

(b) For each pair $i, j \in \Lambda$ the mapping $\phi_j \cdot \phi_i^{-1}$ is a differentiable mapping of $\phi_i(U_i \cap U_j)$ onto $\phi_j(U_i \cap U_j)$;

(c) The collection $(U_i,\phi_i)_{i\in\Lambda}$ is a maximal family of open charts for which (a) and (b) hold.

A ***differentiable manifold*** (or C^∞-manifold or simply manifold) of dimension n is a Hausdorff space with differentiable structure of

dimension n. If M is a manifold, a ***local coordinate system*** on M (or a ***local chart*** on M) is by definition a pair (U_i, ϕ_i). If p is a point in U_i and $\phi_i(p) = (x^1(p), \ldots, x^n(p))$, the set U_i is called a ***coordinate neighborhood*** of p and the numbers $x_j(p)$ are called ***local coordinates*** of p. The mapping $\phi_i : q \to (x^1(q), \ldots, x^n(q))$, $q \in U_i$, is often denoted by $\{x^1, \ldots, x^n\}$.

We notice that the condition (c) is not essential in the definition of a manifold. In fact, if only (a) and (b) are satisfied, the family $(U_i, \phi_i)_{i \in \Lambda}$ can be extended in a unique way to a larger family of open charts such that (a), (b) and (c) are all fulfilled. This is easily seen by defining the larger family as the set of all open charts (V, ϕ) on M satisfying: (i) $\phi(V)$ is an open set in R^n, (ii) for each $i \in \Lambda$, $\phi_i \cdot \phi^{-1}$ is a diffeomorphism of $\phi(V \cap U_i)$ onto $\phi_i(V \cap U_i)$.

An ***analytic structure*** of dimension n is defined in a similar fashion. In (b) we just replace "differentiable" by "analytic". In this case M is called an ***analytic manifold***.

In order to define a ***complex manifold*** of (complex) dimension n we replace R^n in the definition of differentiable manifold by n-dimensional complex number space C^n. The condition (b) is replaced by the condition that the n coordinates of $\phi_j \cdot \phi_i^{-1}(p)$ should be holomorphic functions of the coordinates of p.

Given two manifolds M and M', a mapping $f : M \to M'$ is said to be ***differentiable*** (or C^∞ differentiable), if for every chart (U_i, ϕ_i) of M and every chart (V_i, ψ_i) of M' such that $f(U_i) \subset V_j$, the mapping $\psi_j \cdot f \cdot \phi_i^{-1}$ of $\phi_i(U_{i)}$ into $\psi_j(V_j)$ is differentiable. A ***differentiable function*** on M is a differentiable mapping of M into R. If M is analytic, the function f on M is said to be ***analytic*** if for every chart (U_i, ϕ_i), the function $f \cdot \phi_i^{-1}$ is an analytic function on $\phi_i(U_i)$. The definition of a ***holomorphic*** (or ***complex analytic***) mapping or function is similar.

Let M and N be two manifolds of dimension n and m, respectively. Let $(U_i, \phi_i)_{i \in \Lambda}$ and $(U_a, \psi_a)_{a \in \Lambda}$, be collections of open charts on M and N, respectively. For $i \in \Lambda$, $a \in \Lambda'$, let $\phi_i \times \psi_a$ denote the mapping $(p, q) \to (\phi_i(p), \psi_a(q))$ of the product set $U_i \times V_a$ into R^{n+m}. Then the collection $(U_i \times V_a, \phi_i \times \psi_a)_{i \in \Lambda, a \in \Lambda'}$ of open charts on the product space $M \times N$ satisfies (a) and (b), and hence $M \times N$ can be turned into a manifold the ***product*** of M and N.

By a ***differentiable curve*** in a manifold M, we shall mean a differentiable mapping of a closed interval $[a,b]$ of R into M. We shall now define a ***tangent vector*** (or singly a ***vector***) at a point p of M. Let $\widetilde{F}(p)$ be the algebra of differentiable functions defined in a neighborhood of p. Let $\tau(t)$ $(a \le t \le b)$ be a curve such that $\tau(t_0) = p$. The vector tangent to the curve $\tau(t)$ at p is a mapping $X : \widetilde{F}(p) \to R$ defined by

$$Xf = \left(\frac{df(\tau(t))}{dt}\right)_{t_0}.$$

In other words, Xf is the derivative of f in the direction of the curve $\tau(t)$ at $t = t_0$. The vector X satisfies the following conditions:

(1) X is a linear mapping of $\widetilde{F}(p)$ into R;

(2) $X(fg) = (Xf)g(p) + f(p)(Xg)$ for $f, g \in \widetilde{F}(p)$.

The set of mappings X of $\widetilde{F}(p)$ into R satisfying the preceding two conditions forms a real vector space. Let $x^1, \dots, x^n$ be local coordinates in a coordinate neighborhood U of p. For each i, $\left(\frac{\partial}{\partial x^i}\right)_p$ is a mapping of $\widetilde{F}(p)$ into R which satisfies (1) and (2). Given any curve $\tau(t)$ with $p = \tau(t_0)$, let $x^i = \tau^i(t)$, $i = 1,\dots,n$, be its equations in terms of the local coordinates $x^1, \dots, x^n$. Then

$$\left(\frac{df(\tau(t))}{dt}\right)_{t_0} = \sum_i \left(\frac{\partial f}{\partial x^i}\right)_p \left(\frac{d\tau^i(t)}{dt}\right)_{t_0},$$

which proves that every vector at p is a linear combination of $\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^n}\right)_p$. Conversely, given a linear combination $\sum_i \xi^i \left(\frac{\partial}{\partial x^i}\right)_p$, consider the curve defined by

$$x^i = x^i(p) + \xi^i t, \; i = 1,\dots,n.$$

Then the vector tangent to this curve at $t = 0$ is $\sum_i \xi^i \left(\frac{\partial}{\partial x^i}\right)_p$. If we assume $\sum_i \xi^i \left(\frac{\partial}{\partial x^i}\right)_p = 0$, then $0 = \sum_i \xi^i \left(\frac{\partial x^j}{\partial x^i}\right)_p = \xi^j$ for $j = 1,\dots,n$.

Therefore, $\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^n}\right)_p$ are linearly independent and hence these form a basis of the set of vectors at p. The set of tangent vectors at p, denoted by $T_p(M)$, is called the ***tangent space*** of M at p. The n-tuple of

numbers $\xi^1,\ldots,\xi^n$ are ***components*** of the vectors $\sum_i \xi^i \left(\frac{\partial}{\partial x^i}\right)_p$ with respect to the local coordinates $x^1,\ldots,x^n$.

We notice that on a C^∞ differentiable manifold *M* the tangent space $T_p(M)$ coincides with the space of $X:\widetilde{F}(p)\to R$ satisfying the conditions (1) and (2) above.

A ***vector field*** *X* on a manifold *M* is an assignment of a vector X_p to each point *p* of *M*. If *f* is a differentiable function on *M*, then *Xf* is a function on *M* defined by $(Xf)(p)=X_p f$. A vector field *X* is said to be ***differentiable*** if *Xf* is differentiable for every differentiable function *f*. In terms of local coordinates $x^1,\ldots,x^n$, *X* may be expressed by $X=\sum \xi^i \left(\frac{\partial}{\partial x^i}\right)$, where ξ^i are functions defined in the coordinate neighborhood, called the components of *X* with respect to $x^1,\ldots,x^n$. *X* is differentiable if and only if its components ξ^i are differentiable.

We denote by $\widetilde{X}(M)$ the set of all differentiable vector fields on *M*. From now on we shall consider mainly manifolds of class C^∞, mappings of class C^∞ and vector fields of class C^∞.

If *X* and *Y* are vector fields, define the **bracket** [*X*,*Y*] as a mapping from the ring of functions on *M* into itself by

$$[X,Y]f = X(Yf)-Y(Xf).$$

Let $X=\sum \xi^i \left(\frac{\partial}{\partial x^i}\right)$ and $Y=\sum \eta^j \left(\frac{\partial}{\partial x^j}\right)$. Then

$$[X,Y]f=\sum_{i,j}\left\{\xi^j\left(\frac{\partial \eta^i}{\partial x^j}\right)-\eta^j\left(\frac{\partial \xi^i}{\partial x^j}\right)\left(\frac{\partial f}{\partial x^i}\right)\right\}.$$

This means that [*X*,*Y*] is a vector field with components $\sum_j\left\{\xi^j\left(\frac{\partial \eta^i}{\partial x^j}\right)-\eta^j\left(\frac{\partial \xi^i}{\partial x^j}\right)\right\}$, *i* = 1,...,*n*. With respect to this bracket operation, $\widetilde{X}(M)$ is a **Lie algebra** over *R*.

For any vector fields *X*, *Y* and *Z*, we have the Jacobi identity:

$$[[X,Y\},Z]+[[Y,Z],X]+[[Z,X],Y]=0.$$

We may also regard $\widetilde{X}(M)$ as a module over the algebra $\widetilde{F}(M)$ of differentiable functions on *M* as follows. If *f* is a function and *X* is a vector field on *M*, then *fX* is a vector field on *M* defined by

$(fX)_p = f(p)X_p$ for $p \in M$. We also have

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$$

Let $T_p^*(M)$ be the dual space of the tangent space $T_p(M)$ of M at p. An element of $T_p^*(M)$ is called a ***covector*** at p. An assignment of a covector at each point p is called a 1-***form*** (***differential form of degree*** 1). For each function for M, the ***total differential*** $(df)_p$ of f at p is defined by

$$<(df)_p, X> = Xf \quad \text{for} \quad X \in T_p(M),$$

where < , > denotes the value of the first entry on-the second entry as a linear functional on $T_p(M)$. Let (U, x^i) be a local coordinate system at p. Then $(dx^1)_p, \ldots, (dx^n)_p$ form a basis for $T_p^*(M)$. They form the dual basis of the basis $\left(\frac{\partial}{\partial x^1}\right)_p, \ldots, \left(\frac{\partial}{\partial x^n}\right)_p$ for $T_p(M)$. In a neighborhood of p, every 1-form ω can be uniquely written as

$$\omega = \sum_j f_j dx^j,$$

where f_j are functions in U and are called the ***components*** of ω with respect to $x^1, \ldots, x^n$. The 1-form ω is called ***differentiable*** if f_j are differentiable. This condition is independent of the choice of a local coordinate system. We shall only consider differentiable 1-forms.

A 1-form ω can be defined also as an $\widetilde{F}(M)$-linear mapping of the $\widetilde{F}(M)$-module $\widetilde{X}(M)$ into $\widetilde{F}(M)$. The two definitions are related by

$$(\omega(X))_p = <\omega_p, X_p>, \quad X \in \widetilde{X}(M).$$

Let A be a commutative ring with identity element and $E_1, \ldots, E_s$ be A-modules. Then $E_1 \times \cdots \times E_a$ is also an A-module. A mapping $f: E_1 \times \cdots \times E_s \to F$, where F is an A-module, is said to be A-***multilinear*** if it is A-linear in each argument. The set of all A-multilinear mappings of $E_1 \times \cdots \times E_s$ into F is again A-module. Suppose that all the factors E_i coincide. The A-multilinear mapping is called ***alternate*** if $f(X_1, \ldots, X_s) = 0$ whenever at least two X_i

coincide.

Let T_s^r denote the $\widetilde{F}(M)$ -module of all $\widetilde{F}(M)$ -multilinear mapping of

$$\widetilde{X}(M)^* \times \cdots \times \widetilde{X}(M)^* \times \widetilde{X}(M) \times \cdots \times \widetilde{X}(M)$$

$$(\widetilde{X}(M)^* \ r \text{ times}, \ \widetilde{X}(M) \ s \text{ times})$$

into $\widetilde{F}(M)$, where $\widetilde{X}(M)^*$ is the dual $\widetilde{F}(M)$ -module of $\widetilde{X}(M)$. We put $T_0^r = T^r$, $T_s^0 = T_s$ and $T_0^0 = \widetilde{F}(M)$. A ***tensor field*** K on M of type (r,s) is an element of T_s^r. This tensor field K is said to be ***contravariant*** of degree r, ***covariant*** of degree s. In particular, the tensor fields of type (0,0), (1,0) and (0,1) on M are just the differentiable functions on M, the vector fields and the 1-fours on M, respectively. If p is a point in M, we define $T_s^r(p)$ as the set of all R-multilinear mappings of

$$T_p^*(M) \times \cdots \times T_p^*(M) \times T_p(M) \times \cdots \times T_p(M)$$

$$(T_p^*(M) \ r \text{ times}, \ T_p(M) \ s \text{ times})$$

into R. The set $T_s^r(p)$ is a vector space over R and is nothing but the tensor product

$$T_p(M) \otimes \cdots \otimes T_p(M) \otimes T_p^*(M) \otimes \cdots \otimes T_p^*(M)$$

$$(T_p(M) \ r \text{ times}, \ T_p^*(M) \ s \text{ times})$$

or otherwise written

$$T_s^r(p) = \otimes^r T_p(M) \otimes \otimes^s T_p^*(M).$$

We also put $T_0^0(p) = R$.

Let $\{x^1, \ldots, x^n\}$ be a system of coordinates valid on an open neighborhood U of p. Then there exist vector fields $X_1, \ldots, X_n$ and 1-forms $\omega_1, \ldots, \omega_n$ on M and an open neighborhood V of p, $p \in V \in U$ such that on V

$$X_i = \left(\frac{\partial}{\partial x^i}\right), \quad \omega^j(X_i) = \delta_{ij} \quad (1 \le i, j \le n).$$

On V we put

$$Z_i = \sum_k f_{ik} X_k \ , \quad \theta_j = \sum_l g_{jl} \omega^l \ ,$$

where $f_{ik}, g_{jl} \in \widetilde{F}(V)$. Then we have, for $q \in V$,

$$K(\theta^1,\ldots,\theta^r,Z_1,\ldots,Z_s)(q)=\sum_{l_j=1,k_i=1}^{n} g_{1l_1}\cdots g_{rl_r} f_{1k_1}\cdots f_{sk_s} K(\omega^{l_1},\ldots,\omega^{l_r},X_{k_1},\ldots,X_{k_s})(q)$$

This shows that $K(\theta^1,\ldots,\theta^r,Z_1,\ldots,Z_s)(p)=0$ if some θ^j or some Z_i vanishes at p. We can therefore define an element $K_p \in T_s^r(p)$ by

$$K_p((\theta^1)_p,\ldots,(\theta^r)_p,(Z_1)_p,\ldots,(Z_s)_p)=K(\theta^1,\ldots,\theta^r,Z_1,\ldots,Z_s)(p).$$

Thus the tensor field K gives rise to a family K_p, $p\in M$, where $K_p \in T_s^r(p)$. If $K_p=0$ for all p, then $K = 0$. The element K_p depends differentiably on p in the sense that if V is a coordinate neighborhood of p and K_q ($q\in V$) is expressed as above in terms of basis for $\widetilde{X}(V)^*$ and $\widetilde{X}(V)$, then the coefficients are differentiable functions on V. On the other hand, if there is a rule $p\to K_p$ which to each $p\in M$ assigns a member $K(p)$ of $T_s^r(p)$ in a differentiable manner, then there exists a tensor field K of type (r,s) such that $K_p=K(p)$ for all $p\in M$. In the case when M is analytic it is clear how to define analyticity of a tensor field K.

Let T denote the direct sum of the $\widetilde{F}(M)$-modules T_s^r,

$$T=\sum_{r,s=0}^{\infty} T_s^r.$$

Similarly, if $p\in M$ we consider

$$T(p)=\sum_{r,s=0}^{\infty} T_s^r(p).$$

An element of T is of the form $\sum_{r,s} K_s^r$, where $K_s^r \in T_s^r$ are zero except for a finite number of them at each $p\in M$. The vector space $T(p)$ can be turned into an **associative algebra** over R as follows: Let

$$a=e_1\otimes\cdots\otimes e_r\otimes f^1\otimes\cdots\otimes f^s,$$
$$b=e'_1\otimes\cdots\otimes e'_c\otimes f'^1\otimes\cdots\otimes f'^d$$

where e'_i, e'_j are members of a basis for $T_p(M)$, f^k, f'^l are members of a dual basis for $T_p^*(M)$. Then $a\otimes b$ is defined by

$$a\otimes b=e_1\otimes\cdots\otimes e_r\otimes e'_1\otimes\cdots\otimes e'_c\otimes f^1\otimes\cdots\otimes f^s\otimes f'^1\otimes\cdots\otimes f'^d.$$

We put $a\otimes 1=a$, $1\otimes b=b$ and extend the operation $(a,b)\to a\otimes b$ to a bilinear mapping of $T(p)\times T(p)$ into $T(p)$. Then $T(p)$ is an

associative algebra over R. The multiplication in $T(p)$ is independent of the choice of basis.

The tensor product $\otimes$ in T is defined as the $\widetilde{F}(M)$-bilinear mapping $(K, L) \to K \otimes L$ of $T \times T$ into T such that

$$(K \otimes L)_p = K_p \otimes L_p\,, \quad K \in T_s^r\,, \quad L \in T_d^c\,.$$

This turns the $\widetilde{F}(M)$-module T into a ring satisfying

$$f(K \otimes L) = fK \otimes L = K \otimes fL$$

for $f \in \widetilde{F}(M)$, $K, L \in T$. In other words, T is an **associative algebra** over $f \in \widetilde{F}(M)$. The algebras T and $T(p)$ are called the ***mixed tensor algebras*** over M and $T_p(M)$, respectively. The submodules

$$T^* = \sum_{r=0}^{\infty} T^r\,, \quad T_* = \sum_{s=0}^{\infty} T_s$$

are subalgebras of T and the subspaces

$$T^*(p) = \sum_{r=0}^{\infty} T^r(p)\,, \quad T_*(p) = \sum_{s=0}^{\infty} T_s(p)$$

are subalgebras of $T(p)$.

We now define the notion of ***contraction***. Now let r, s be two integers ≥ 1, and let i, j be integers such that $1 \leq i \leq r$, $1 \leq j \leq s$. Consider the R-linear mapping $C_j^i : T_s^r(p) \to T_{s-1}^{r-1}(p)$ defined by

$$C_j^i(e_1 \otimes \cdots \otimes e_r \otimes f^1 \otimes \cdots \otimes f^s) = < e_i, f_j > (e_1 \otimes \cdots \hat{e}_i \cdots \otimes e_r \otimes f^1 \otimes \cdots \hat{f}^j \cdots \otimes f^s)$$

where $\{e_k\}$ is a basis of $T_p(M)$, $\{f^l\}$ is the dual basis of $T_p^*(M)$. (The symbol $\hat{e}_i$ over a letter means that the letter is missing.) We note that C_j^i is independent of the choice of basis. There exists now a unique $\widetilde{F}(M)$-linear mapping $C_j^i : T_s^r \to T_{s-1}^{r-1}$ such that

$$(C_j^i(K))_p = C_j^i(K_p)$$

for all $K \in T_s^r$ and all $p \in M$. This mapping satisfies the relation

$$C_j^i(X_1 \otimes \cdots \otimes X_r \otimes \omega^1 \otimes \cdots \otimes \omega^s) = < X_i, \omega_j > (X_1 \otimes \cdots \hat{X}_i \cdots \otimes X_r \otimes \omega^1 \otimes \cdots \hat{\omega}^j \cdots \otimes \omega^s)$$

for all $X_1, \ldots, X_r \in T^1$, $\omega^1, \ldots, \omega^s \in T_1$. The mapping C_j^i is called the ***contraction*** of the i-th contravariant index and the j-th covariant index.

For the basis $\{e_i\}$ for $T_p(M)$ and the dual basis $\{f_s\}$ for $T_p^*(M)$, every tensor *K* of type (*r*,*s*) can be expressed uniquely as

$$K = \sum_{i_1,\ldots,i_r,j_1,\ldots,j_s} K^{i_1\cdots i_r}_{j_1\cdots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes f^{j_1} \otimes \cdots \otimes f^{j_s} ,$$

where $K^{i_1\cdots i_r}_{j_1\cdots j_s}$ are called the ***components*** of *K* with respect to the above basis. In terms of components, the contraction C^i_j is represented by

$$(C^i_j K)^{i_1\cdots i_{r-1}}_{j_1\cdots j_{s-1}} = \sum_k K^{i_1\cdots k\cdots i_{r-1}}_{j_1\cdots k\cdots j_{s-1}} ,$$

where the superscript *k* appears at the *i*-th position and the subscript *k* appears at the *j*-th position.

Let $\Lambda T_p^*(M)$ be the exterior algebra over $T_p^*(M)$. An *r*-form ω is an assignment of an element of degree *r* in $\Lambda T_p^*(M)$ to each point *p* of *M*. In terms of a local coordinate system $x^1,\ldots,x^n$, *r*-form ω can be expressed uniquely as

$$\omega = \sum_{i_1<\cdots<i_r} f_{i_1\cdots i_r} dx^{i_1} \cdots dx^{i_r} .$$

The *r*-form ω is said to be ***differentiable*** if the components $f_{i_1\cdots i_r}$ are all differentiable. By an *r*-form we shall mean a differentiable *r*-form. An *r*-form ω can be defined also as a skew-symmetric *r*-linear mapping over $\widetilde{F}(M)$ of $\widetilde{X}(M)\times\cdots\times\widetilde{X}(M)$ (*r* times) into $\widetilde{F}(M)$. The two definitions are related as follows. If $\omega_1,\ldots,\omega_r$ are 1-forms and $X_1,\ldots,X_r$ are vector fields, then $(\omega_1\wedge\cdots\wedge\omega_r)(X_1,\ldots,X_r)$ is $\frac{1}{r!}$ times the determinant of the matrix $(\omega_j(X_k))_{j,k=1,\ldots,r}$ of degree *r*.

Let $D^r = D^r(M)$ be the totality of *r*-forms on *M* for each *r* = 0,1,…,*n*. Then $D^0 = \widetilde{F}(M)$. Each D^r is a real vector space and can be also considered as an $\widetilde{F}(M)$-module: for $f \in \widetilde{F}(M)$ and $\omega \in D^r$, $f\omega$ is an *r*-form defined by $(f\omega)_p = f(p)\omega_p$, $p \in M$. We set

$$D = D(M) = \sum_{r=0}^{n} D^r(M) .$$

With respect to the exterior product, *D*(*M*) forms an algebra over *R*.

Exterior differentiation d can be characterized as follows:

(1) d is an R-linear mapping of $D(M)$ into itself such that $d(D^r) \subset D^{r+1}$;

(2) For $f \in D^0$, df is the total differential;

(3) If $\omega_1 \in D^r$ and $\omega_2 \in D^s$, then

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2;$$

(4) $d^2 = 0$.

Let V be an m-dimensional real vector space. We define a V-valued r-form ω on M as an assignment to each point p of M a skew-symmetric r-linear mapping of $T_p(M) \times \cdots \times T_p(M)$ (r times) into V. If we take a basis $e_1, \ldots, e_m$ for V, we can write ω uniquely as $\omega = \sum_j \omega^j e_j$, where ω^j are usual r-forms on M. If ω^j are all differentiable, ω is said to be ***differentiable***. The exterior derivative $d\omega$ is defined to be $\sum_j d\omega^j e_j$, which is a V-valued (r+l)-form.

Let f be a mapping of a manifold M into another manifold M'. We define the ***differential*** $f_* : T_p(M) \to T_{f(p)}(M')$ at p of f as follows: For each $X \in T_p(M)$, choose a curve $\tau(t)$ in M such that X is the vector tangent to $\tau(t)$ at $p = \tau(t_0)$. Then $f_*(X)$ is the vector tangent to the curve $f(\tau(t))$ at $f(p) = f(\tau(t_0))$. If g is a function in a neighborhood of $f(p)$, then $(f_*(X))g = X(g \cdot f)$. When it is necessary to specify the point p, we write $(f_*)_p$. Then there is no danger of confusion, we may singly write f instead of f_*. The transpose of $(f_*)_p$ is a linear mapping of $T^*_{f(p)}(M')$ into $T^*_p(M)$. For any r-form ω' on M', we define an r-form $f^*\omega'$ on M by

$$(f^*\omega')(X_1, \ldots, X_r) = \omega'(f_*X_1, \ldots, f_*X_r), \quad X_1, \ldots, X_r \in T_p(M).$$

We also have $d(f^*\omega') = f^*(d\omega')$, that is, d commutes with f^*.

A mapping f of M into M' is said to be of ***rank*** r at $p \in M$ if the dimension of $f_*(T_p(M))$ is r. If the rank of f at p is equal to n = dim M, $(f_*)_p$ is **injective** and dim $M \leq$ dim M'. If the rank of f at p is equal to n' = dim M', $(f_*)_p$ is **surjective** and dim $M \geq$ dim M'. We notice that the following proposition is hold (see Chevalley; pp.79-80)).

PROPOSITION 1.1. *Let f be a mapping of M into M' and p be a point of M.*

(1) *If* $(f_*)_p$ *is injective, there exist a local coordinate system* $\{U; x^i\}$ *of p and a local coordinate system* $\{U'; y^a\}$ *of f(p) such that*

$$y^i(f(q)) = x^i(q) \quad for \quad q \in U,\ i=1,\ldots,n.$$

In particular, f is a homeomorphism of U onto f(U);

(2) *If* $(f_*)_p$ *is surjective, there exist a local coordinate system* $\{U; x^i\}$ *of p and a local coordinate system* $\{U'; y^a\}$ *of f(p) such that*

$$y^a(f(q)) = x^a(q) \quad for \quad q \in U,\ a=1,\ldots,n'.$$

In particular, $f : U \to M'$ *is open;*

(3) *If* $(f_*)_p$ *is a linear isomorphism of* $T_p(M)$ *onto* $T_{f(p)}(M')$*, then f defines a homeomorphism of a neighborhood U of p onto a neighborhood U' of f(p) and its inverse* $f^{-1} : U' \to U$ *is also differentiable.*

A mapping f of M into M' is called an ***immersion*** if $(f_*)_p$ is injective for every point p of M. Then M is called an ***immersed submanifold*** of M'. If the immersion f is injective, it is called an ***imbedding*** of M into M'. We say then that M (or the image $f(M)$) is an ***imbedded submanifold*** of M'. When there is no danger of confusion, we say simply that M is a submanifold of M' instead of that M is an immersed submanifold of M' or an imbedded submanifold of M'.

A homeomorphism f of M onto M' is called a ***diffeomorphism*** if both f and f^{-1} are differentiable. A diffeomorphism of M onto itself is called a ***differentiable transformation*** (or simply a ***transformation***) of M. A transformation ϕ of M induces an automorphism ϕ^* of the algebra $D(M)$ of differential forms on M and, in particular, an automorphism of the algebra $\widetilde{F}(M) : (\phi^* f)(p) = f(\phi(p))$ for $f \in \widetilde{F}(M)$, $p \in M$. From this we have an automorphism ϕ_* of the Lie algebra $\widetilde{X}(M)$ by $(\phi_* X)_p = (\phi_*)_q (X_q)$, where $X \in \widetilde{X}(M)$, $p = \phi(q)$. They are related by $\phi^*((\phi_* X) f) = X(\phi^* f)$. Although any mapping ϕ of M into M' carries a differential form ω' on M' into a differential form $\phi^*(\omega')$ on M, ϕ does not send a vector field on M into a vector field on M' in general. We say that a vector field X on M is ***ϕ-related*** to X' on M' if $(\phi_*)_p X_p = X'_{\phi(p)}$ for all p. If X and Y are ϕ-related to X' and Y', respectively, then $[X,Y]$ is ϕ-related to $[X',Y']$.

A ***1-parameter group of*** (***differentiable***) ***transformations*** of M is a mapping of $R \times M$ into M, $(t, p) \in R \times M \to \phi_t(p) \in M$, which satisfies the following conditions:

(1) For each $t \in R$, $\phi_t : p \to \phi_t(p)$ is a transformation of M;

(2) For all $t, s \in R$ and $p \in M$, $\phi_{t+s}(p) = \phi_t(\phi(s))$.

A curve $\tau(t)$ in M is called an ***integral curve*** of a vector field X if the vector $X_{\tau(t)}$ is tangent to $\tau(t)$ for every t. For any point p of M, there is a unique integral curve $\tau(t)$ of a vector field X, defined for $|t| < \varepsilon$ for some $\varepsilon > 0$, such that $p = \tau(0)$.

Each 1-parameter group of transformations ϕ_t induces a vector field X as follows. For any point $p \in M$, X_p is the vector tangent to the curve $\tau(t) = \phi_t(p)$, called the ***orbit*** of p, at $p = \phi_0(p)$. The orbit $\phi_t(p)$ is an integral curve of X starting at p. A ***local 1-parameter group of local transformations*** can be defined in the same way, except that $\phi_t(p)$ is defined only for t in a neighborhood of 0 and p in an open set of M. A local 1-parameter group of local transformations defined on $I_\varepsilon \times U$ is a mapping of $I_\varepsilon \times U$ into M which satisfies the following conditions:

(1') For each $t \in I_\varepsilon$, $\phi_t : p \to \phi_t(p)$ is a diffeomorphism of U onto the open set $\phi_t(U)$ of M;

(2') If $t, s, ts \in I_\varepsilon$ and if $p, \phi_s(p) \in U$, then $\phi_{t+s}(p) = \phi_t(\phi_s(p))$.

As in the case of a 1-parameter group of transformations, ϕ_t induces a vector field X defined on U. We also have the converse, that is, we can prove the following: Let X be a vector field on M. For each point p_0 of M, there exist a neighborhood U of p_0, a positive number ε and a local 1-parameter group of transformations $\phi_t : U \to M$, $t \in I_\varepsilon$, which induces the given X (see Kobayashi-Nomizu [1;p.13]). We say that X generates a local 1-parameter group of local transformations ϕ_t in a neighborhood of p_0. If there exists a (global) 1-parameter group of transformations of M which induces X, then we say that X is ***complete***. If $\phi_t(p)$ is defined on $I_\varepsilon \times M$ for some ε , then X is complete. Thus, if M is compact, every vector field X is complete.

We notice here that the differential ϕ_* of a transformation ϕ of M gives a linear isomorphism of $T_{\phi^{-1}(X)}(M)$ onto $T_X(M)$. This linear isomorphism can be extended to an isomorphism of the tensor algebra $T(\phi^{-1}(X))$ onto the tensor algebra $T(X)$, which we denote by the same ϕ . Given a tensor field K, we define a tensor field K by

$$(\phi K)_X = \phi(K_{\phi^{-1}(X)}) .$$

In this way, every transformation ϕ of M induces an algebra automorphism of T which preserves type and commutes with contractions.

Let X be a vector field on M and ϕ_t a global 1-parameter group of transformations of M. For each t, ϕ_t is an automorphism of the algebra T. For any tensor field K on M, we set

$$(L_X K)_X = \lim_{t\to 0}\frac{1}{t}[K_X - (\phi_t K)_X].$$

The mapping L_X of T into itself which sends K into $L_X K$ is called the ***Lie differentiation*** with respect to X. It will be no difficulty in modifying the definition of Lie differentiation when X is not complete, that is, when ϕ_t is a local 1-parameter group of transformations generated by X. For the tie differentiation we have

PROPOSITION 1.2. *Lie differentiation L_X with respect to a vector field X satisfies the following conditions*:

(a) L_X *is a derivative of T, that is, it is linear and satisfies*

$$L_X(K\otimes K') = (L_X K)\otimes K' + K\otimes (L_X K'), \quad K, K' \in T;$$

(b) L_X *is type-preserving*: $L_X(T_s^r) \subset T_s^r$;

(c) L_X *commutes with every contraction of a tensor field*;

(d) $L_X f = Xf$, $f \in \widetilde{F}(M)$;

(e) $L_X Y = [X, Y]$, $Y \in \widetilde{X}(M)$.

Proof. It is clear L_X is linear. Moreover, we have

$$\begin{aligned} L_X(K\otimes K') &= \lim_{t\to 0}\frac{1}{t}[K\otimes K' - \phi_t(K\otimes K')] \\ &= \lim_{t\to 0}\frac{1}{t}[K\otimes K' - (\phi_t K)\otimes K'] + \lim_{t\to 0}\frac{1}{t}[(\phi_t K)\otimes K' - (\phi_t K)\otimes(\phi_t K')] \\ &= \left(\lim_{t\to 0}\frac{1}{t}[K - (\phi_t K)\otimes K']\right)\otimes K' + \lim_{t\to 0}(\phi_t K)\otimes\left(\frac{1}{t}[K' - (\phi_t K')]\right) \\ &= (L_X K)\otimes K' + K\otimes(L_X K') \end{aligned}$$

Since ϕ_t preserves type and commutes with contractions, so does L_X. If f is a function on M, then

$$(L_X f)(x) = \lim_{t\to 0}\frac{1}{t}\left[f(x) - f(\phi_t^{-1}(x)\right] = -\lim_{t\to 0}\frac{1}{t}\left[f(\phi_t^{-1}(x) - f(x)\right].$$

We see that $\phi_t^{-1} = \phi_{-t}$ is a local 1-parameter group of local transformations generated by -X, and hence we have $L_X f = -(-X)f = Xf$.

Let f be a function on M. We can take a function g_t such that $f\cdot\phi_t = f + t\cdot g_t$ and $g_0 = Xf$ (see Kobayashi-Nomizu [1;p.15]). We

put $p(t)=\phi_t^{-1}(p)$. Then

$$((\phi_t)_* Y)_p f=(Y(f\cdot\phi_t))_{p(t)}=(Yf)_{p(t)}+t(Yg_t)_{p(t)}$$

and

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}[Y-(\phi_t)_* Y]_p f&=\lim_{t\to 0}\frac{1}{t}[(Yf)_p-(Yf)_{p(t)}]-\lim_{t\to 0}(Yg_t)_{p(t)}\\&=X_p(Yf)-Y_p g_0\\&=[X,Y]_p f\end{aligned}.$$

Therefore we have (e). (QED)

By a ***derivative*** of T, we shall mean a mapping of T into itself which satisfies conditions (a), (b) and (c) of Proposition 1.2.

We now prepare some basic formulas for later use. For the proof see Kobayashi-Nomizu [1].

Let X and Y be vector fields on M. Then

$$L_{[X,Y]}=[L_X,L_Y].$$

Let K be a tensor field of type (1, r). Then we have

$$(L_X K)(Y_1,\ldots,Y_r)=[X,K(Y_1,\ldots,Y_r)]-\sum_{i=1}^{r}K(Y_1,\ldots,[X,Y_i],\ldots,Y_r).$$

We notice that a tensor field K is invariant by ϕ_t for every t if and only if $L_X K=0$.

A ***derivation*** (resp. ***skew-derivation***) of $D(M)$ is a linear mapping A of $D(M)$ into itself which satisfies

$$A(\omega\wedge\omega')=A\omega\wedge\omega'+\omega\wedge A\omega',\quad \omega,\omega'\in D(M)$$

(resp. $A(\omega\wedge\omega')=A\omega\wedge\omega'+(-1)^r\omega\wedge A\omega'$, $\omega\in D^r(M)$, $\omega'\in D(M)$).
A derivation or a skew-derivation A of $D(M)$ is said to be of ***degree*** k if it maps $D^r(M)$ into $D^{r+k}(M)$ for every r. The **exterior differentiation** d is a. skew-derivation of degree 1 and the **Lie differentiation** L_X is a derivation of degree 0. Indeed, the formula

$$(L_X\omega)(Y_1,\ldots,Y_r)=X(\omega(Y_1,\ldots,Y_r))-\sum_{i=1}^{r}\omega(Y_1,\ldots,[X,Y_i],\ldots,Y_r)$$

implies that $L_X(D^r(M))\subset D^r(M)$ and, for $\omega,\omega'\in D(M)$,

$$_X(\omega\wedge\omega')=L_X\omega\wedge\omega'+\omega\wedge L_X\omega'.$$

Moreover, L_X commutes with d. If ω is an r-form, then

$$\begin{aligned}(d\omega)(X_0,X_1,\ldots,X_r)&=\frac{1}{r+1}\sum_{i=0}^{r}(-1)^i X_i(\omega(X_0,\ldots,\hat{X}_i,\ldots,X_r)\\&+\frac{1}{r+1}\sum_{0\le i<j\le r}(-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots X_r)\end{aligned}$$

where $\hat{X}_i$ means that the term is omitted. If ω is a **1-form**, then

$$2(d\omega)(X,Y,Z) = X(\omega(Y,Z)) + Y(\omega(Z,X)) + Z(\omega(X,Y)) - \omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y).$$

If ω is a **2-form**, then

$$3(d\omega)(X,Y,Z) = X(\omega(Y,Z)) + Y(\omega(Z,X)) + Z(\omega(X,Y)) - \omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y).$$

Remark. Sometimes, we define $d\omega$ without the coefficients $1/(r+1)$ of the right hand side of the above equation to simplify the notation.

In the next place, we give the definition of a ***distribution*** on a manifold. A distribution S of dimension r on M is an assignment to each point p of M an r-dimensional subspace S_p of $T_p(M)$. S is called ***differentiable*** if every point p has a neighborhood U and r differentiable vector fields, say, $X_1,\ldots,X_r$, which form a basis of S_q at every $q \in U$. The set $X_1,\ldots,X_r$ is called a ***local basis*** for S in U. A vector field X is said to ***belong to*** S if $X_p \in S_p$ for all $p \in M$. S is called ***involutive*** if $[X,Y] \in S$ for any $X, Y \in S$. By a distribution we shall always mean a differentiable distribution.

A connected submanifold N of M is called an ***integral manifold*** of S if $f_*(T_p(N)) = S_p$ for all $p \in N$, f being the imbedding of N into M. If there is no other integral manifold of S which contains N, N is called a ***maximal integral manifold*** of S. The classical **theorem of Frobenius** can be formulated as follows (cf. Chevalley [1;p.94]).

PROPOSITION 1.3. *Let S be an involutive distribution on a manifold M. Through every point* $p \in M$*, there passes a unique maximal integral manifold N(p) of S. Any integral manifold through p is an open submanifold of N*(p).

2. CONNECTIONS AND COVARIANT DIFFERENTIATIONS

First of all we give the definition of affine connection on a manifold M.

Definition. An affine connection on a manifold M is a rule ∇ which assigns to each $X \in \widetilde{X}(M)$ a linear mapping ∇_X of the vector space $\widetilde{X}(M)$ into itself satisfying the following two conditions:

(1) $\nabla_{fX+gY} = f\nabla_X + g\nabla_Y$;

(2) $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$,

for $f, g \in \widetilde{F}(M)$, $X, Y \in \widetilde{X}(M)$. The operator ∇_X is called the ***covariant differentiation*** with respect to X.

LEMMA 2.1. *Suppose M has the affine connection* ∇ *and let U be an*

open submanifold of M. Let $X, Y \in \widetilde{X}(M)$. *If X or Y vanishes identically on U*, *then so does* $\nabla_X Y$.

Proof. Suppose Y vanishes identically on U. Let $p \in U$ and $g \in \widetilde{F}(M)$. To prove that $((\nabla_X Y)g)(p) = 0$, we select $f \in \widetilde{F}(M)$ such that $f(p) = 0$ and $f = 1$ outside U. Then $fY = Y$ and

$$(\nabla_X Y)g = (\nabla_X (fY))g = (Xf)(Yg) + f(\nabla_X Y)g$$

which vanishes at p. The statement about X follows similarly. (QED)

By Lemma 2.1, an affine connection ∇ on M induces an affine connection on an arbitrary open submanifold U of M, which will be denoted by the same ∇. In fact, let X, Y be two vector fields on U. For each $p \in U$ there exist vector fields X', Y' on M which agree with X and Y in an open neighborhood V of p. We then put $(\nabla_X Y)_q = (\nabla_{X'} Y')_q$ for $q \in V$. By Lemma 2.1, the right-hand side of this equation is independent of the choice of X', Y'. It follows immediately that the rule $\nabla : X \to \nabla_X$ ($X \in \widetilde{X}(U)$) is an affine connection on U.

In particular, suppose U is a coordinate neighborhood with coordinate system $\{x^1, \ldots, x^n\}$. To simplify the notation, we write ∇_i instead of $\nabla_{\frac{\partial}{\partial x^i}}$. We define the functions Γ_{ij}^k on U by

$$\nabla_i\left(\frac{\partial}{\partial x^j}\right) = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}. \tag{2.1}$$

If $\{y^1, \ldots, y^n\}$ is another coordinate system valid on U, we get another set of functions $\overline{\Gamma}_{ab}^c$ by

$$\nabla_a\left(\frac{\partial}{\partial y^b}\right) = \sum_c \overline{\Gamma}_{ab}^c \frac{\partial}{\partial y^c}.$$

From (1) and (2) in the definition of affine connection we find easily

$$\overline{\Gamma}_{ab}^c = \sum_{i,j,k} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \frac{\partial y^c}{\partial x^k} \Gamma_{ij}^k + \sum_j \frac{\partial^2 x^j}{\partial y^a \partial y^b} \frac{\partial y^c}{\partial x^j}. \tag{2.2}$$

On the other hand, suppose that there is given an open covering of M by open coordinate neighborhoods U and in each U a system of functions Γ_{ij}^k such that (2.2) holds whenever two of these neighborhoods overlap.

Then we can define ∇_i by (2.1) and thus we get an affine connection ∇ in each U. We define an affine connection $\overline{\nabla}$ on M as follows: Let $X, Y \in \widetilde{X}(M)$ and $p \in M$. If U is a coordinate neighborhood containing p, let

$$(\overline{\nabla}_X Y)_p = (\nabla_{X'} Y')_p$$

if X' and Y' are the vector fields on U induced by X and Y, respectively. Then $\overline{\nabla}$ is an affine connection on M which on each U induces the connection.

Let $\{x^1,\ldots,x^n\}$ be a coordinate system valid on U of p. On the set U we have $X=\sum f_i \frac{\partial}{\partial x^i}$ where $f_i \in \widetilde{F}(U)$ and $f_i(p)=0$ $(1 \le i \le n)$.

Using Lemma 2.1 we find $(\nabla_X Y)_p = \sum f_i(p)(\nabla_i Y)_p = 0$.

LEMMA 2.2. *Let* $X,Y \in \widetilde{X}(M)$. *If X vanishes at a point p of M, then so does* $\nabla_X Y$.

Definition. Suppose ∇ is an affine connection on M and f is a diffeomorphism of M. A new affine connection ∇' can be defined on M by

$$\nabla'_X Y = f_*^{-1}(\nabla_{f_* X} f_* Y), \quad X,Y \in \widetilde{X}(M).$$

The affine connection ∇ is called ***invariant*** under f if $\nabla' = \nabla$. In this case f is called an ***affine transformation*** of M. Similarly we can define an affine transformation of one manifold onto another.

Let $\gamma : t \to \gamma(t)$ $(t \in I)$ be a curve in M defined on an open interval $I \in R$. Differentiation with respect to the parameter will often denoted by a dot ('). Let J be a closed subinterval of I such that $\gamma_J : t \to \gamma(t)$ $(t \in J)$ has no double points and such that $\gamma(J)$ is contained in a coordinate neighborhood U. We need the following.

LEMMA 2.3. *Let* $g(t)$ *be a differentiable function on an open interval containing J. Then there exists a function* $G \in \widetilde{F}(M)$ *such that*

$$G(\gamma(t)) = g(t) \quad (t \in J).$$

We put $X(t) = \dot{\gamma}(t)$ $(t \in I)$. By Lemma 2.3 it is easy to see that there exist vector fields $X,Y \in \widetilde{X}(M)$ such that ($Y(t)$ being an associated vector to $t \in I$)

$$X_{\gamma(t)} = X(t), \quad Y_{\gamma(t)} = Y(t) \quad (t \in J).$$

The family $Y(t)$ $(t \in J)$ is said to be ***parallel*** with respect to Y_J (or parallel along Y_J) if

(2.3) $$(\nabla_X Y)_{\gamma(t)} = 0 \quad \text{for all } t \in J.$$

We show that this definition is independent of the choice of X and Y. We express (2.3) in the coordinates $\{x^1,\ldots,x^n\}$. Then X and Y are given by

$$X = \sum_i X^i \frac{\partial}{\partial x^i}, \quad Y = \sum_j Y^j \frac{\partial}{\partial x^j} \quad \text{on } U.$$

For simplicity we put $x^i(t) = x^i(\gamma(t))$, $X^i(t) = X^i(\gamma(t))$ and $Y^i(t) = Y^i(\gamma(t))$, $t \in J$, $1 \le i \le n$. Then $X^i(t) = \dot{x}^i(t)$. Since

$$\nabla_X Y = \sum_k \left(\sum_i X^i \frac{\partial Y^k}{\partial x^i} + \sum_{i,j} X^i Y^j \Gamma_{ij}^k \right) \frac{\partial}{\partial x^k} \quad \text{on U,}$$

we obtain

(2.4) $$\frac{dY^k}{dt} + \sum_{i,j} \Gamma_{ij}^k \frac{dx^i}{dt} Y^j = 0 \quad (t \in J).$$

This equation involves X and Y only through their values on the curve. Consequently, condition (2.3) for parallelism is independent of the choice of X and Y. It is well now obvious how to define parallelism with respect to any finite curve segment γ_J and finally with respect to the entire curve γ.

Definition. Let $\gamma : t \to \gamma(t)$ $(t \in I)$ be a curve in *M*. The curve γ is called a ***geodesic*** if the vector field $X(t) = \dot{\gamma}(t)$ defined along γ is parallel with respect to γ, that is, $\nabla_X X = 0$ for all *t*. A geodesic γ is called ***maximal*** if it is not a proper restriction of any geodesic.

From (2.4), if γ_J is a geodesic segment, then

(2.5) $$\frac{d^2 x^k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx^i}{dt} \frac{dx^j}{dt} = 0 \quad (t \in J).$$

If we change the parameter on the geodesic and put $t = f(s)$, $(f'(s) \neq 0)$, then we get a new curve $s \to \gamma_J(f(s))$. This curve is a geodesic if and only if *f* is a linear function, as (2.5) shows.

We notice that the two following propositions hold.

PROPOSITION 2.1. *Let p and q be two points in M and* γ *a curve segment from p to q. The parallelism* τ *with respect to* γ *induces an isomorphism* $T_p(M)$ *onto* $T_q(M)$.

PROPOSITION 2.2. *Let M be a manifold with an affine connection. Let p be any point in M and let* $X \neq 0$ *in* $T_p(M)$. *Then there exists a unique maximal geodesic* $\gamma : t \to \gamma(t)$ *in M such that* $\gamma(0) = p$ *and* $\dot{\gamma}(0) = X$.

The geodesic with properties in Proposition 2.2 will be denoted by γ_X. If $X = 0$, we put $\gamma_X(t) = p$ for all $t \in R$. We then have (cf. Helgason [1])

THEOREM 2.1. *Let M be a manifold with an affine connection. Let p be*

any point in M. Then there exists an open neighborhood N_0 *of* 0 *in* $T_p(M)$ *and an open neighborhood* N_p *of p in M such that the mapping* $X \to \gamma_X(1)$ *is a diffeomorphism of* N_0 *onto* N_p.

The mapping $X \to \gamma_X(1)$ described in Theorem 2.1 is called the ***exponential mapping*** at *p* and will be denoted by *Exp* (or Exp_p).

Let *M* be a manifold with an affine connection and *p* be a point in *M*. An open neighborhood N_0 of the origin $T_p(M)$ is said to be ***normal*** if: (1) *Exp* is a diffeomorphism of N_0 onto an open neighborhood N_p of *p* in *M*; (2) if $X \in N_0$, $0 \le t \le 1$, then $tX \in N_0$.

A neighborhood N_p of *p* in *M* is called a ***normal neighborhood*** of *p* if $N_p = ExpN_0$, where N_0 is a normal neighborhood of 0 in $T_p(M)$. Assuming this to be the case, and letting $X_1, \ldots, X_n$ denote some basis of $T_p(M)$, the inverse mapping

$$Exp(a^1 X_1 + \cdots + a^n X_n) \to (a^1 + \cdots + a^n)$$

of N_p into R^n is called a ***system of normal coordinates*** at *p*.

We now give a useful refinement of Theorem 2.1 (cf. Helgason [1]).

THEOREM 2.2. *Let M be a manifold with an affine connection. Then each point* $p \in M$ *has a normal neighborhood p which is a normal neighborhood of each of its points.*

In the next place, we shall define **covariant derivatives** of arbitrary tensor fields. We first prove

THEOREM 2.3. *Let M be a manifold with an affine connection. Let* $p \in M$ *and let X, Y be two vector fields on M. Assume* $X_p \neq 0$. *Let* $s \to \phi(s)$ *be an integral curve of X through* $p = \phi(0)$ *and* τ_t *the parallel translation from p to* $\phi(t)$ *with respect to the curve* ϕ. *Then*

$$(\nabla_X Y)_p = \lim_{s \to 0} \frac{1}{s}({\tau_s}^{-1} Y_{\phi(s)} - Y_p).$$

Proof. Consider a fixed $s > 0$ and the family $Z_{\phi(t)}$ ($0 \le t \le s$) which is parallel with respect to the curve ϕ such that $Z_{\phi(0)} = {\tau_s}^{-1} Y_{\phi(s)}$. We can put

$$Z_{\phi(t)} = \sum_i Z^i(t)\left(\frac{\partial}{\partial x^i}\right)_{\phi(t)}, \quad Y_{\phi(t)} = \sum_i Y^i(t)\left(\frac{\partial}{\partial x^i}\right)_{\phi(t)}$$

and we have the relations

$$\dot{Z}^k(t) + \sum_{i,j} \Gamma_{ij}^k \dot{x}^i(t) Z^j(t) = 0 \quad (0 \le t \le s),$$

$$Z^k(s) = Y^k(s) \quad (1 \le k \le n).$$

By the mean value theorem $Z^k(s) = Z^k(0) + s\dot{Z}^k(t^*)$ for a suitable number t^* between 0 and s. Hence the k-th component of $\frac{1}{s}(\tau_s{}^{-1}Y_{\phi(s)} - Y_p)$ is

$$\begin{aligned}\frac{1}{s}(Z^k(0) - Y^k(0)) &= \frac{1}{s}(Z^k(s) - s\dot{Z}^k(t^*) - Y^k(0)) \\ &= \sum_{i,j} \Gamma^k_{ij}(\phi(t^*))\dot{x}^i(t^*)Z^j(t^*) + \frac{1}{s}(Y^k(s) - Y^k(0))\end{aligned}.$$

As $s \to 0$ this expression has the limit

$$\frac{dY^k}{ds} + \sum_{i,j} \Gamma^k_{ij} \frac{dx^i}{ds} Y^j .$$

Let this last expression be denoted by A_k. It was shown earlier that

$$(\nabla_X Y)_p = \sum_k A_k \left(\frac{\partial}{\partial x^k} \right)_p .$$

This proves the theorem. (QED)

By using Theorem 2.3 it is now possible to define **covariant derivatives** of arbitrary tensor fields.

Let p and q be two points of M and γ a curve segment in M from p to q. Let τ be the parallel translation along γ. If $F \in T_p^*(M)$, we define $\tau \cdot F \in T_q^*(M)$ by $(\tau \cdot F)(A) = F(\tau^{-1} \cdot A)$ for each $A \in T_q(M)$.

If K is a tensor field on M of type (r, s), we define $\tau \cdot K_q \in T_s^r(q)$ by

$$(\tau \cdot K_p)(F_1, \ldots, F_r, A_1, \ldots, A_s) = K_p(\tau^{-1}F_1, \ldots, \tau^{-1}F_r, \tau^{-1}A_1, \ldots, \tau^{-1}A_s)$$

for $A_i \in T_q(M)$ and $F_j \in T_q^*(M)$. Let $X \in \widetilde{X}(M)$ and p be any point in M where $X_p \neq 0$. With the notation of Theorem 2.3 we put

(2.6) $$(\nabla_X K)_p = \lim_{s \to 0} \frac{1}{s}(\tau_s{}^{-1}K_{\phi(s)} - K_p).$$

For each point $q \in M$ where $X_q = 0$ we put $(\nabla_X K)_q = 0$ in accordance with Lemma 2.2. For a function f on M we put

$$(\nabla_X f)_p = \lim_{s \to 0} \frac{1}{s}(f(\phi(s)) - f(p)),$$

if $X_p \neq 0$, otherwise we put $(\nabla_X f)_p = 0$. Then we have $\nabla_X f = Xf$. Finally ∇_X is extended to a linear crapping of T into itself.

THEOREM 2.4. *The operator* ∇_X *has the following properties*:

(1) ∇_X *is a derivation of the mixed tensor algebra* T;

(2) ∇_X *preserves type of tensors*;

(3) ∇_X *commutes with contractions*.

We now give the **structure equations** of Cartan. For this purpose we define the torsion **tensor field** and the **curvature tensor field**.

On a manifold M with an affine connection, we put

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

where X, Y are vector fields on M. Note that $T(X,Y) = -T(Y,X)$ and $R(X,Y) = -R(Y,X)$. It is easy to verify that $T(fX,gY) = fgT(X,Y)$ and $R(fX,gY)hZ = fghR(X,Y)Z$ for all $f,g,h \in \widetilde{F}(M)$, $X,Y,Z \in \widetilde{X}(M)$. The mapping $(\omega, X, Y) \to \omega(T(X,Y))$ is an $\widetilde{F}(M)$-multilinear mapping $\widetilde{X}(M)^* \times \widetilde{X}(M) \times \widetilde{X}(M)$ into $\widetilde{F}(M)$ and is an element of $T_2^1(M)$. This element is called the ***torsion tensor field*** and is also denoted by T. Similarly, the mapping $(\omega, Z, X, Y) \to \omega(R(X,Y)Z)$ is an $\widetilde{F}(M)$-multilinear mapping $\widetilde{X}(M)^* \times \widetilde{X}(M) \times \widetilde{X}(M) \times \widetilde{X}(M)$ into $\widetilde{F}(M)$ and therefore is an element of $T_3^1(M)$. This element is called the ***curvature tensor field*** and is denoted by R. The tensor fields T and R is of type (1, 2) and (1, 3), respectively.

Let p be a point of M and suppose $X_1, \ldots, X_n$ is a basis for the vector fields in some open neighborhood N_p of p, that is, each vector field X on N_p can be written as $X = \sum_i f_i X_i$ where $f_i \in \widetilde{F}(N_p)$.

We define the functions Γ_{ij}^k, T_{ij}^k, R_{lij}^k on N_p by the formulas

$$\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k ,$$

$$T(X_i, X_j) = \sum_k T_{ij}^k X_k ,$$

$$R(X_i, X_j) X_k = \sum_l R_{kij}^l X_l .$$

Let ω^i, ω_j^i ($1 \le i,j \le n$) be the 1-forms on N_p determined by

$$\omega^i(X_j) = \delta_j^i, \quad \omega_j^i = \sum_k \Gamma_{kj}^i \omega^k .$$

That is, we put

$$\omega_j^i(X_k) = \Gamma_{kj}^i .$$

Thus the forms ω_j^i determine the functions Γ_{kj}^i on N_p and thereby the connection ∇. On the other hand, as the next theorem shows, the forms ω_j^i are described by the torsion and curvature tensor fields.

THEOREM 2.5. (the ***structure equations*** of Cartan).

$$d\omega^i = -\sum_j \omega_j^i \wedge \omega^j + \frac{1}{2}\sum_{j,k} T_{jk}^i \omega^j \wedge \omega^k , \tag{2.7}$$

$$d\omega_j^i = -\sum_k \omega_k^i \wedge \omega_j^k + \frac{1}{2}\sum_{k,l} R_{jkl}^i \omega^k \wedge \omega^l . \tag{2.8}$$

Proof. If we define the functions c_{jk}^i by $[X_j, X_k] = \sum_i c_{jk}^i X_i$, we obtain

$$\begin{aligned} d\omega^i(X_j, X_k) &= \frac{1}{2}\left\{X_j\omega^i(X_k) - X_k\omega^i(X_j) - \omega^i([X_j, X_k])\right\} \\ &= -\frac{1}{2}c_{jk}^i \end{aligned}.$$

As for the right-band side of this equation, we have

$$\begin{aligned} T(X_j, X_k) &= \nabla_{X_j} X_k - \nabla_{X_k} X_j - [X_j, X_k] \\ &= \sum_i (\Gamma_{jk}^i - \Gamma_{kj}^i - c_{jk}^i) X_i \end{aligned},$$

from which

$$T_{jk}^i = \Gamma_{jk}^i - \Gamma_{kj}^i - c_{jk}^i .$$

We also have

$$\begin{aligned} \left(-\sum_l \omega_l^i \wedge \omega^l\right)(X_j, X_k) &= -\frac{1}{2}\sum_l \left\{\omega_l^k(X_j)\omega^l(X_k) - \omega^l(X_j)\omega_l^i(X_k)\right\} \\ &= \frac{1}{2}(\Gamma_{kj}^i - \Gamma_{jk}^i) \end{aligned}.$$

From these equations we have (2.7).

Similarly, we find

$$R_{jkl}^i = \sum (\Gamma_{lj}^p \Gamma_{kp}^i - \Gamma_{kj}^p \Gamma_{lp}^i) + X_k \Gamma_{lj}^i - X_l \Gamma_{kj}^i - c_{kl}^p \Gamma_{pj}^i ,$$

$$d\omega_j^i(X_k, X_l) = \frac{1}{2}(X_k \Gamma_{lj}^i - X_l \Gamma_{kj}^i - c_{kl}^p \Gamma_{pj}^i),$$

$$\omega_p^i \wedge \omega_j^p (X_k, X_l) = -\frac{1}{2}(\Gamma_{lj}^p \Gamma_{kp}^i - \Gamma_{kj}^p \Gamma_{lp}^i) .$$

From these equations we have (2.8). (QED)

We now define a ***Riemannian metric*** on M. It is a tensor field g of type (0, 2) which satisfies the following two conditions:

(i) It is symmetric: $g(X,Y) = g(Y,X)$ for any $X, Y \in \widetilde{X}(M)$;

(ii) It is positive-definite: $g(X,X) \geq 0$ for every $X \in \widetilde{X}(M)$ and $g(X,X) = 0$ if and only if $X = 0$.

A manifold M with Riemannian metric g is called a ***Riemannian manifold***. A Riemannian metric gives rise to an **inner product** on each tangent space $T_x(M)$ to M at x. Let $\{x^1, \ldots, x^n\}$ be a local coordinate system in M. The components g_{ij} of g with respect to this local coordinate system are given by

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \ i,j=1,\ldots,n.$$

We call g_{ij} the ***covariant components*** of g. The ***contravariant components*** g^{ij} of g are defined by

$$g^{ij} = g(dx^i, dx^j), \ i,j=1,\ldots,n.$$

We have then

$$g_{ij} g^{jk} = \delta_i^k, \quad \delta_i^k = \begin{cases} 1, \_ \ k = i \\ 0 \_ \ k \neq i \end{cases},$$

where, here and in the sequel, if we need, we use the ***Einstein convention***, that is, repeated indices, one upper index and the other lower index, denotes summation over its range. If X^i are components of a vector field X with respect to $\{x^1, \ldots, x^n\}$, that is, $X = X^i \dfrac{\partial}{\partial x^i}$, then the components X_i of the corresponding **covector** or the corresponding **1-form** are related to X^i by

$$X^i = g^{ij} X_j, \quad X_i = g_{ij} X^j .$$

The **inner product** g in the tangent space $T_x(M)$ and in its dual space $T_x^*(M)$ can be extended to an inner product, denoted also by g, in the tensor space T_s^r at x for each type (r, s). If K and L are tensors at x of type (r, s) with components

$$K_{j_1 \cdots j_s}^{i_1 \cdots i_r} \quad \text{and} \quad L_{j_1 \cdots j_s}^{i_1 \cdots i_r}$$

with respect to $\{x^1, \ldots, x^n\}$, then the inner product $g(K,L)$ of K and L is

defined to be

$$g(K,L)=g_{i_1k_1}\cdots g_{i_rk_r}g^{j_1t_1}\cdots g^{j_st_s}K^{i_1\cdots i_r}_{j_1\cdots j_s}L^{k_1\cdots k_r}_{t_1\cdots t_s}.$$

We put

$$|K|=g(K,K)^{1/2},$$

which is called the ***length of the tensor field*** *K* with respect to *g*.

On a Riemannian manifold *M*, the arc length of a differentiable curve $\gamma:t\to\gamma(t)$, $a\le t\le b$, is defined by

$$L(\gamma)=\int_a^b g(\dot\gamma(t),\dot\gamma(t))^{1/2}dt.$$

This definition can be generalized to a piecewise differentiable curve in an obvious manner. The ***distance*** $d(x,y)$ between two points *x* and *y* of *M* is defined by the infimum of the lengths of all piecewise differentiable curves joining *x* and *y*. Then we have

$$d(x,y)=d(y,x),$$
$$d(x,y)+d(y,z)\ge d(x,z),$$
$$d(x,y)\ge 0,$$
$$d(x,y)=0 \quad \text{if and only if} \quad x=y.$$

We can see that the topology defined by the distance function (metric) *d* is the sane as the manifold topology of *M*.

THEOREM 2.6. *On a Riemannian manifold M there exists one and only one affine connection satisfying the following two conditions*:

(1) *The torsion tensor T vanishes*, *i.e*.,

$$T(X,Y)=\nabla_XY-\nabla_YX-[X,Y]=0;$$

(2) *g is parallel, i.e*., $\nabla_Xg=0$.

Proof. *Existence*: Given vector fields *X* and *Y* on *M*, we define ∇_XY by setting

(2.9)

$$\begin{aligned}2g(\nabla_XY,Z)&=Xg(Y,z)+Yg(X,Z)-Zg(X,Y)\\&\quad+g([X,Y],Z)+g([Z,X],Y)+g(X,[Z,Y])\end{aligned}$$

for any vector field *Z* on *M*. Then the mapping $(X,Y)\to\nabla_XY$ defines an affine connection on *M*. From the above definition of ∇_XY, we have $T(X,Y)=0$ and

$$Xg(Y,Z)=g(\nabla_XY,Z)+g(Y,\nabla_XZ),$$

which shows that $\nabla_Xg=0$, that is, ∇ is a ***metric connection*** on *M*.

Uniqueness: By a straightforward computation, we can see that, if ∇_XY satisfies $\nabla_Xg=0$ and $T(X,Y)=0$, then it satisfies the equation which defines ∇_XY. (QED)

The connection ∇ given by (2.9) is called the ***Riemannian connection*** (sometimes called the ***Levi-Civita connection***).

Putting $X = \frac{\partial}{\partial x^j}$, $Y = \frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^k}$ in (2.9), the components Γ^i_{jk} of the Riemannian connection with respect to a local coordinate system $\{x^1, \ldots, x^n\}$ are given by

$$\sum_l g_{lk} \Gamma^l_{ji} = \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^k} \right).$$

Let M and M' be Riemannian manifolds with Riemannian metrices g and g', respectively. A mapping $f : M \to M'$ is called ***isometric*** at a point x of M if $g(X,Y) = g'(f_* X, f_* Y)$ for all $X, Y \in T_x(M)$. In this case, f_* is **injective** at x, because $f_* X = 0$ implies $X = 0$. A mapping f which is isometric at every point of M is thus an **immersion**, which we call an ***isometric immersion***. If, moreover, f is 1 : 1, then it is called an ***isometric imbedding*** of M into M'. If f maps M 1 : 1 onto M', then f is called an ***isometry*** of M onto M'. In this case, the differential of the isometry f commutes with the parallel translation. Moreover, if f is an isometric immersion of M into M' and if $f(M)$ is open in M', then the differential of f commutes the parallel translation and every geodesic of M is mapped by f into a geodesic of M'.

A Riemannian manifold M or a Riemannian metric g on M is said to be ***complete*** if the metric function d is complete, that is, all Cauchy sequences converge. It is well-known that the following conditions on M are equivalent:

(1) M is complete;

(2) Every bounded subset of M with respect to d is relatively compact;

(3) All geodesic arc can be extended in two directions indefinitely with respect to the arc length.

It is also well-known that any two points x and y in M can be joined by a geodesic arc whose length is equal to $d(x,y)$ (Hopf-Rinow [1], de Rham [2]). We can also see that every compact Riemannian manifold is complete.