CHAPTER I. ELEMENTS OF GAS DYNAMICS AND CLASSICAL THEORY OF SHOCK WAVES

1. Continuous Flow of an Inviscid and Non-Thermally-Conducting Gas

§ 1. Equations of Gas Dynamics

For high compression of liquids (and solid bodies) there are needed pressures of hundreds of thousands of atmospheres and above. Therefore, under usual conditions a liquid can be considered as an incompressible medium. Velocities of flow of liquid during small changes of density are much less than speed of sound, which is the scale of velocity characterizing a continuous medium. During small changes of density and motions which are slow as compared to speed of sound, gas also can be considered to be incompressible, and its motion can be described with the help of hydrodynamics of an incompressible fluid. However, large changes of density and velocity of flow comparable with speed of sound in gasses, in distinction from liquids, are attained comparatively easily; at pressure drops of order of magnitude of the actual pressure, i.e., at $\Delta p \sim 1$ atm, if initial pressure of gas is atmospheric. Under such conditions it is necessary to consider compressibility of the substance. Equations of gas dynamics thus differ from equations of hydrodynamics of an incompressible fluid in that in them there is considered possibility of large changes of density of the substance.

State of moving gas with known thermodynamic properties is determined by specifying speed, density, and pressure as functions of coordinates and time. For finding these functions there serves the system of equations of gas dynamics, which is composed, in differential form, of the general laws of conservation of mass, momentum and energy of a substance.

Let us write these equations without derivation, which can be found, for instance, in book of L. D. Landau and Ye. M. Lifshits [1]. We will disregard action of body forces (gravity), and also viscosity and thermal conduction of substance.[1] Let us designate by $\frac{\partial}{\partial t}$ partial derivative with respect to time referred to a given point of space, the local

[1] Equation of gas dynamics taking into account viscosity and thermal conductivity will be considered in § 20.

derivative, and by $\frac{D}{Dt}$ the particle derivative, which characterizes change in time of some quantity, connected with a given moving particle of substance. If $\boldsymbol{u}$ is velocity vector of particle with components u_x, u_y, u_z or u_i, where $i = 1, 2, 3$, then

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \qquad (1.1)$$

First equation is **continuity equation**; it indicates conservation of mass of the substance, i.e., to the fact that change of density ρ in given element of volume occurs due to inflow (or outflow) of substance into this element:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{u} = 0 . \qquad (1.2)$$

With help of definition (1.1), continuity equation can be written in the form

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{u} = 0 . \qquad (1.3)$$

In the particular case of an incompressible liquid, when $\rho = const$, continuity equation is simplified:

$$\nabla \cdot \boldsymbol{u} = 0 . \qquad (1.4)$$

Second equation expresses **Newton's law** and does not differ from equation of motion of an incompressible liquid (p is pressure):

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p , \qquad (1.5)$$

or, in the form of Euler's equation,

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p . \qquad (1.6)$$

As it is easy to verify by means of direct calculation, equation of motion together with continuity equation is equivalent to law of conservation of momentum, written in a form analogous to equation (1.2):

$$\frac{\partial}{\partial t} \rho u_i - -\frac{\partial \Pi_{ik}}{\partial x_k} , \qquad (1.7)$$

where Π_{ik} is tensor of momentum flux density

$$\Pi_{ik} = \rho u_i u_k + p \delta_{ik} . \qquad (1.8)$$

Equation (1.7) expresses the fact that change of i-th component of momentum at given point of space is connected with outflow (inflow) of momentum together with mass (ifsulmto ooo

forces of pressure (second term).[2]

Third equation is essentially new as compared to hydrodynamics of an incompressible liquid and is equivalent to **first law of thermodynamics**, i.e., the law of conservation of energy. It is possible to read it thus: change of specific internal energy ε of a given particle of substance occurs due to work of compression, which is produced on it by its surrounding medium, and also due to energy release from outside sources:

$$\frac{D\varepsilon}{Dt} + p\frac{DV}{Dt} = Q. \tag{1.9}$$

Here $V = 1/\rho$ is specific volume, and Q is energy release per second per gram of substance from external sources (Q can also be negative if there are non-mechanical losses of energy, for instance due to radiation).

With help of continuity equation and equation of motion, the energy equation also can be reduced to a form similar to (1.2), (1.7)

$$\frac{\partial}{\partial t}\left(\rho\varepsilon + \frac{\rho u^2}{2}\right) = -\nabla \cdot \left[\rho u\left(\varepsilon + \frac{u^2}{2}\right) + pu\right] + \rho Q. \tag{1.10}$$

Physical meaning of this equation is that change of total energy of unit of volume at given point of space occurs due to outflow (inflow) of energy during motion of the substance, work of forces of pressure and energy release from external sources.

The continuity, motion and energy equations form a system of five equations (equation of motion is a vector equation and equivalent to three scalar equations) in five unknown functions of coordinates and time: ρ, u_x, u_y, u_z, p. External sources of energy Q are considered to be given, and internal energy ε can be expressed in terms of density and pressure, since thermodynamic properties of substance are also assumed to be known: $\varepsilon = \varepsilon(p, \rho)$.

If energy, as this frequently happens, is known not as a function of pressure and density, but either as a function of temperature T and density or temperature and pressure, then to this system we should add the equation of state of substance $p = f(T, \rho)$. Equation of state of ideal gas has the form

$$pV = RT, \quad p = R\rho T, \tag{1.11}$$

[2] In the right side of formula (1.7) there is produced summation over the twice met index k ($k = 1, 2, 3$); $\delta_{ik} = 1$ when $i = k$ and $\delta_{ik} = 0$ when $i \neq k$.

where R is the gas constant per unit mass.[3]

Energy equation (1.9) has general significance and is valid even when substance is not in thermodynamic equilibrium. In that particular case, which is most important in practice, when substance is in thermodynamic equilibrium, it is possible to write it in different form with help of the second law of thermodynamics

$$TdS = ds + pdV\ , \tag{1.12}$$

where S is specific entropy. In absence of external sources of heat, the third equation of gas dynamics is equivalent to equation of constancy of entropy of a particle, i.e., to condition of adiabatic flow

$$\frac{DS}{Dt} = 0\ . \tag{1.13}$$

In an ideal gas with constant heat capacity, entropy is especially simply expressed in terms of pressure and density (specific volume)

$$S = c_v \ln pV^{\gamma} + const\ , \tag{1.14}$$

where γ is adiabatic index, equal to specific heat ratio at constant pressure and constant volume $\gamma = \dfrac{c_p}{c_v} = 1 + \dfrac{R}{c_v}$. In this case adiabatic equation (1.13) (or energy equation) can be directly written in form of differential equation relating pressure and density (pressure and volume)

$$\frac{1}{p}\frac{Dp}{Dt} + \gamma\frac{1}{V}\frac{DV}{Dt} = 0\ . \tag{1.15}$$

To this system of differential equations of gas dynamics must be added corresponding initial and boundary conditions.

§ 2. Lagrangian Coordinates

Equations in which gas dynamic quantities are considered as functions of spatial coordinates and time are called equations in Eulerian form or ***equations in Eulerian coordinates***.

In the case of one-dimensional motions, i.e., plane, cylindrical and spherically symmetric, we frequently use other, ***Lagrangian coordinates***. In distinction from an Euler coordinate, a Lagrangian coordinate is connected not with a fixed point of space, but with a definite particle of substance. Gas dynamic quantities expressed as functions of Lagrangian coordinates characterize changes of density, pressure and velocity of every particle of substance with flow of time.

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[3] $R = \widetilde{R}/\mu_0$, where $\widetilde{R}$ is universal gas constant, and μ_0 is molecular weight.

Lagrangian coordinates are especially convenient in examining **internal processes** occurring in a substance which do not go beyond the bounds of a given particle; let us say a **chemical reaction**, the flow of which with passage of time depends on change of temperature and density of the particle. Introduction of Lagrangian coordinates in a number of cases permits us to more briefly and easily find exact solutions of equations of gas dynamics, or makes numerical integration of the latter more convenient.

Time derivative in Lagrangian coordinates is equivalent simply to particle derivative $\frac{D}{Dt}$. Particle can be characterized by mass of substance, which distinguishes it from some other fixed particle, or by its coordinate at initial moment of time.

Introduction of Lagrangian coordinates is especially simple in the plane case, when motion depends only on one cartesian coordinate *x*. Let us designate current Eulerian coordinate of considered particle by *x*, and coordinate of some fixed particle by x_1. (as the fixed particle there may be, for instance, selected a particle near a solid wall or near boundary between gas and vacuum, if such exist in the problem). Then the mass of a column of unit cross section between considered particle and fixed particle is equal to

$$m = \int_{x_1}^{x} \rho dx\,, \tag{1.16}$$

and increment of mass upon transition from the particle to a neighboring one is

$$dm = \rho dx\,. \tag{1.17}$$

Quantity *m* can be selected as a Lagrangian coordinate.

If at initial moment, as this frequently happens, gas is at rest, and its density is constant, $\rho(x,0) = \rho_0$, then as Lagrangian coordinate it is convenient to take initial coordinate of particle measured from point x_1; we will designate it by *a*. Then

$$a = \int_{x_1}^{x} \frac{\rho}{\rho_0} dx\,, \quad da = \frac{\rho}{\rho_0} dx\,. \tag{1.18}$$

Equations of **plane** motion of gas in Lagrangian coordinates takes on a simple form. Equation of continuity, written with respect to specific volume $V = 1/\rho$ and unique *x* component of velocity *u* is

$$\frac{\partial V}{\partial t} = \frac{\partial u}{\partial m} \quad \text{or} \quad \frac{1}{V_0}\frac{\partial V}{\partial t} = \frac{\partial u}{\partial a}\,. \tag{1.19}$$

Here, as in subsequent equations, time derivative is particle derivative $\frac{D}{Dt}$, but it is better to write it in the form of partial derivative $\frac{\partial}{\partial t}$ in order to stress that it is taken at *m* and *a* = const, i.e., for a given particle with definite Lagrangian coordinate *m* or *a*. Equation of motion in Lagrangian coordinates has the form

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial m} \text{ or } \frac{\partial u}{\partial t} = -V_0 \frac{\partial p}{\partial a}. \qquad (1.20)$$

The equation of energy written in form (1.9) or in form of adiabatic condition (1.13) (in the absence of external sources of heat and dissipative processes - viscosity and thermal conduction), they retain their form; it is necessary only to replace designation $\frac{D}{Dt}$ by $\frac{\partial}{\partial t}$. In an ideal gas with constant heat capacity, condition of adiabaticity (1.13) gives

$$pV^{\gamma} = f[S(m)], \qquad (1.21)$$

where function *f* depends only on entropy of given particle *m*. In so-called **isentropic motion**, when entropies of all particles are identical and do not change in time, *f* = const, where equation $pV^{\gamma} = const$ is valid in Lagrangian as well as in Eulerian coordinates.

It is essential that in the plane case, Euler coordinate *x* in explicit form is not contained in equation. After the Lagrangian equations are solved and the function *V*(*m*, *t*) is solved, the dependence of flow variables on the Eulerian cooedinate may be obtained by integrating (1.17)

$$dx = V(m,t)dm, \quad x(m,t) = \int_0^m V(m,t)dm + x_1(t). \qquad (2.22)$$

In **cylindrical** and **spherical** cases, equations of gas dynamics in Lagrangian coordinates are somewhat more complicated than in the plane case, since now in the equations there is contained in explicit form the Eulerian coordinate, and in the system of equations there is included an additional equation relating Lagrangian and Eulerian coordinates. For instance, in spherical case, Lagrangian coordinate can be defined as mass included inside spherical volume near center of symmetry:

$$m = \int_0^r 4\pi r^2 \rho dr, \quad dm = 4\pi r^2 \rho dr, \qquad (2.23)$$

If at initial moment, density of gas is constant, it is possible to take as

Lagrangian coordinate initial radius r_0 of the "particle", considered as an elementary spherical shell:

$$\frac{4\pi r_0^{\,3}}{3}\rho_0 = \int_0^r 4\pi r^2 \rho dr\ ,\quad dr_0 = \frac{r^2}{r_0^{\,2}}\frac{\rho}{\rho_0}dr\ . \qquad (2.24)$$

Equation of continuity in spherical Lagrangian coordinates is

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial m}4\pi r^2 u \quad \text{or} \quad \frac{1}{V_0}\frac{\partial V}{\partial t} = \frac{1}{r_0^{\,2}}\frac{\partial}{\partial r_0}r^2 u\ , \qquad (2.25)$$

Equation of motion

$$\frac{\partial u}{\partial t} = -4\pi r^2 \frac{\partial p}{\partial m} \quad \text{or} \quad \frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{r^2}{r_0^{\,2}}\frac{\partial p}{\partial r_0}\ . \qquad (2.26)$$

Energy equation or adiabatic equation remain the same as in the plane case. As an additional equation, in the system there is included differential (or integral) relationship (1.23) or (1.24), which relates m and r or r_0 and r.

Equations for cylindrical case are formed fully analogously to the spherical case. It is necessary to note that in two-dimensional and three-dimensional flows, transition to Lagrangian coordinates, as a rule, is not advantageous, since equations are then greatly complicated.

§ 3. Sound Waves

Speed of sound is included in equations of gas dynamics as the speed of propagation of small perturbations. In the limiting case, when change of density $\Delta\rho$ and pressure Δp during fluid motion are very small as compared to mean values of density ρ_0 and pressure p_0, and velocities are small as compared to speed of sound c, equations of gas dynamics are transformed into equations of acoustics describing the propagation of sound waves.

Let us write density and pressure in the form $\rho = \rho_0 + \Delta\rho$, $p = p_0 + \Delta p$, and consider quantities $\Delta\rho$, Δp, and also velocity u as small quantities. Disregarding quantities of second order of smallness, we will transform Eulerian equations of continuity and motion for the plane case. Equation of continuity gives

$$\frac{\partial \Delta\rho}{\partial t} = -\rho_0 \frac{\partial u}{\partial x}\ . \qquad (1.27)$$

The equation of motion takes the form

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} = -\left(\frac{\partial p}{\partial \rho}\right)_S \frac{\partial \Delta\rho}{\partial x}\ . \qquad (1.28)$$

In the last transformation it is taken into account that motion in sound

wave is adiabatic. Therefore, small change of pressure is associated with small change of density through the adiabatic derivative: $\Delta p = \left(\frac{\partial p}{\partial \rho}\right)_S \Delta\rho$. This derivative constitutes, as we will now see, the square of the speed of sound

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_S \quad (1.29)$$

and corresponds to unperturbed state of fluid.

Differentiating (1.27) with respect to time, and (1.28) with respect to space coordinate, we will eliminate cross derivative $\frac{\partial^2 u}{\partial t}\partial x$ and obtain wave equation for change of density

$$\frac{\partial^2 \Delta\rho}{\partial t^2} = c^2 \frac{\partial^2 \Delta\rho}{\partial x^2}. \quad (1.30)$$

The similar equation is satisfied by the magnitude of change of pressure Δp, which is proportional to $\Delta p = c^2 \Delta\rho$, and also by velocity u and all other parameters of the fluid, for instance, temperature.[4] Wave equation of type (1.30) admits two families of solutions:

$$\Delta\rho = \Delta\rho(x - ct), \quad \Delta p = \Delta p(x - ct), \quad \Delta u = \Delta u(x - ct) \quad (1.31)$$

and

$$\Delta\rho = \Delta\rho(x + ct), \quad \Delta p = \Delta p(x + ct), \quad \Delta u = \Delta u(x + ct) \quad (1.32)$$

(by c we mean the positive root $c = +\sqrt{\left(\frac{\partial p}{\partial \rho}\right)_S}$).

The first solution describes perturbation propagating in the direction of positive x axis, and the second describes perturbation propagating in the opposite direction. In the first case, for instance, given value of density corresponds to a definite value of the argument x - ct, i.e., with flow of time it goes in the direction of positive x with velocity c. Thus, c is velocity of propagation of sound waves.

Noticing that $\frac{\partial}{\partial x} u(x \mp ct) = \mp \frac{1}{c}\frac{\partial}{\partial t} u(x \mp ct)$, and taking into

[4] In order to obtain wave equation for velocity, we will differentiate equation (1.30) with respect to time and use equations (1.27) and (1.28):

$$\frac{\partial^3 \Delta\rho}{\partial t^3} = c^2 \frac{\partial^3 \Delta\rho}{\partial x^2 \partial t} = -c^2 \rho_0 \frac{\partial}{\partial x}\frac{\partial^2 u}{\partial t^2} = -c^2 \rho_0 \frac{\partial}{\partial x}\frac{\partial^2 u}{\partial x^2},$$

from which $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f(t)$. Noticing that before the wave in unperturbed fluid ahead of the wave $u = 0$, we will find that $f(t) = 0$.

account the fact that in undisturbed gas before the wave $u = 0$, $\Delta\rho = 0$ (see footnote), we will find with help of equation (1.27) the relation between the particle velocity of gas u and changes of density or pressure:

$$u = \pm \frac{c}{\rho_0} \Delta\rho = \pm \frac{\Delta p}{\rho_0 c}, \quad \Delta p = c^2 \Delta\rho = \pm \rho_0 c u . \qquad (1.33)$$

Upper sign refers to wave travelling in the direction of positive x, and the lower sign to wave travelling in the direction of negative x. In both cases particle velocity is in the direction of propagation of the wave where the fluid is compressed, and in the opposite direction, where it is rarefied.

General solution of wave equations for $\Delta\rho$ and u is composed of two particular solutions, which correspond to waves travelling in positive and negative directions of the x axis. According to (1.31), (1.32) and (1.33), solutions for density and velocity can be written in the following form:

$$\Delta\rho = \frac{\rho_0}{c} f_1(x - ct) + \frac{\rho_0}{c} f_2(x + ct) , \qquad (1.34)$$

$$u = f_1(x - ct) - f_2(x + ct) , \qquad (1.35)$$

where f_1 and f_2 are arbitrary functions of their arguments which are determined by initial distributions of density and velocity:

$$f_1 = \frac{1}{2}\left[\frac{c}{\rho_0}\Delta\rho(x,0) + u(x,0)\right],$$

$$f_2 = \frac{1}{2}\left[\frac{c}{\rho_0}\Delta\rho(x,0) - u(x,0)\right].$$

For instance, if at initial moment there is a rectangular perturbation

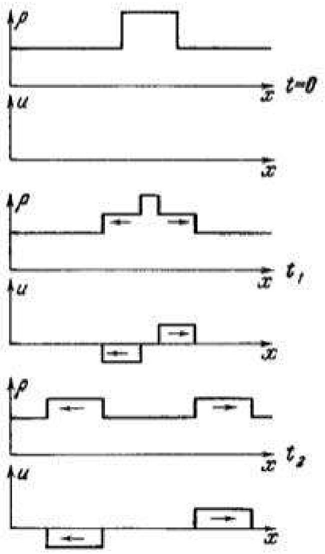

Fig.1.1 Propagation of a rectangular density and pressure pulse in linear acoustics.

of density, and gas everywhere is motionless, then to the right and to the left there begin to travel rectangular perturbations, as shown in Fig. 1.1. If at initial moment, distributions of density and velocity have the form depicted in Fig. 1.2, where $u = \frac{c}{\rho_0} \Delta\rho$, so that $f_2 = 0$, then rectangular pulses will travel only in one direction. (Such a perturbation

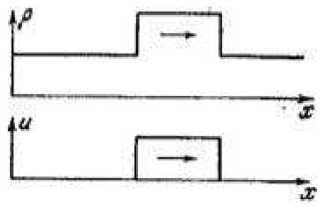

Fig.1.2 Propagation of a rectangular density and pressure pulse in linear acoustics.

can be created by a piston which at initial moment starts to be thrust into gas at rest with constant velocity *u*, and after a certain time is "instantaneously" stopped. If length of rectangular pulse is equal to *L*, then, obviously, time of action of piston is $t_1 = L/c$).

Special importance for acoustics is presented by **monochromatic** sound waves, in which all quantities are periodic functions of time of the type

$$f = A\cos\left(\frac{\omega}{c}x - \omega t\right),$$

or, in complex form,

$$f = A\exp\left[-i\omega\left(t - \frac{x}{c}\right)\right].$$

where $\nu = \frac{\omega}{2\pi}$ is frequency of sound, and $\lambda = \frac{c}{\nu}$ is wave length. Any perturbation can be expanded in a Fourier integral, i.e., can be represented in the form of a set of monochromatic waves with different frequencies.

Sounds perceived by human ear have frequency ν from 20 to 20,000 cps (oscillations per second) and wave lengths corresponding to speed of sound in atmospheric air c = 350 m/sec,[5] from 15 m to 1.5 cm.

For an idea of the numerical values of different quantities in a sound wave, we will indicate that for the strongest sound, which is 10^5 times

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[5] Adiabatic index (specific heat ratio) of air under normal conditions is

$$\gamma = 1.4 \quad \text{and} \quad c = \left(\frac{\partial p}{\partial \rho}\right)_S^{1/2} = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2} = (\gamma R T_0)^{1/2}$$

(since $p \sim \rho^\gamma$ for constant *S*).

more intense[6] than the fortissimo of an orchestra, amplitude of change of air density in wave is 0.4% of normal density; amplitude of change of pressure is 0.56% of atmospheric; amplitude of velocity is 0.4% of velocity of sound, i.e., 1.3 m/sec. Amplitude of displacement of particles of air is of the order of $\Delta x \approx \dfrac{u}{2\pi\nu} = \dfrac{u}{c}\dfrac{\lambda}{2\pi} \approx 6\times 10^{-4}\lambda$ ($\Delta x \approx 0.036$ cm for $\nu = 500$ cps).

Let us find energy connected with small a perturbation which is propagated through a gas at rest. Increase of specific internal energy of perturbed substance with accuracy up to terms of the second order of smallness with respect to $\delta\rho$ (or Δp, or u) is

$$\varepsilon - \varepsilon_0 = \left(\frac{\partial\varepsilon}{\partial\rho}\right)_0 \Delta\rho + \frac{1}{2}\left(\frac{\partial^2\varepsilon}{\partial\rho^2}\right)_0 (\Delta\rho)^2 .$$

Since the motion is isentropic, the derivatives are taken at constant entropy. It is possible to calculate them with help of thermodynamic relationship: $d\varepsilon = TdS - pdV = \dfrac{p}{\rho^2}d\rho$. We obtain

$$\varepsilon - \varepsilon_0 = \frac{p_0}{\rho_0^{\,2}}\Delta\rho + \frac{c^2}{2\rho_0^{\,2}}(\Delta\rho)^2 - \frac{p_0}{\rho_0^{\,3}}(\Delta\rho)^2 .$$

The increment of internal energy in unit volume to the same accuracy is equal to

$$\begin{aligned}\rho\varepsilon - \rho_0\varepsilon_0 &= (\rho_0 + \Delta\rho)(\varepsilon - \varepsilon_0) + \varepsilon_0\Delta\rho \\ &= \left(\varepsilon_0 + \frac{p_0}{\rho_0}\right)\Delta\rho + \frac{c^2}{2\rho_0}(\Delta\rho)^2 , \\ &= h_0\Delta\rho + \frac{c^2}{2\rho_0}(\Delta\rho)^2\end{aligned}$$

where $h = \varepsilon + p/\rho$ is the specific enthalpy.

The **density of internal energy** connected with the perturbation, in the first approximation, is proportional to $\Delta\rho$. The **density of kinetic energy** $\dfrac{\rho u^2}{2} \approx \dfrac{\rho_0 u^2}{2}$ is a quantity of the second order of smallness. From relationship (1.33), which holds for a plane travelling wave, it is clear that the term of the second order in internal energy density and the kinetic energy are exactly equal to each other, so that **total energy**

[6] As will be shown below, energy or intensity of sound is proportional to square of amplitude of changes of pressure or density. Loudness of sound is measured in decibels, in logarithmic scale. As zero is taken average threshold of sensitivity of the human ear. Increase of loudness by n decibels signifies increase of energy of sound by $10^{n/10}$ times. Increase of loudness from rustle of leaves or whisper (~10 db) to orchestra fortissimo (~80 db) corresponds to increase of energy of sound by 10^7 times.

density of perturbation is

$$E = h_0 \Delta\rho + \frac{c^2}{2\rho_0}(\Delta\rho)^2 + \frac{\rho_0 u^2}{2} \\ = h_0 \Delta\rho + \rho_0 u^2 . \tag{1.36}$$

The first-order change in the above expression is related to the changes in total volume of the gas that occurred as a result of the disturbance. If perturbation was created in such a way that volume of gas on the whole was not changed, then energy of perturbation of all of the gas is a quantity of the second order with respect to $\Delta\rho$, since during integration over volume, the term proportional to $\Delta\rho$ vanishes. Such, for instance, is the situation in a wave packet which propagates through gas occupying an infinite space, where at infinity the gas is not perturbed (Fig. 1.3). Changes of density in regions of compression, with accuracy up to terms of the second order are compensated by changes in regions of rarefaction.

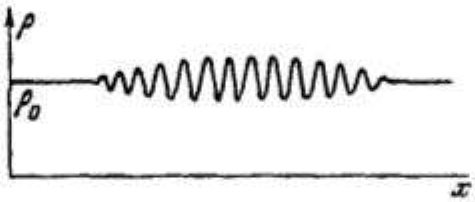

Fig.1.3 Distribution of density in a wave packet.

Thus, energy of sound is a quantity of the second order of smallness which is proportional to square of amplitude:[7]

$$E_{SW} = \rho_0 u^2 . \tag{1.37}$$

If perturbation was created in such a way that volume of gas was changed, then in energy of perturbation there remains a term which is proportional to first power of $\Delta\rho$. However, this main fraction of energy, which is proportional to $\Delta\rho$, may be "returned by the gas," if source of perturbation returns to its own initial position. Energy then remaining in the perturbed gas will constitute only a quantity of the second order of smallness.

Let us explain this situation in a simple example. Assume that at the initial moment a piston begins to move into the undisturbed gas with a constant velocity u (much smaller than speed of sound, $u \ll c$). At time

[7] Expression (1.37) should be averaged over time or space:

$E_{SW} = \rho_0 \overline{u^2}$ ($\overline{u} \sim \overline{\Delta\rho} \sim \overline{\Delta p} = 0$, while $\overline{u^2} \sim \overline{(\Delta\rho)^2} \sim \overline{(\Delta p)^2} > 0$).

t_1 the piston stops "instantaneously". A compression pulse of length $(c-u)t_1 \approx ct_1$, whose energy is equal to the work done by the external force in moving the piston, $put_1 = (p_0 + \Delta p)ut_1 \approx p_0 ut_1$, will travel through the gas (this case was considered above and is illustrated in Fig. 1.4). Energy in the first approximation is proportional to "amplitude" of

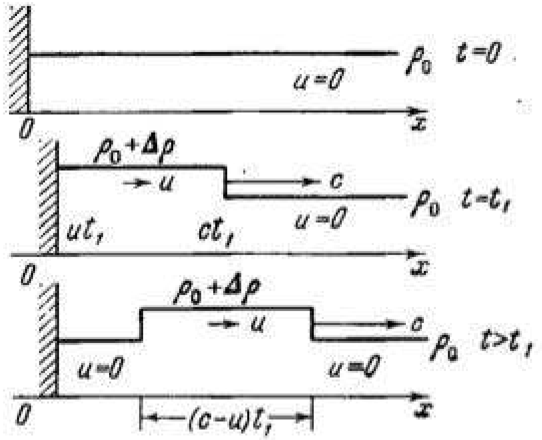

Fig. 1.4 Propagation of a compression pulse created by a piston moving in a gas.

wave u, $\Delta\rho$, Δp and time of compression (i.e., length of the perturbation). Let us now give the gas the possibility to return the piston to its place in such a way that at time t_1, velocity of piston u "instantaneously" changes to the opposite, (-u), and at the time $t_2 = 2t_1$, the piston, which has returned to the initial position, "instantaneously" stops. Perturbation will now have the form depicted in Fig. 1.5, where there are shown states at moments $t = 0$, t_1, t_2 and $t > t_2$. It is easy to verify by direct calculation that in the second period, from t_1 to t_2, the gas performed on the piston work which in first approximation is exactly equal to work which was accomplished by the piston on the gas in the first period from zero to t_1. Lengths of positive and negative regions of the pulse in first approximation are also identical and are equal to $ct_1 = c(t_2 - t_1)$. Thus, if we sum the energies in compressed and rarefied regions of the pulse, then terms of first order will cancel out. If we carry out all calculations taking into account terms of following order,[8] then in the energy there will remain term of the second order, where perturbation energy density will be expressed by general formula (1.37).

§ 4. Spherical Sound Waves

In the absence of absorption (i.e., without taking into account viscosity and thermal conduction; see § 22), amplitude and density of

[8] In particular, lengths of pulses of compression and rarefaction will differ by the amount $2ut_1$ (for $t_2 - t_1 = t_1$).

energy of **plane waves** do not decrease with flow of time. For instance, pulses depicted in Fig. 1.4 and 1.5 depart to "infinity," without changing their shape and amplitude.

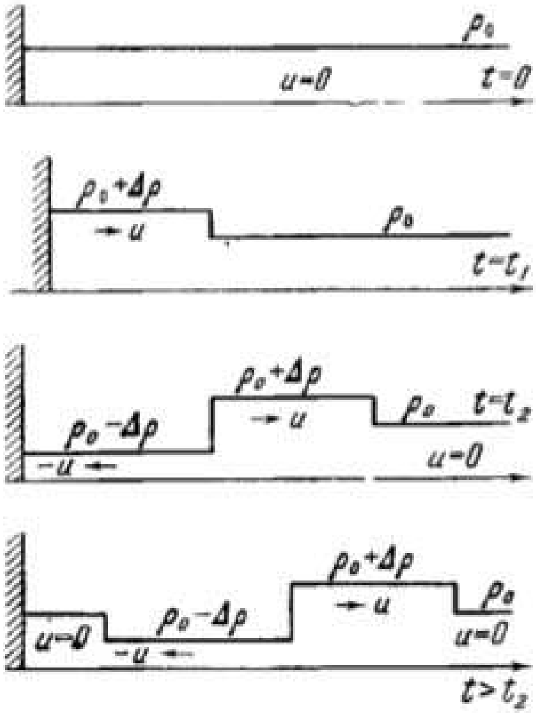

Fig.1.5 Propagation of pulses of compression and rarefaction from a piston which was first thrust into the gas, and then returned to its original position.

In **spherical wave** this is no longer so. By linearizing equation of continuity in the spherically symmetric case, we will obtain

$$\frac{\partial \Delta\rho}{\partial t} = -\frac{\rho_0}{r^2}\frac{\partial}{\partial r} r^2 u .$$

The linearized equation of motion is the same as (1.28)

$$\frac{\partial u}{\partial t} = -\frac{c^2}{\rho_0}\frac{\partial \Delta\rho}{\partial r} .$$

Hence, as in the plane case, we obtain wave equation in the spherical case for $\Delta\rho$, solution of which, which describes the wave going out from the center, is

$$\Delta\rho = \frac{f(r-ct)}{r} . \tag{1.38}$$

If we consider short pulses, of length much less than r, then it is possible to say that shape of pulse given by function $f(r - ct)$ does not change, and amplitude of wave decreases proportionally to $1/r$. This is completely natural. Let us assume that from the center there proceeds a pulse of finite width Δr. With propagation of the pulse, the mass of substance involved in motion, which is equal approximately to $\rho_0 4\pi r^2 \Delta r$, increases proportionally to r^2. Acoustical energy of unit

of volume is proportional to $(\rho)^2$. Since the total energy is conserved, $(\Delta\rho)^2 r^2 = const$, and the amplitude should decrease as $\Delta\rho \sim 1/r$.

The spherical wave differs from a plane wave in yet one more respect. Let us substitute solution (1.38) in equation of motion

$$\frac{\partial u}{\partial t} = -\frac{c^2}{\rho_0}\left[\frac{f7(r-ct)}{r} - \frac{f(r-ct)}{r^2}\right],$$

and integrate the resulting equation with respect to time. We obtain the following solution for velocity

$$u = \frac{c}{\rho_0}\left[\frac{f(r-ct)}{r} - \frac{\int^{r-ct} f(\xi)d\xi}{r^2}\right] = \frac{c}{\rho_0}\left[\Delta\rho - \frac{\phi(r-ct)}{r^2}\right], \tag{1.39}$$

which differs from formula for plane case (1.33) by the presence of an additional term. In the plane wave in region of perturbation, the fluid can be only compressed, as this occurs in the case depicted in Fig, 1.4. In a **spherical** wave this is impossible; behind the region of compression there necessarily follows a region of rarefaction.

Indeed, behind the region of perturbation, $\Delta\rho$ and u become zero. In the plane case, in virtue of proportionality $u \sim \Delta\rho$, this condition is satisfied automatically, independently of shape of pulse. In **spherical** wave, for this it is necessary that behind region of perturbation $\phi(r-ct) = 0$, i.e., that integral over entire region of perturbation is equal to zero

$$\phi(r-ct) = \int f(\xi)d\xi = \int r\Delta\rho dr = 0.$$

Hence it is clear that $\Delta\rho$ in spherical wave changes sign, i.e., behind region of compression there follows a region of rarefaction.

The additional fluid included in a spherical wave is equal to $\int \Delta\rho 4\pi r^2 dr$. But since $\Delta\rho \sim 1/r$; the additional mass in the compression wave increases as wave goes out from the center. The quantity of compressed substances increasing in process of propagations causes the appearance of a wave of lowered density following behind the wave of raised density.

Change of pressure in spherical wave is proportional to change of density, as in the plane wave. Velocity, as can be seen from formula (1.39) is not proportional to $\Delta\rho$ or Δp. In fact, the velocity and

change of density change sign at various points, so that in a wave propagating from the center, profiles of density and speed have the form depicted in Fig. 1.6.

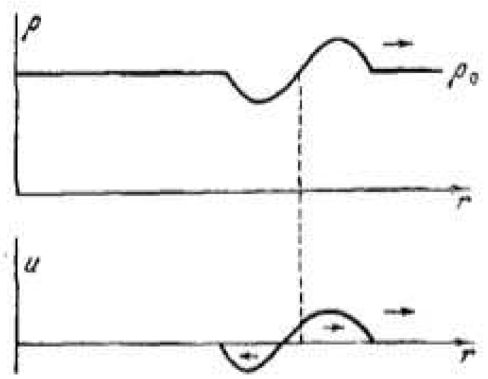

Fig. 1.6 Distribution of density and velocity in spherical sound wave.

§ 5. Characteristics

In § 3 it was shown that if at initial moment t_0 at some point x_0 of motionless gas whose density and pressure everywhere are identical, we create arbitrary small perturbations of velocity and pressure (or density[9]), then from this point in both directions with speed of sound there travel two waves carrying the perturbations. In the wave propagating in the direction of positive x, to the right, small changes of all quantities are related with each other by the relationships[10]

$$\Delta_1 u = \frac{\Delta_1 p}{\rho_0 c} = \frac{c}{\rho_0}\Delta_1 \rho = f_1(x - ct) .$$

For a wave propagating to the left, these relationships are

$$\Delta_2 u = -\frac{\Delta_2 p}{\rho_0 c} = -\frac{c}{\rho_0}\Delta_2 \rho = -f_2(x + ct) .$$

Arbitrary perturbations Δu and Δp, which appear at initial moment, can always be broken-up into two components: $\Delta u = \Delta_1 u + \Delta_2 u$, $\Delta p = \Delta_1 p + \Delta_2 p$, which obey these relationships, so that, in general, initial perturbation is propagated in different directions in the form of two waves. If initial perturbations Δu and Δp are not arbitrary, but already are related to each other by one of the relationships, then the perturbation travels in one of the directions (this corresponds to vanishing of one of the functions f_1 or f_2).

If gas is not at rest, but moves as a whole with constant velocity u, then the picture does not change, with only the exception that now the

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[9] Since the flow is isentropic, the changes of density and pressure are not independent, but always are related to each other by the thermodynamic relationship $\Delta p = c^2 \Delta\rho$.

waves are carried by the flow, so that velocities of their propagation relative to a motionless observer become equal to $u + c$ (to the right) and $u - c$ ("to the left"[11]). This can easily be verified if we go over in equations of gas dynamics to a new system of coordinates moving together with the gas at velocity u.

Let us assume now that in arbitrary plane isentropic flow of gas, described by functions $u(x, t)$, $p(x, t)$ (or $\rho(x, t)$, see the first footnote in § 5), at the time t_0 at point x_0 there appeared arbitrary small perturbations of velocity and pressure. Considering small region near point x_0 and small intervals of time (small neighborhood of point x_0, t_0 on x, t plane), it is possible in the first approximation to disregard changes of unperturbed functions $u(x, t)$, $p(x, t)$ and consequently, $\rho(x, t)$ and $c(x, t)$ in this neighborhood, and to consider them to be constant and equal to values at point x_0, t_0. The entire above described picture of propagation of perturbations can be transferred to this case. If perturbations $\Delta u(x_0, t_0)$, $\Delta p(x_0, t_0)$ are arbitrary, then they also are broken up into two components, one of which will start to propagate to the right with velocity $u_0 + c_0$, and the other "to the left" with velocity $u_0 - c_0$, whereby u_0 and c_0 here one should understand local values of these quantities at point x_0, t_0.

Since u and c change from point to point, then for a long period of time, paths of propagation of perturbations on x, t-plane, which are described by equations $\frac{dx}{dt} = u + c$ and $\frac{dx}{dt} = u - c$ will be curved. These lines on x, t-plane along which small perturbations propagate are called ***characteristics***. During plane isentropic flow of gas, as we can see, there exist **two families of characteristics**, which are described by equations

$$\frac{dx}{dt} = u + c, \quad \frac{dx}{dt} = u - c.$$

They are termed the C_+ and C_- characteristics.

Through every point on the x, t-plane it is possible to draw two characteristics, which belong to C_+ and C_- families. In general, characteristics are curvilinear, as is shown in Fig. 1.7. In region of constant flow, where u, p, c, ρ are constant in space and time, characteristics of both families are straight lines.

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[10] We write here Δu instead of u for the purpose of consistency of designations.

[11] We include the word "to the left" in quotes: if u > c, then the wave also travels to the right, but, of course, slower than the first.

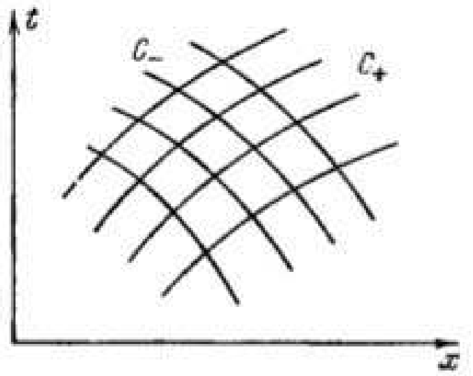

Fig. 1.7 Network of two families of characteristics in the isentropic case.

If flow is not isentropic, but only adiabatic, i.e., if entropies of different particles of gas do not change in time, but differ from each other, there are possible perturbations of entropy. Since the motion is adiabatic $\frac{DS}{Dt} = 0$, i.e., any perturbation of entropy not accompanied by perturbations of other quantities (p, ρ, u) remains localized in the particle and moves together with the particle along the streamline. Consequently, in the case of non-isentropic flow these lines are also characteristics. They are described by equation $\frac{dx}{dt} = u$ and are called C_0 characteristics. In non-isentropic flow, through every point x, t there pass **three** characteristics, and the x, t plane is covered with a network of three families of characteristics C_+, C_- and C_0 (Fig. 1.8).

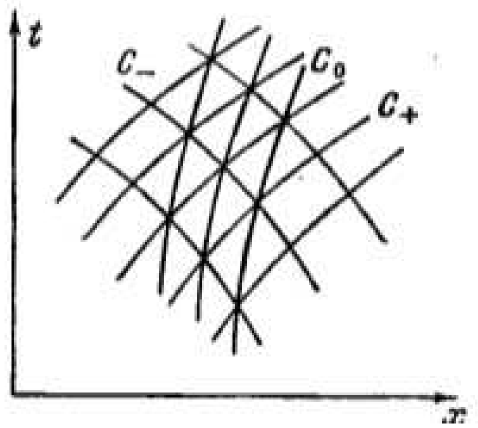

Fig. 1.8 Network of three families of characteristics in the non-isentropic case.

Till now we have spoken about characteristics as lines on the x, t-plane along which small perturbations propagate. However, this does not exhaust the significance of characteristics. Equations of gas dynamics can be transformed to such a form that they contain

derivatives of gas dynamic quantities only along characteristics. As will be shown in the following section, in isentropic flow, along characteristics there move not only small perturbations, but also definite combinations of gas dynamic quantities.

As it is known, a function of two variables $f(x,\ t)$ can be differentiated with respect to time along a definite curve $x=\phi(t)$ on the x, t-plane. The time derivative of function $f(x,\ t)$ along arbitrary curve $x=\phi(t)$ is determined by slope of tangent to curve at given point $\frac{dx}{dt}=\phi'$ and is equal to

$$\left(\frac{df}{dt}\right)_{\phi}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\frac{dx}{dt}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\phi' .$$

We are already acquainted with two particular cases of differentiation along a curve: these are the partial derivative with respect to time $\frac{\partial}{\partial t}$ (along curve x = const, $\phi'=0$) and the particle derivative $\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}$ (along path of motion of particle or along streamline $\frac{dx}{dt}-\phi'=u$).

Let us transform equations of plane adiabatic motion to a form such that they contain derivatives of gas dynamic quantities only along characteristics. For this we will eliminate from equation of continuity

$$\frac{D\rho}{Dt}+\rho\frac{\partial u}{\partial x}=0$$

the derivative of density, replacing it by derivative of pressure. Since density is thermodynamically related to pressure and entropy $\rho=\rho(p,S)$, and since $\frac{DS}{Dt}=0$, we have

$$\frac{D\rho}{Dt}=\left(\frac{\partial\rho}{\partial p}\right)_S\frac{Dp}{Dt}+\left(\frac{\partial\rho}{\partial S}\right)_p\frac{DS}{Dt}=\frac{1}{c^2}\frac{Dp}{Dt} .$$

By substituting this expression into the continuity equation and multiplying the equation by c/ρ, we find

$$\frac{1}{\rho c}\frac{\partial p}{\partial t}+\frac{u}{\rho c}\frac{\partial p}{\partial x}+c\frac{\partial u}{\partial x}=0 .$$

We add this equation with the equation of motion

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0$$

and obtain

$$\left[\frac{\partial u}{\partial t}+(u+c)\frac{\partial u}{\partial x}\right]+\frac{1}{\rho c}\left[\frac{\partial p}{\partial t}+(u+c)\frac{\partial p}{\partial x}\right]=0 .$$

By subtracting one equation from the other, we will find analogously,

$$\left[\frac{\partial u}{\partial t}+(u-c)\frac{\partial u}{\partial x}\right]-\frac{1}{\rho c}\left[\frac{\partial p}{\partial t}+(u-c)\frac{\partial p}{\partial x}\right]=0 .$$

The first of these equations contains derivatives only along C_+ characteristics, and the second, only along C_- characteristics. Noticing that adiabatic equation $\frac{dS}{dt}=0$ can be considered as an equation along C_0 characteristics, we will write equations of gas dynamics in the form

$$du+\frac{1}{\rho c}dp=0 \quad \text{along} \quad C_+ : \quad \frac{dx}{dt}=u+c , \tag{1.40}$$

$$du-\frac{1}{\rho c}dp=0 \quad \text{along} \quad C_- : \quad \frac{dx}{dt}=u-c , \tag{1.41}$$

$$dS=0 \text{ along } \quad C_- : \quad \frac{dx}{dt}=u . \tag{1.42}$$

In Lagrangian coordinates, equations of characteristics take the form

$$\left.\begin{aligned} C_+ &: \frac{da}{dt}=\frac{\rho c}{\rho_0} \\ C_- &: \frac{da}{dt}=-\frac{\rho c}{\rho_0} \\ C_0 &: \frac{da}{dt}=0 \end{aligned}\right\} .$$

Equations along characteristics do not differ from equations (1.40) to (1.42).

In **spherically symmetric** flow, equations of characteristics in Eulerian coordinates are the same as in the plane case (only coordinate *x* must be replaced by radius *r*). Equations along characteristics $C_\pm$ contain additional terms depending on the functions themselves, and not on their derivatives

$$du\pm\frac{1}{\rho c}dp=\mp\frac{2uc}{r}dt \quad \text{along} \quad C_\pm : \frac{dr}{dt}=u\pm c .$$

In a number of cases, equations of gas dynamics written in characteristic form are more convenient for numerical integration than usual equations.

§ 6. Plane Isentropic Flow, Riemann Invariants

In isentropic flow, entropy, which is constant in space and time, in general, drops out of equations. All flow is described by two functions: by velocity $u(x, t)$ and by some one of the thermodynamic variables: $p(x, t)$, $p(x, t)$ or $c(x, t)$. The latter are uniquely related with each other at every point by purely thermodynamic relationships: $\rho = \rho(p)$, $c = c(\rho)$ or $p = p(\rho)$, $c = c(\rho)$; $c^2 = \dfrac{dp}{d\rho}$.

Differential expressions $du + \dfrac{dp}{\rho c}$ and $du - \dfrac{dp}{\rho c}$ now constitute total differentials of quantities

$$\left.\begin{aligned} J_+ &= u + \int \frac{dp}{\rho c} = u + \int c \frac{d\rho}{\rho} \\ J_- &= u - \int \frac{dp}{\rho c} = u - \int c \frac{d\rho}{\rho} \end{aligned}\right\} \tag{1.43}$$

which are called ***Riemann invariants***.[12] With the help of thermodynamic relationships, integral quantities $\int \dfrac{dp}{\rho c} = \int c \dfrac{d\rho}{\rho}$ in principle can be expressed in terms of one of the thermodynamic variables, let us say, the speed of sound c. For instance, in an perfect gas with constant heat capacity

$$p = const \times \rho^{\gamma}, \quad c^2 = \gamma \cdot const \times \rho^{\gamma - 1},$$

and

$$J_{\pm} = u \pm \frac{2}{\gamma - 1} c. \tag{1.44}$$

Riemann Invariants are determined with accuracy up to the arbitrary constant, which in those cases when it is convenient can be completely omitted, as this is done in formula (1.44).

Equations (1.40), (1.41) indicate that in isentropic flow, Riemann invariants are constant along characteristics

$$dJ_+ = 0, \quad J_+ = const \quad \text{along} \quad C_+ : \frac{dx}{dt} = u + c$$

$$dJ_- = 0, \quad J_- = const \quad \text{along} \quad C_- : \frac{dx}{dt} = u - c. \tag{1.45}$$

This situation can be considered as a generalization of relationships which are accurate for the case of propagation of acoustic waves through a gas with constant velocity, density and pressure. The latter

[12] During non-isentropic flow, ρ and c depend on two variables p and S, and the expressions $du \pm dp/\rho c$ no longer are total differentials. Combinations (1.43) in this case do not have a definite meaning.

are obtained from general equations as a first approximation. If we assume that $u = u_0 + \Delta u$, $p = p_0 + \Delta p$, then in the first approximation we obtain

$$J_{\pm} = u_0 + \Delta u \pm \int \frac{d\Delta p}{\rho_0 c_0} = \Delta u \pm \frac{\Delta p}{\rho_0 c_0} + const\ . \qquad (1.46)$$

The equations of characteristics in first approximation are written in the form

$$\frac{dx}{dt} = u_0 \pm c_0\,, \quad x = (u_0 \pm c_0)t + const\ .$$

Thus, along path $x = (u_0 + c_0)t + const$ there is kept the quantity $\Delta u + \frac{\Delta p}{\rho_0 c_0}$, from which it is clear that it can be represented in the form of a function of the constant in the equation $x = (u_0 \pm c_0)t + const$ in the following way:

$$\Delta u + \frac{\Delta p}{\rho_0 c_0} = 2 f_1 [x - (u_0 + c_0)t]\ .$$

Along path $x = (u_0 - c_0)t + const$ there is kept the quantity

$$\Delta u - \frac{\Delta p}{\rho_0 c_0} = -2 f_2 [x - (u_0 - c_0)t]\ .$$

Changes of velocity and pressure are represented in the form of superposition of two waves f_1 and f_2, which travel in opposite directions: $\Delta u = f_1 - f_2$, $\Delta p = \rho_0 c_0 (f_1 + f_2)$, where in each of them quantities are related to each other by relationships already known to us:

$$\Delta_1 u = \frac{\Delta_1 p}{\rho_0 c_0} = f_1\,, \quad \Delta_2 u = -\frac{\Delta_2 p}{\rho_0 c_0} = -f_2\ .$$

The Riemann invariants J_+ and J_- can be considered as new functions describing motion of gas in exchange for old variables; velocity of gas u and one of thermodynamic quantities, for instance, speed of sound c. They are uniquely related to variables u and c by equations (1.43). By solving these equations for u and c, it is possible to return from functions J_+, J_- to functions u and c. For instance, for a perfect gas with constant heat capacity, by formulas (1.44)

$$u = \frac{J_+ + J_-}{2}\,; \quad c = \frac{\gamma - 1}{4}(J_+ - J_-)\ .$$

Considering invariants as functions of independent variables x and t, equations of characteristics can be written in the form

$$C_+ : \frac{dx}{dt} = F_+(J_+, J_-)\ ; \quad C_- : \frac{dx}{dt} = F_-(J_+, J_-)\ . \qquad (1.47)$$

where F_+ and F_- are known functions, whose form is determined only by thermodynamic properties of the fluid. For a perfect gas with constant specific heats, we have

$$F_+ = \frac{\gamma+1}{4} J_+ + \frac{3-\gamma}{4} J_-\ ; \quad F_- = \frac{3-\gamma}{4} J_+ + \frac{\gamma+1}{4} J_-\ .$$

As can be seen from equations (1.45), characteristics have the property to transfer constant values of one of the invariants. Since along a definite C_+ characteristic $J_+ = const$, change of slope of characteristic is determined by change of only one quantity, that is J_-. In exactly the same way, along the C_- characteristic J_- is constant, and change of slope during transition from one point of the *x*, *t* plane to another is determined by change of invariant J_+.

Equations written in characteristic form make the casual, relationship of phenomena in gas dynamics very graphic. Let us consider any plane isentropic flow of gas in an infinite space. Let us assume that at initial moment $t = 0$ there are given distributions of gas dynamic quantities over *x* coordinate: $u(x, 0)$ and $c(x, 0)$, or which is equivalent, there are given distributions of invariants $J_+(x,0)$ and $J_-(x,0)$. On the plane of *x*, *t* (Fig, 1.9) there exists a network of C_+

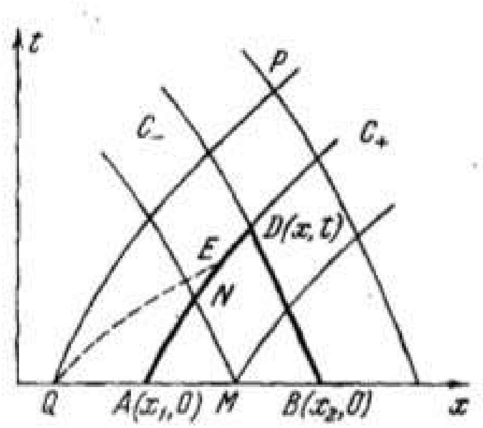

Fig. 1.9 An x, t diagram illustrating the domain of dependence.

and C_- characteristics, which go out from different points of the *x*-axis.[13] Values of gas dynamic quantities at any point $D(x, t)$ (at coordinate point *x* at the moment of time *t*) are determined only by values of quantities at initial points $A(x_1,0)$ and $B(x_2,0)$:

$$J_+(x,t) = J_+(x_1,0)\ ; \quad J_-(x,t) = J_-(x_2,0)\ .$$

For instance, for a perfect gas with constant heat capacity, by solving these equations for *u* and *c*, it is possible to write physical variables at

[13] It is possible to construct this network after there is found the solution of the problem.

point D in explicit form

$$\left.\begin{aligned} u(x,t) &= \frac{u_1+u_2}{2} + \frac{2}{\gamma-1}\frac{c_1-c_2}{2} \\ c(x,t) &= \frac{c_1+c_2}{2} + \frac{\gamma-1}{2}\frac{u_1-u_2}{2} \end{aligned}\right\}, \qquad (1.48)$$

where u_1 and c_1 are values at point $A(x_1,0)$ and u_2 and c_2 are values at point $B(x_2,0)$.

It is impossible, of course, to say that state of gas at point D depends on assignment of initial conditions only at two initial points A and B, since the actual position of point D, as the place where C_+ and C_- characteristics, going out from points A and B intersect, depends on path of these characteristics. These paths are determined by assignment of initial conditions on all of segment AB of axis x. For instance, slope of C_+ characteristic AD at intermediate point N (see Fig. 1.9) is determined not only by invariant $J_+(A)$, but also by value of invariant $J_-(M)$, which is transferred to N from intermediate point M of segment AB.

But state of gas at D is completely determined by assignment of initial conditions on segment AB of axis x, and absolutely does not depend on initial values of quantities outside of this segment. If, let us say, we somewhat change initial values at point Q, then this in no way will affect state of gas at D, simply because perturbation due to this change will not succeed in reaching coordinate point x by the moment t. It will arrive at this coordinate point later (at point P along C_+ characteristic QP).

Analogously, initial state of gas on segment AB of axis x affects state of gas at subsequent moments of time only at those points which are located inside region bounded by C_- characteristic AP and C_+ characteristic BQ (Fig. 1.10). It does not affect state at M, since

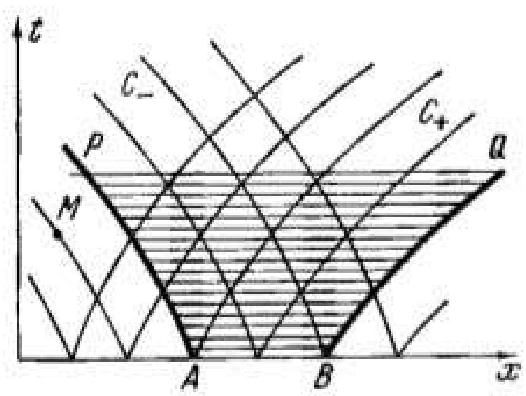

Fig. 1.10 An x, t diagram illustrating the region of influence.

"signals" from initial conditions on segment AB will not succeed in reaching coordinate point x_M at the time t_M.

We will stress that the presented considerations about causal relationship of phenomena are valid only under the condition that characteristics of one family do not intersect with each other. For instance, if C_+ characteristic from Q (see Fig. 1.9) went along dotted path QE, then state of gas at Q would influence state at D. But in region of continuous flow, characteristics belonging to one family indeed never intersect. Intersection would lead to non-single-valuedness of gas dynamic quantities. Indeed, at point of intersection of two C_+ characteristics x, t, invariants J_+ would have two different values, corresponding to each of the two characteristics. Meanwhile, to every point of plane x, t there belongs only one value each of J_+ and J_-, which are related with the unique values of velocity of gas and speed of sound at this point. As we will see below, intersection of characteristics of one family leads to disturbance of continuity of flow and appearance of discontinuities of gas dynamic quantities, i.e., **shock waves**.

It is possible to draw lines of characteristics on all of plane x, t only if we know solution of gas dynamic problem. If solution is unknown, then it is impossible to indicate exactly the position of point D in Fig. 1.9 at which characteristics going out from A and B intersect.

However, it is possible approximately to find place of intersection by replacing true curvilinear paths AD and BD by straight lines whose slopes correspond to initial values of u_1, c_1 and u_2, c_2 at points A and B (or $J_+(A)$, $J_-(B)$) (Fig. 1.11). Selecting points A and B

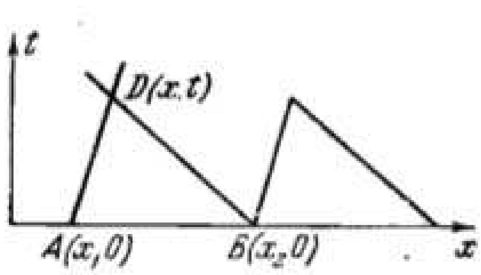

Fig. 1.11 Local approximation of the characteristics by straight lines.

sufficiently close to each other in such a manner that error due to replacement of true paths of characteristics by straight lines is small, we find position of point of intersection from equation

$$x - x_1 = (u_1 + c_1)t\ ,\quad x - x_2 = (u_2 - c_2)t\ .$$

Values of u and c at place of intersection are determined by formulas (1.48). Such an operation, in essence, constitutes the simplest scheme of

numerical integration of equations (1.45). Covering plane x, t by a network of triangles analogous to ADB, it is possible successively, step by step, to advance solution of equations forward in time, proceeding from initial conditions $u(x, 0)$, $c(x, 0)$ or $J_+(x,0)$, $J_-(x,0)$.

§ 7. Plane Isentropic Flow of Gas in a Bounded Space

Let us consider some plane isentropic flow of gas in a bounded space. Let us assume that the gas occupies space between two plane surfaces - pistons, which move according to given laws $x_1 = \psi_1(t)$, $x_2 \sim \psi_2(t)$, where at initial moment t = 0 coordinates of pistons are equal to x_{10} and x_{20}. At initial moment there are given distributions of velocity u and thermodynamic variable c over coordinate x on segment $x_{10} < x < x_{20}$; $u(x,0)$, $c(x,0)$ or, which is equivalent, there are given distributions of invariants $J_+(x,0)$ and $J_-(x,0)$.

Let us draw on plane x, t a network of characteristics and lines of pistons (Fig. 1.12). Points of type F, through which there pass C_+ and C_- characteristics going out from points lying inside segment O_1O_2 of axis x do not at all differ from points during motion of gas in unbounded space. Just as there, to these points there are transferred initial values of Invariants $J_+(x,0)$ and $J_-(x,0)$.

We will consider a point lying on line of piston, for definiteness, point D of the left piston. To point D from the "past"' there is transferred only one invariant J_-; it is transferred along the C_- characteristic coming from point A of initial segment O_1O_2 so that $J_-(D) = J_-(A)$. Second invariant J_+ is not brought to D, since C_+ characteristic does not arrive at D (from the "past"). C_+ characteristic only goes out from D (into the "future"), taking with it the value of invariant J_+ "formed" at this point. State of gas at point D is determined by value of introduced invariant J_- and a second quantity - velocity u, which in virtue of boundary condition coincides with known velocity of piston at point D: $u_1(D)$. This pair of quantities $J_-(D) = J_-(A)$ and $u = u_1(D)$ replaces now the pair of quantities J_+, J_-, which arrive at points of gas, which do not touch the pistons. Second invariant J_+ is composed in D of quantities $J_-(D)$ and $u_1(D)$: $J_+(D) = 2u_1(D) - J_-(D)$, and is transferred by the C_+ characteristic. For Instance, to point E arrives C_- characteristic going out from point B of initial segment of x-axis and carrying invariant $J_-(B)$: $J_-(E) = J_-(B)$. C_+ characteristic arrives from line of

piston, from D, and brings invariant J_+, which is equal to $J_+(D)$: $J_+(E) = J_+(D)$.

State of gas at E depends on initial conditions on segment O_1B of axis x and velocities of left piston on segment O_1D of line of piston.

Thus, during flow in bounded space, state of gas at any point may depend not only on initial conditions, but also on boundary conditions.

In general, the state at arbitrary point of plane x, t is determined by assignment of values of u and c or J_+, J_- on segment of arbitrary curve cut off by C_+ and C_- characteristics passing through the considered point. For instance, state at Q is determined by state on segment MN of curve S (see Fig, 1. 12).

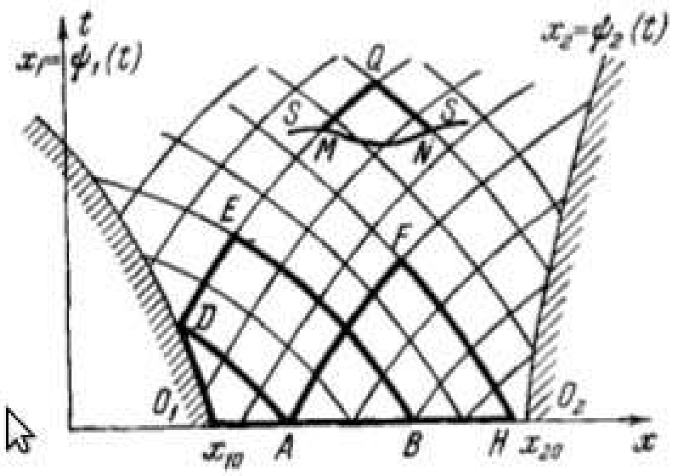

Fig. 1.12 Sketch of characteristics for plane isentropic gas flow between two pistons.

Analogously to the preceding, onto the right piston from the "past" along C_+ characteristics are transferred invariants J_+, and C_- characteristics themselves start from points of line of piston and carry into the "future" invariants J_-, which are composed of the introduced invariants J_+ and values of velocity of piston u_2, with which velocities of layer of gas adjacent to piston coincide.

Pressure on piston is uniquely determined by the one introduced invariant and velocity of piston. Let us consider for example point D on left piston. Let us assume that gas is perfect with constant specific heats. Let us designate by u_A, c_A initial velocity of gas and speed of sound at point A, and by u_{1D} velocity of piston at point D. We have for speeds of gas and sound at D

$$u_D = u_{1D}, \quad J_- = u_D - \frac{2}{\gamma - 1} c_D = u_A - \frac{2}{\gamma - 1} c_A,$$

whence

$$c_D = c_A + \frac{\gamma - 1}{2}(u_{10} - u_A)$$

or in terms of the invariant

$$c_D = \frac{\gamma - 1}{2}\left[u_{10} - J_-(A)\right].$$

Pressure on piston p_D is related with speed of sound c_D purely thermodynamically, $p_D = const \times c_D{}^{2\gamma/(\gamma-1)}$.

Presented considerations permit us to give a clear **graphic physical meaning to Riemann invariants**. Let us take the following experiment. Let us introduce at a particular time t at point x a flat plate parallel to the surface of piston. Let us assume that on one, the left side of the plate there is a pressure indicator, which reacts to pressure of gas on the left of the plate.

By moment t at x from the left onto the indicator there arrives invariant $J_+ = u + \int \frac{dp}{\rho c} = u + w(p)$, where u and p are velocity and pressure of gas unperturbed by the plate ($w(p)$ is function of pressure, depending only on thermodynamic properties of gas and its entropy). At the time t, the gas is decelerated near plate and stops, inasmuch as the plate is at rest. New pressure on the left of the plate corresponding to the stopped gas (u = 0) we will designate by p_1. Then $J_+ = u + w(p) = w(p_1)$. Indicator will register pressure of repulsion $-p_1$. Since the function w is known, scale of indicator can be calibrated in such a manner that reading of indicator directly gives magnitude of invariant J_+. Analogously, pressure indicator placed on right side of plate measures invariant J_- arriving from the right.

If we place a very thin plate perpendicular to surfaces of pistons, parallel to velocity of flow, in such a manner that gas freely flows around the plate without changing velocity, the indicator will register pressure of unperturbed flow p. Since it is calibrated to directly give magnitude of $w(p)$, the indicator will measure combination of invariant

$$w(p) = \frac{1}{2}(J_+ - J_-)\,.$$

§ 8. Simple Waves

§ 9. Distortion of Profiles in Travelling Wave of Finite Amplitude. Some Properties of Simple Waves

§ 10. Rarefaction Wave

§ 11. Centered Rarefaction Wave as an Example of Self-Similar Motion of Gas