5. FIBRE BUNDLES AND COVERING SPACES

Let G be a differentiable manifold with a countable basis. If G is a group such that the group operation $(a,b) \in G \times G \to ab^{-1} \in G$ is a differentiable mapping, then G is called a ***Lie group***.

A Lie group is clearly a locally compact group with a countable basis. Let G_0 be the connected component of G containing the identity element of G. We see that G_0 is a closed normal subgroup of G_0. Moreover, since G_0 is locally connected, G_0 is an open submanifold of G. G_0 is also a Lie group.

We denote by L_a (resp. R_a) the left (resp. right) translation of G by an element $a \in G : L_a x = ax$ (resp. $R_a x = xa$) for every $x \in G$. For $a \in G$, *ad* a is the inner automorphism of G defined by $(ad\_a)x = axa^{-1}$ for every $x \in G$. A vector field X on G is called ***left invariant*** (resp. ***right invariant***) if it is invariant by all L_a, i.e., $(L_a)_* X = X$ for all $a \in G$ (resp. R_a, i.e., $(R_a)_* X = X$ for all $a \in G$). Let $\widetilde{T}$ be the set of all left invariant vector fields on G. We call $\widetilde{T}$ the ***Lie algebra*** of the Lie group G. In fact, $\widetilde{T}$ is closed for the usual addition, scalor multiplication and bracket operation. As a vector space, $\widetilde{T}$ is isomorphic with the tangent space $T_e(G)$ at the identity, the isomorphism being given by the mapping which sends $X \in \widetilde{T}$ into X_e, the value of X at e. Thus $\widetilde{T}$ is a Lie subalgebra of dimension n (= dim G) of the Lie algebra of vector fields $\widetilde{X}(G)$.

Every $A \in \widetilde{T}$ generates a (global) 1-parameter group of transformations of G.

We say that a Lie group G is a ***Lie transformation group*** on a manifold M or that G acts on M if the following conditions are satisfied:

(1) Every element $a \in G$ induces a transformation of M, denoted by $x \to xa$, where $x \in M$;

(2) $(a,x) \in G \times M \to xa \in M$ is a differentiable mapping;

(3) $x(ab) = (xa)b$ for all $a, c \in G$ and $x \in M$.

We also write $R_a x$ for xa and say that G acts on M on the right. If we write ax and assume $(ab)x = a(bx)$ instead of (3), we say that G acts on M on the left and write $L_a x$ for ax also. Note that $R_{ab} = R_a \cdot R_b$ and $L_{ab} = L_a \cdot L_b$. From (3) and from the fact that each R_a or L_a is 1 : 1 on M, it follows that R_e and L_e are the identity transformation of M.

We say that G acts ***effectively*** (resp. ***freely***) on M if $R_a x = x$ for all $x \in M$ (resp. for same $x \in M$) implies that $a = e$.

Definition. Let M be a manifold and G a Lie group. A (***differentiable***) ***principal fibre bundle*** over M with group G consists of a manifold P and an action of G on P satisfying the following conditions:

(1) G acts freely on P on the right: $(u,a) \in P \times G \to ua = R_a u \in P$;

(2) M is the quotient space of P by the equivalence relation induced by G, $M = P/G$, and the canonical projection $\pi : P \to M$ is differentiable;

(3) P is locally trivial, that is, every point x of M has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic with $U \times G$ in the sense that there is a diffeomorphism $\psi : \pi^{-1}(U) \to U \times G$ such that $\psi(u) = (\pi(u), \phi(u))$ where ϕ is a mapping of $\pi^{-1}(U)$ into G satisfying $\phi(ua) = (\phi(u))a$ for all $u \in \pi^{-1}(U)$ and $a \in G$.

A principal fibre bundle will be denoted by $P(M,G,\pi)$, $P(M,G)$ or simply P. We call P the ***total space*** or the ***bundle space***, M the ***base space***, G the ***structure group*** and π the ***projection***. For each point x of M, $\pi^{-1}(x)$ is a closed submanifold of P, called the ***fibre*** over x. If u is a point of $\pi^{-1}(x)$, then $\pi^{-1}(x)$ is the set of points ua, $a \in G$, and is called the ***fibre*** through u. Every fibre is diffeomorphic to G.

Given a Lie group G and a manifold M, G acts freely on $P = M \times G$ on the right as follows. For each $b \in G$, R_b maps $(x,a) \in M \times G$ into $(x,ab) \in M \times G$. The principal fibre bundle $P(M,G)$ thus obtained is called ***trivial***.

From local triviality of $P(M,G)$ we see that if W is a submanifold of M, then $\pi^{-1}(W)(W,G)$ is a principal fibre bundle. We call it the ***restriction*** of P to W and denote it by $P|W$.

Given a principal fibre bundle $P(M,G)$, the action of G on P induces a homeomorphism σ of the Lie algebra $\widetilde{T}$ of G into the Lie algebra $\widetilde{X}(P)$ of vector fields on P. σ can be defined as follows: For every u, let σ_u be the mapping $a \in G \to ua \in P$. Then $(\sigma_u)_* A_e = (\sigma A)_u$. For each A, $A^* = \sigma(A)$ is called the ***fundamental vector field*** corresponding to A. Since the action of G sends each fibre into itself, A_u^* is tangent to the fibre at each $u \in P$. As G acts freely on P, A^* never vanishes on P if $A \neq 0$. The dimension of each fibre being equal to that of $\widetilde{T}$, the mapping $A \to (A^*)_u$ of $\widetilde{T}$ into $T_u(P)$ is a linear isomorphism of $\widetilde{T}$ onto the tangent space at u of the fibre through. We also see that for each $a \in G$, $(R_a)_* A^*$ is the fundamental vector field corresponding to $(ad(a^{-1}))A \in \widetilde{T}$.

We now give the concept of **transitive functions**. For a principal fibre bundle $P(M,G)$, we can choose an open covering $\{U_i\}$ of M, each $\pi^{-1}(U_i)$ provided with a diffeomorphism $u \to (\pi(u), \phi_i(u))$ of $\pi^{-1}(U_i)$ onto $U_i \times G$ such that $\phi_i(ua) = (\phi_i(u))a$. If $u \in \pi^{-1}(U_i \cap U_j)$, then $\phi_j(ua)(\phi_i(ua))^{-1} = \phi_j(u)(\phi_i(u))^{-1}$, which shows that $\phi_j(u)(\phi_i(u))^{-1}$ depends only on $\pi(u)$, not on u.

We define a mapping $\psi_{ji} : U_i \cap U_j \to G$ by $\psi_{ji}(\phi(u)) = \phi_j(u)(\phi_i(u))^{-1}$. The family of mappings ψ_{ji} are called ***transitive functions*** of the bundle $P(M,G)$ corresponding to the open covering $\{U_i\}$ of M. It is easy to verify that

(5.1) $\qquad \psi_{ki}(x) = \psi_{kj}(x) \cdot \psi_{ji}(x)$ for $x \in U_i \cap U_j \cap U_k$.

Conversely, we have (cf. Kobayashi-Nomizu [1]):

PROPOSITION 5.1. *Let M be a manifold,* $\{U_i\}$ *an open covering of M and G a Lie group. Given a mapping* $\psi_{ji} : U_i \cap U_j \to G$ *for every non-empty* $U_i \cap U_j$, *in such a way that the relation* (5.1) *are satisfied, we can construct a* (*differentiable*) *principal fibre bundle P*(*M*,*G*) *with transitive functions* ψ_{ji}.

A homomorphism f of a principal fibre bundle $P'(M',G')$ into another principal fibre bundle $P(M,G)$ consists of a mapping $f' : P' \to P$ and a homomorphism $f'' : G' \to G$ such that $f'(u'a') = f'(u')f''(a')$ for all $u' \in P'$ and $a' \in G'$. For the sake of simplicity, we shall denote f' and f'' by the same letter f. Every homomorphism $f : P' \to P$ maps each fibre of P' into fibre of P and hence induces a mapping of M' into M, which will be also denoted by f. A homomorphism $f : P'(M',G') \to P(M,G)$ is called an ***imbedding*** or ***injection*** if the induced mapping $f : M' \to M$ is an imbedding and if $f : G' \to G$ is a monomorphism. By identifying P' with $f(P')$, G' with $f(G')$ and M' with $f(M')$, we say that $P'(M',G')$ is a ***subbundle*** of $P(M,G)$. If, moreover, $M' = M$ and the induced mapping $f : M' \to M$ is the identity transformation of M, $f : P'(M',G') \to P(M,G)$ is called a ***reduction*** of the structure group G of $P(M,G)$ to G'. The subbundle $P'(M',G')$ is called a ***reduced bundle***. Given $P(M,G)$ and a Lie subgroup G' of G, we say that G is reducible to G' if there is a reduced bundle $P'(M',G')$.

Let $P(M,G)$ be a principal fibre bundle and F a manifold on which G acts on the left: $(a,\xi) \in G \times F \to a\xi \in F$. We shall construct a fibre

bundle $E(M,F,G,P)$ associated with P with standard fibre F. On the product manifold $P \times F$, we let G act on the right as follows: $a \in G$ maps $(u,\xi) \in P \times F$ into $(ua, a^{-1}\xi) \in P \times F$. The quotient space of $P \times F$ by this group action is denoted by $E = P \times$. The mapping $P \times F \to M$ which maps (u,ξ) into $\pi(u)$ induces a mapping π_E, called the ***projection***, of E onto M. For each $x \in M$, $\pi_E{}^{-1}(x)$ is called the ***fibre*** of E over x. Every point x of M has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic to $U \times G$. Identifying $\pi^{-1}(u)$ with $U \times G$, we see that the action of G on $\pi^{-1}(U) \times F$ on the right is given by

$$(x,a,\xi) \to (x,ab,b^{-1}\xi) \quad \text{for} \quad (x,a,\xi) \in U \times G \times F \quad \text{and} \quad b \in G.$$

It follows that the isomorphism $\pi^{-1}(U) \approx U \times G$ induces an isomorphism $\pi_E{}^{-1}(U) \approx U \times F$. We can therefore introduce a differentiable structure in E by the requirement that $\pi_E{}^{-1}(U)$ is an open submanifold of E which is diffeomorphic with $U \times F$ under the isomorphism $\pi_E{}^{-1}(U) \approx U \times F$. The projection π_E is then a differentiable mapping of E onto M. We call $E(M,F,G,P)$ or simply E the ***fibre bundle*** over the **base space** M, with **standard fibre** F and **structure group** G, which is associated with the **principal fibre bundle** P.

We recall here some results on **covering spaces**. Given a connected, locally arcwise connected topological space M, a connected space E is called a ***covering space*** over M with projection $p : E \to M$ if every point x of M has a connected open neighborhood U such that each connected component of $p^{-1}(U)$ is open in E and is mapped homeomorphically onto U by p.

Two covering spaces $p : E \to M$ and $p' : E' \to M$ are isomorphic if there exists a homeomorphism $f : E \to E'$ such that $p' \cdot f = p$. A covering space $p : E \to M$ is a ***universal covering space*** if E is simply connected. If M is a manifold, every covering space has a unique structure of manifold such that p is differentiable.

PROPOSITION 5.2. (1) *Given a connected manifold* M, *there is a unique* (*unique up to an isomorphism*) *universal covering manifold*, *which will be denoted by* $\hat{M}$.
(2) *The universal covering manifold* M *is a principal fibre bundle over* M *with group* $\pi_1(M)$ *and the projection* $p : \hat{M} \to M$, *where* $\pi_1(M)$ *is the first homotopy group of* M.

For the proof, see Steenrod [1;pp.67-71].

PROPOSITION 5.3. *Let* M *be a Riemannian manifold with metric* g.

Let $p: E \to M$ be a covering manifold of M. Then p^*g is a Riemannian metric on E. Moreover, E is complete if and only if M is complete.

Example 5.1. Bundle of linear frames: Let M be an n-dimensional manifold. A linear frame u at a point x of M is an ordered basis $X_1,\ldots,X_n$ of $T_x(M)$. Let $L(M)$ be the set of all linear frames u at all points of M and let π be the mapping of $L(M)$ onto M which maps a linear frame u at x into x. The general linear group $GL(n;R)$ acts on $L(M)$ on the right as follows. Let $a = (a^i_j) \in GL(n;R)$ and $u = (X_1,\ldots,X_n)$ be a linear frame at x. Then ua is the frame $(Y_1,\ldots,Y_n)$ at x defined by $Y_i = \sum_j a^j_i X_j$. It is clear that $GL(n;R)$ acts freely on $L(M)$ and $\pi(u) = \pi(v)$ if and only if $v = ua$ for some $a \in GL(n;R)$. Let $(x^1,\ldots,x^n)$ be a local coordinate system in a coordinate neighborhood U in M. Every frame u at $x \in U$ can be expressed uniquely in the form $u = (X_1,\ldots,X_n)$ with $X_i = \sum_k X^k_i \dfrac{\partial}{\partial x^k}$, where (X^k_i) is a non-singular matrix. This shows that $\pi^{-1}(U)$ is in 1 : 1 correspondence with $U \times GL(n;R)$. We can make $L(M)$ into a differentiable manifold by taking (x^j) and (X^k_i) as a local coordinate system in $\pi^{-1}(U)$. It is easy to verify that $L(M)(M,GL(n;R))$ is a principal fibre bundle. We call $L(M)$ the ***bundle of linear frames*** over M. A linear frame u at $x \in M$ can be defined as a non-singular linear mapping of R^n onto $T_x(M)$. The two definitions are related to each other as follows. Let $e_1,\ldots,e_n$ be the natural basis for R^n: $e_1 = (1,0,\ldots,0),\ldots,$ $e_n = (0,\ldots,0,1))$. A linear frame $u = (X_1,\ldots,X_n)$ at x can be given as a linear napping $u: R^n \to T_x(M)$ such that $ue_i = X_i$ for i = 1,...,n. The action of $GL(n;R)$ on $L(M)$ can be accordingly interpreted as follows. Consider $a = (a^i_j) \in GL(n;R)$ as a linear transformation of R^n which maps e_j into $\sum_i a^i_j e_i$. Then $ua: R^n \to T_x(M)$ is the composite of the following two mappings:

$$R^n \xrightarrow{a} R^r \xrightarrow{u} T_x(M).$$

Example 5.2. **Tangent bundle**: Let $GL(n;R)$ act on R^n as above. The tangent bundle $T(M)$ over M is the bundle associated with $L(M)$ with standard fibre R^n. The fibre of $T(M)$ over $x \in M$ may be considered as $T_x(M)$.

Example 5.3. **Tensor bundle**: Let T_s^r be the tensor space of type (r, s) over the vector space R^n. The group $GL(n;R)$ can be regarded as a group of linear transformations of T_s^r. With this standard fibre T_s^r, we obtain the tensor bundle $T_s^r(M)$ of type (r, s) over M which is associated with $L(M)$. It is easy to see that the fibre of $T_s^r(M)$ over $x \in M$ may be considered as the tensor space over $T_x(M)$ of type (r, s).

Example 5.4. **Vector bundle**: Let F be either the real number field R or the complex number field C, F^n the vector space of all n-tuples of elements of F and $GL(n;F)$ the group of all (n,n)-non-singular matrices with entries from F. $GL(n;F)$ acts on F^n on the left in a natural manner; if $a = (a_j^i) \in GL(n;F)$ and $\xi = (\xi^1, \ldots, \xi^n) \in F^n$, then $a\xi = (\sum_j a_j^1 \xi^j, \ldots, \sum_j a_j^n \xi^j) \in F^n$. Let $P(M,G)$ be a principal fibre bundle and ρ a representation of G into $GL(n;F)$. Let $E(M, F^n, G, P)$ be the associated bundle with standard fibre F^n on which G acts through ρ. We call E a ***real*** or ***complex vector bundle*** over M according as $F = R$ or $F = C$. Each fibre $\pi_E^{-1}(x)$, $x \in M$, of E has the structure of a vector space such that every $u \in P$ with $\pi(u) = x$, considered as a mapping of F^n onto $\pi_E^{-1}(x)$, is a linear isomorphism of F^n onto $\pi_E^{-1}(x)$.

We give examples of universal covering manifolds.

Example 5.5. Let $e_1, \ldots, e_n$ be any basis of R^n, and let G be the subgroups of R^n generated by $e_1, \ldots, e_n$: $G = \{\sum m_i e_i : m_i\text{\_integers}\}$.

The action of G on R^n is properly discontinuous and R^n is the universal covering manifold of R^n/G. The quotient manifold R^n/G is called an n-dimensional ***torus***.

Example 5.6. Let $S^n = \{(x^1, \ldots, x^{n+1}) \in R^{n+1} : \sum (x^i)^2 = 1\}$ and G be

the group consisting of the identity transformation of S^n and the transformation of S^n which maps (x^i) into $(-x^i)$. Then S^n ($n \geq 2$) is the universal covering manifold of S^n/G. The quotient manifold S^n/G is called the n-dimensional ***real projective space***.