

3. Viscosity and Heat Conduction in Gas Dynamics

§ 20. Equations of One-Dimensional Gas Flow

The dissipative processes - viscosity (internal friction) and heat conduction - are connected with existence of molecular structure of a fluid. These processes create additional, non-hydrodynamic transfer of momentum and energy and lead to non-adiabatic character of motion and to thermodynamically irreversible transformation of mechanical energy into heat. Viscosity and heat conduction appear only in the presence of large gradients of hydrodynamic quantities, which occur, for instance, in boundary layer in flows past solid bodies or within the shock front. In this book we shall consider viscosity and heat conduction basically from the point of view of their effects on internal structure of shock fronts in gases. During the study of this structure, flow can be considered to depend on one coordinate x (plane flow), since the thickness of shock front is always considerably less than the radius of curvature of its surface. Therefore, we will not dwell on derivation of general equation of motion of a viscous liquid (gas), which can be found, for instance, in the book of L. D. Landau and E. M. Lifshits [1]. We will explain only how there can be obtained equation for one-dimensional, plane case.

We will write equation of conservation of momentum for an inviscid gas (1.7) in the plane case, where all quantities depend only on one coordinate x , and velocity has only one component of u

$$\frac{\partial}{\partial t}(\rho u) = -\frac{\partial \Pi_{xx}}{\partial x}, \quad \Pi_{xx} = p + \rho u^2.$$

We will take into account now the fact that gas consists of molecules colliding with each other. Let us imagine an area of unit cross section perpendicular to x axis. This area is penetrated from both sides by molecules moving in definite directions after having suffered their last collisions. Molecules emerge after the last collision from layers of thickness of the order of mean free path of molecules l . These layers are adjacent to both sides of the area (Fig. 1.38). If n is number of molecules in unit volume and \bar{v} is their average thermal velocity, then $n\bar{v}$ molecules will pass per unit time through the area from left to right. Each molecule carries through the area the hydrodynamic momentum mu , where m is mass of a molecule. The total flux density of hydrodynamic momentum from left to right is, therefore, of the order of $n\bar{v} \cdot mu$. Analogously, the flux of hydrodynamic momentum from

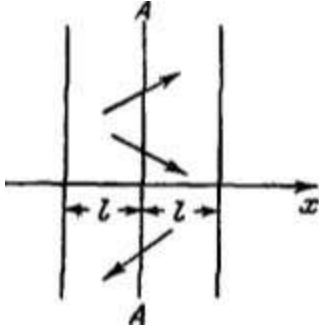


Fig. 1.38 Schematic illustration for the derivation of the equation for the molecular transport of momentum.

right to left is equal approximately to $n\bar{v} \cdot m(u + \Delta u)$, where Δu is the increment of hydrodynamic velocity during transition from left layer to right: that is $\Delta u \approx \frac{\partial u}{\partial x} l$. The flux density of the x component of

momentum in x direction connected with molecular transfer is equal to the difference between the fluxes from left to right and from right to left, i.e., $-n\bar{v}ml \frac{\partial u}{\partial x}$. This quantity corresponds to additional transfer of momentum due to internal friction; it must be added to momentum flux density $\Pi_{xx} = p + \rho u^2$.

A more rigorous treatment, based on three-dimensional considerations, shows that a numerical coefficient of the order unity should be introduced into the above expressions. Namely, the equation of conservation of momentum, taking into account viscosity, in the plane case has the form

$$\frac{\partial}{\partial t}(\rho u) = -\frac{\partial \Pi_{xx}}{\partial x}, \quad \Pi_{xx} = p + \rho u^2 - \sigma', \quad \sigma' = \frac{4}{3}\eta \frac{\partial u}{\partial x}, \quad (1.90)$$

where η is coefficient of viscosity, which for gases (in the absence of relaxation processes; see below) in order of magnitude is equal to

$$\eta \sim n\bar{v}ml = \rho \bar{v}l.$$

The quantity σ' constitutes the xx component of viscous stress tensor. Its appearance of it in formula for flux of momentum is equivalent to the appearance of additional "pressure" created by internal friction forces. With the help of continuity equation, equation (1.90) can be easily transformed to the equation of motion

$$\rho \frac{Du}{Dt} = -\frac{\partial}{\partial x}(p - \sigma'). \quad (1.91)$$

where $\frac{\partial \sigma'}{\partial x}$ is force of internal friction calculated for unit volume of gas.

In the presence of dissipative processes, additional terms also appear in energy equation. The additional energy flux is related to the added "viscous" pressure. A quantity $-\sigma'u$, analogous to pu , should be added to expression for the energy flux density, the divergence of which appears in (1.10). In addition, the energy flux transferred by heat conduction

$$J = -\kappa \frac{\partial T}{\partial x}, \quad (1.92)$$

where κ is the coefficient of thermal conductivity, must be introduced into this expression. Expression (1.92) is easy to obtain by the same approach used in deriving the viscous momentum flux. Thus it turns out that in gases, the coefficient of thermal conductivity is of the order of $\rho c_p \bar{v} l$.

Taking into account the two dissipation terms, energy equation (1.10) for the plane case becomes

$$\frac{\partial}{\partial t} \left(\rho \varepsilon + \frac{\rho u^2}{2} \right) = - \frac{\partial}{\partial x} \left[\rho u \left(\varepsilon + \frac{u^2}{2} \right) + pu - \sigma'u + J \right]. \quad (1.93)$$

Rearranging this equation with the help of continuity equation, equation of motion and thermodynamic identity $TdS = d\varepsilon + pdV$, we obtain an equation for rate of change of entropy of the fluid particle

$$\rho T \frac{DS}{Dt} = \sigma' \frac{Kdu}{\partial x} - \frac{\partial J}{\partial x} = \frac{4}{3} \mu \left(\frac{\partial u}{\partial x} \right)^2 + \frac{\partial}{\partial x} \left(\kappa \frac{\partial T}{\partial x} \right). \quad (1.94)$$

First term in the right-hand side of this equation constitutes mechanical energy dissipated per unit volume due to viscosity. This term is always positive, since $\mu > 0$ and $\left(\frac{\partial u}{\partial x} \right)^2 > 0$; consequently, the internal friction forces lead to local increase in fluid entropy.

The second term corresponds to heating or cooling of the fluid by heat conduction. It can be either positive or negative, since heat conduction leads to transfer of heat from hotter regions into cooler ones. However, the entropy of the fluid as a whole always increase due to heat conduction. This can be shown if we divide equation (1.94) by T and integrate over the entire volume. The change of entropy of the fluid occupying a volume bounded by surfaces x_1 and x_2 due to heat conduction is equal to

$$\int_{x_1}^{x_2} \frac{1}{T} \frac{\partial}{\partial x} \left(\kappa \frac{\partial T}{\partial x} \right) dx = \frac{1}{T} \kappa \frac{\partial T}{\partial x} \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{\kappa}{T^2} \left(\frac{\partial T}{\partial x} \right)^2 dx.$$

If the fluid is thermally insulated on boundaries x_1 and x_2 , then fluxes of heat on boundaries disappear and there remains only the second term in the right-hand side, which is always positive ($\kappa > 0$).

Equations of gas dynamics, written taking into account viscosity and heat conduction, permit us to determine the condition under which the role of these dissipative processes become important. Let us compare inertial forces in equation of motion with viscosity forces. If U is the scale of velocity, and d is a characteristic dimensions of the flow region, then the scale of time is of the order of d/U , and inertial term

$\rho \frac{Du}{Dt}$ is of the order of $\frac{\rho U^2}{d}$. The viscosity term in equation $\frac{\partial}{\partial x} \left(\frac{4}{3} \mu \frac{\partial u}{\partial x} \right)$ is of the order of $\frac{\rho U}{d^2}$, and the ratio of it to the inertial term is of the order of

$$\frac{1}{\text{Re}} = \frac{\mu}{\rho U d} = \frac{\nu}{U d} \sim \frac{l}{d} \frac{c}{U}.$$

The reciprocal of this ratio has the name **Reynolds number**

($\nu = \frac{\mu}{\rho} \sim l \bar{v} \sim l c$ is the kinematic viscosity and $c \sim \bar{v}$ is speed of sound). In an analogous way, by comparing heat transfer by the heat conduction with mechanical energy transfer, we will find that their ratio is of the order of

$$\frac{1}{\text{Pe}} = \frac{\kappa}{\rho c_p U d} \sim \frac{\chi}{U d} \sim \frac{l}{d} \frac{c}{U},$$

where Pe is **Peclet number**, which is close in gases to Reynolds number,

since coefficient of molecular thermal diffusivity $\chi = \frac{\kappa}{\rho c_p}$ is

numerically close to the coefficient of kinematic viscosity ν . (For instance, in air under normal conditions $\nu \approx \chi \approx 0.15 \text{ cm}^2 / \text{sec}$.)

Thus, viscosity and heat conduction can be neglected for $\text{Re} \approx \text{Pe} \gg 1$. If we consider motion with velocities less than or close to the speed of sound, dimensions of the system for this have to be much larger than mean free path of molecules: $d / l \gg 1$. This condition, as we will see, is not satisfied, in particular, in region of shock front, the thickness of which is comparable with mean free path of molecules.

Therefore, inside the shock front, dissipative processes turn out to be essential. Namely these processes are responsible for the increase in entropy across a shock wave.

§ 21. Remarks on the Second Viscosity Coefficient

§ 22. Remarks on the Absorption of Sound

§ 23. The Structure and Thickness of a Weak Shock Front

Let us consider the internal structure and thickness of the thin layer representing the shock wave across which the gas undergoes the transition from the initial state to the final state; we refer this layer as the *shock front*. In this layer there occur sharp increase in density of the fluid, pressure and velocity changes, and, as has been shown by calculations based only on the application of laws of conservation of mass, momentum, and energy, there occurs an increase in entropy. The entropy increase indicates that in the transition layer there occurs dissipation of mechanical energy and that an irreversible transformation of mechanical energy into heat. Therefore, in order to understand the detailed nature of the shock transition it is necessary to take into consideration the dissipative processes – viscosity and heat conduction.

Let us consider one-dimensional plane flow of a viscous and heat-conducting gas in a coordinate system in which the shock front is at rest. The thickness of the front is very small as compared to the characteristic scales of length for all of the gas dynamic process on the whole, for instance, as compared to distance from front of shock wave to piston pushing the gas and creating the wave.

Even if piston moves with variable speed and the strength of the shock wave changes in time, the strength of the wave will remain practically unchanged during the small time interval Δt required to traverse a distance of the order of the front width Δx . Therefore, for the period of a certain time, which is small as compared to total time scale of gas dynamic process, but large in comparison with Δt , the whole pattern of the distribution of gas dynamic quantities across the wave front propagates through the gas in "frozen" form as a whole. In other words, in a system of coordinates in which the front is at rest, flow of gas can at every given moment be considered to be steady.

Let us write equations of continuity, momentum, and entropy, taking into account viscosity and thermal conduction for the plane steady case. Since the process is steady-state, we can drop terms in

partial derivative with respect to time $\frac{\partial}{\partial t}$, and can replace the partial derivative with respect to coordinate $\frac{\partial}{\partial x}$ by total derivative $\frac{d}{dx}$. We obtain

$$\left. \begin{aligned} \frac{d}{dx}(\rho u) &= 0 \\ \frac{d}{dx}\left(p + \rho u^2 - \frac{4}{3}\mu \frac{du}{dx}\right) &= 0 \\ \rho u T \frac{dS}{dx} &= \frac{4}{3}\mu \left(\frac{du}{dx}\right)^2 + \frac{d}{dx}\left(\kappa \frac{dT}{dx}\right) \end{aligned} \right\}. \quad (1.96)$$

With help of second law of thermodynamics $TdS = dh - Vdp$ and equations of continuity and momentum, we can rewrite the entropy equation in form of an energy equation

$$\frac{d}{dx}\left[\rho u\left(h + \frac{u^2}{2}\right) - \frac{4}{3}\mu u \frac{du}{dx} - \kappa \frac{dT}{dx}\right] = 0. \quad (1.97)$$

We now impose appropriate boundary conditions on the solution of these equations, by requiring the gradients of all quantities to vanish ahead of the front at $x = -\infty$ and behind the front at $x = +\infty$. At these limits the variables assume their initial and final values, designated, as before, by the subscripts "0" and "1" (Fig. 1.39).

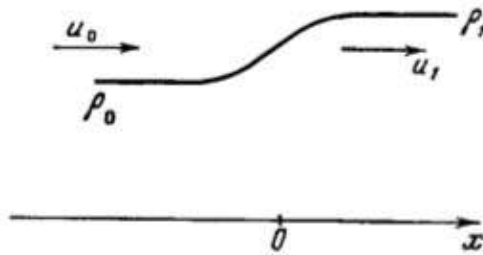


Fig. 1.39. Sketch illustrating the structure of a shock front.

First integrals of system of equations of mass, momentum and energy are obtained immediately

$$\rho u = \rho_0 u_0, \quad (1.98)$$

$$p + \rho u^2 - \frac{4}{3}\mu \frac{du}{dx} = p_0 + \rho_0 u_0^2, \quad (1.99)$$

$$\rho u \left(h + \frac{u^2}{2}\right) - \frac{4}{3}\mu u \frac{du}{dx} - \kappa \frac{dT}{dx} = \rho_0 u_0 \left(h_0 + \frac{u_0^2}{2}\right). \quad (1.100)$$

The constants of integration here are expressed in terms of initial

values of quantities p , ρ , T , u and are considered as functions of current x coordinate.¹

From equation (1.99) it is clear that due to presence of viscosity, i.e., term containing du/dx , distribution of quantities over x in wave front should be continuous (otherwise gradient du/dx would go to infinity, which is incompatible with finiteness of the quantities themselves).

In order to understand better the roles of viscosity and heat conduction, we will first consider two particular cases of structure of front:

- (1) when viscosity is absent and there is only heat conduction;
- (2) when there is only viscosity and heat conduction is absent.

We will not look for exact solutions of equations here (this problem will be considered in Chapter VII, which is specially dedicated to the study of structure of shock wave fronts). Let us limit ourselves only to clarifying qualitative picture of phenomenon and estimates the width of the front.

(1) Heat conduction is present, but there is no viscosity: $\nu = 0$.

This case is remarkable due to the fact that the equation of momentum becomes of the form

$$p + \rho u^2 = p_0 + \rho_0 u_0^2,$$

which is analogous to the equation relating the final and initial values of these quantities. This equation, however, now describes all the intermediate states in the wave front. With the help of continuity equation (1.98) we obtain

$$p = p_0 + \rho_0 u_0^2 \left(1 - \frac{V}{V_0} \right). \quad (1.101)$$

Thus, the point describing the state of gas within the shock front goes from the initial point A in the p , V -plane to the final point B along the straight line AB . The line AB has already been discussed in the description of a Hugoniot curve.

Let us draw the isentropes (Poisson adiabats) through points of initial and final states in the p , V -plane (Fig. 1.40); the Hugoniot curve is not shown). If we plot on the plane a whole series of isentropes (Poisson adiabats) with various values of entropy, it is evident that one

¹ At $x = +\infty$, $\frac{du}{dx} = 0$, $\frac{dT}{dx} = 0$, $p = p_1$, $\rho = \rho_1$, $u = u_1$, and we arrive at the laws of conservation of mass, momentum, and energy across the discontinuity, (1.61), (1.62) and (1.64).

of the curves will be tangent to straight line AB at a certain point M , as shown in Fig. 1.40. At this point the entropy along the straight line AB is at its maximum ($S_0 < S_1 < S_M$). It follows from equations (1.98) and (1.101) that the gas velocity u at the point of tangency M is exactly equal to the local speed of sound ($u = c$ at the point M ; we note that $u_0 > c_0$ at point A and $u_1 < c_1$ at point B).

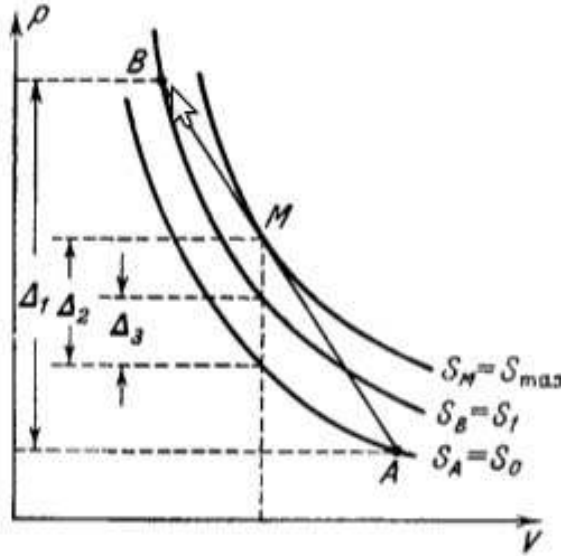


Fig. 1.40. p, V diagram illustrating the structure of shock fronts in the absence of viscosity. The state of the gas in the wave changes along the straight line AB . Segments Δ_1 , Δ_2 and Δ_3 are of first, second and third order, respectively, with respect to the wave strength.

Let us find the maximum value of the entropy S_{\max} from the condition of tangency of isentrope (Poisson adiabat) $S = S_{\max}$, and the straight line AB . As we will soon see, the quantity $S_{\max} - S_0$ is proportional to either $(V_1 - V_0)^2$ or $(p_1 - p_0)^2$; therefore, we shall write the equations for the family of isentropes $p(V, S)$ and for the straight line AB in the form of expansion about the point A , neglecting third-order terms (in this approximation the isentropes S_0 and S_1 coincide; see §18). The equation for the isentrope is

$$p - p_0 = \left(\frac{\partial p}{\partial V} \right)_{S_A} (V - V_0) + \frac{1}{2} \left(\frac{\partial^2 p}{\partial V^2} \right)_{S_A} (V - V_0)^2 + \left(\frac{\partial p}{\partial S} \right)_{V_A} (S - S_0).$$

The equation for the straight line is

$$\begin{aligned} p - p_0 &= \frac{p_1 - p_0}{V_1 - V_0} (V - V_0) \\ &= \left(\frac{\partial p}{\partial V} \right)_{S_A} (V - V_0) + \frac{1}{2} \left(\frac{\partial^2 p}{\partial V^2} \right)_{S_A} (V_1 - V_0)(V - V_0) \end{aligned}$$

The condition of tangency is expressed by the equality

$$\left(\frac{\partial p}{\partial V} \right)_{isent} = \left(\frac{\partial p}{\partial V} \right)_{str.line},$$

which gives the equation for determination of volume V_M at the point of tangency M . It can be shown that the point M is situated exactly half

way between points A and B , i.e., $V_M - V_0 = \frac{1}{2}(V_1 - V_0)$. Substituting

this expression into the equation for the straight line, we find the pressure p_M at the point M . Next, substituting p_M and V_M into the equation for the isentrope and solving for the entropy at the point M , we obtain

$$S_M - S_0 = S_{\max} - S_0 = \frac{1}{8} \frac{\left(\frac{\partial^2 p}{\partial V^2} \right)_{S_A}}{\left(\frac{\partial p}{\partial S} \right)_{V_A}} (V_1 - V_0)^2.$$

Thus, maximum entropy change within the shock front, taking into account heat conduction alone, is a second-order quantity with respect to the quantity $V - V_0$ or $p - p_0$, in contrast to the total entropy change $S_1 - S_0$ which is of the third order of smallness. This can also be shown on the bases of geometrical arguments; the maximum distance of the straight line AB from the isentropes (Poisson adiabat) $S = S_0$ in the p, V -plane is proportional to $(V_1 - V_0)^2$ or $(p_1 - p_0)^2$. Thus, the difference between pressures at point M and on the isentrope S_A (or S_B) at the same volume V_M is equal to

$$\begin{aligned} p_M(V_M) - p_{S_A}(V_M) &= \frac{1}{2} \left(\frac{\partial^2 p}{\partial V^2} \right)_{S_A} (V_M - V_0)(V_1 - V_M) \\ &= \frac{1}{8} \left(\frac{\partial^2 p}{\partial V^2} \right)_{S_A} (V_1 - V_0)^2 \end{aligned} \quad (1.102)$$

(the pressure difference between points on isentropes S_B and S_A at the same volume V_M is a third-order of smallness).

The presence of maximum of entropy within the shock front indicates that profile of temperature $T(x)$ is inflected at the point of maximum entropy; thus the temperature and entropy distributions in a weak shock wave, considering heat conduction only, can be described by the curves shown in [Fig. 1.41](#). This conclusion follows from the entropy equation (1.96), which, in the absence of viscosity, takes the form

$$\rho u T \frac{dS}{dx} = \frac{d}{dx} \kappa \frac{dT}{dx} = \kappa \frac{d^2 T}{dx^2} \quad (1.103)$$

(in a weak wave the temperature changes only very slightly, so that coefficient of thermal conductivity can be considered to be constant).

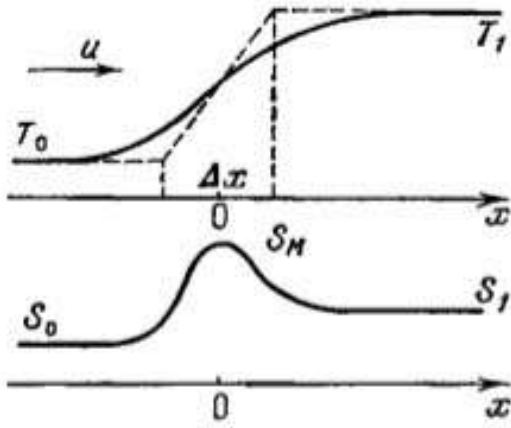


Fig. 1.41. Temperature and entropy distributions in a weak shock front in the absence of viscosity. Δx is the effective front thickness.

The existence of the maximum of entropy is connected with the fact that conductive heat transfer takes place from region with higher temperature to region with lower temperature. Therefore, the gas flowing into the wave is first heated by heat conduction (with an increase in entropy), and then cooled (with decrease in entropy). In the end state, as compared to the initial state, the entropy obviously increases. This is illustrated in Fig. 1.41, where flow along the x axis with a velocity $u(x)$ corresponds to the change in state of a given gas particle with time.

Let us now estimate the thickness of the shock front. For this we divide equation (1,103) by T and integrate it with respect to x from the initial state A ($x = -\infty$), where $\frac{dT}{dx} = 0$, to some point x in the wave

(we also use the fact that $\rho u = \rho_0 u_0 = \text{const}$)

$$\rho_0 u_0 (S - S_0) = \kappa \int_{-\infty}^x \frac{1}{T} \frac{d^2 T}{dx^2} dx = \kappa \left\{ \frac{1}{T} \frac{dT}{dx} + \int_{T_0}^T \frac{1}{T^2} \frac{dT}{dx} dT \right\}. \quad (1.104)$$

If we apply this equation to the final state B ($x = +\infty$), where $\frac{dT}{dx} = 0$,

the first term in brackets vanishes and

$$\rho_0 u_0 (S_1 - S_0) = \kappa \int_{T_0}^{T_1} \frac{1}{T^2} \frac{dT}{dx} dT.$$

We now define the **effective thickness of the shock front** Δx , in which there is only heat conduction, by

$$\frac{T_1 - T_0}{\Delta x} = \left| \frac{dT}{dx} \right|_{\max},$$

where the geometric meaning is clear from Fig. 1.41. In order to

estimate the integral we set $\frac{dT}{dx} \sim \frac{T_1 - T_0}{\Delta x}$, from which we find

$$\rho_0 u_0 (S_1 - S_0) \sim \kappa \frac{1}{T_0^2} \frac{(T_1 - T_0)^2}{\Delta x}.$$

Expressing the temperature jump in terms of the pressure jump, we obtain

$$T_1 - T_0 = \left(\frac{\partial T}{\partial p} \right)_S (p_1 - p_0) = \frac{V_0}{c_p} (p_1 - p_0),$$

where c_p is the specific heat at constant pressure. Using formula (1.89) for the entropy jump and recalling that approximately $\left(\frac{\partial^2 V}{\partial p^2} \right)_S \sim \frac{V_0}{p_0^2}$, $\kappa \sim \rho_0 c_p l c_0$, and also that $u_0 \approx c_0$, we obtain from

(1.104) for an estimate of the **thickness of the shock front**

$$\Delta x \sim l \frac{p_0}{p_1 - p_0}. \quad (1.105)$$

The thickness of the shock front is inversely proportional to the wave strength, and the scale factor, as shown by (1.105), is the molecular mean free path l .

From equation (1.104) it is also possible to estimate the maximum increase of the entropy. At the point of maximum entropy $\frac{dS}{dx} = 0$ and

the gradient $\frac{dT}{dx}$ is also a maximum. Thus, the dominant term in the

expression in braces (1.104) is the first term which is proportional to

$\frac{\Delta T}{\Delta x} \sim \frac{\Delta p}{\Delta x} \sim (\Delta p)^2$, while the second term is proportional to

$\frac{(\Delta T)^2}{\Delta x} \sim (\Delta p)^3$. Hence it is clear that $S_{\max} - S_0 \sim (\Delta p)^2$, while

$S_1 - S_0 \sim (\Delta p)^3$.

Considering the internal structure of the shock front (taking into account heat conduction alone), we can only claim that the temperature in the wave changes continuously, while other quantities, such as density, velocity and pressure may, in general, be discontinuous. Indeed, the consideration of the structure of shock waves without taking into account of viscosity shows that it is impossible to construct a continuous distribution for all quantities in the wave for a sufficiently strong wave. This difficulty had already pointed out by Rayleigh (detail about this see Chapter VII, §3), and indicates the important role of

viscosity in realization of the irreversible shock compression of a fluid across a shock wave.

Let us consider now the second particular case.

(2) Viscosity is present but there is no heat conduction: $\kappa = 0$.

In this case we must retain the complete form of the momentum equation (1.99). A point describing the state of the gas in the wave moves in the p, V -plane, from point A to point B along some curve (shown in Fig. 1.42 by a dashed line) rather than along the straight line AB . From the entropy equation without the heat conduction term, we find

$$\rho u T \frac{dS}{dx} = \frac{4}{3} \mu \left(\frac{du}{dx} \right)^2, \quad (1.106)$$

which shows that the wave entropy increases monotonically from the initial value $S_0 = S_A$ to the final value $S_1 = S_B$; thus the dashed line is entirely contained between the isentropes (Poisson adiabats) S_0 and S_1 (see Fig. 1.42). Since the isentropes are convex downwards

$\left(\frac{\partial^2 p}{\partial V^2} \right)_S > 0$, the dashed line lies entirely below the straight line AB .²

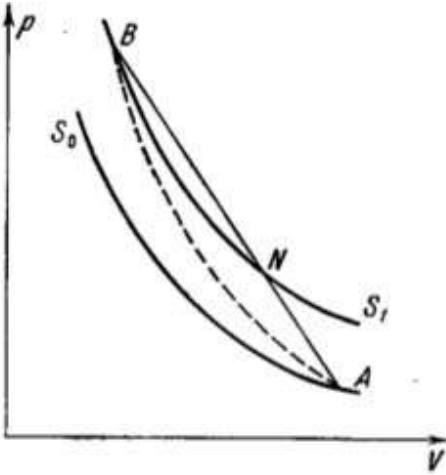


Fig. 1.42. p, V diagram for the structure of a shock front neglecting heat conduction. The state of the gas in the wave changes along the dashed curve AB .

The equation of the curve describing the transition from point A to point B is given by

$$p = p_0 + \rho_0 u_0^2 \left(1 - \frac{V}{V_0} \right) + \frac{4}{3} \mu \frac{du}{dx}. \quad (1.107)$$

Since the curve lies entirely below the straight line, then $\frac{du}{dx} < 0$ at

² Actually, the vertical distance between the isentropes S_1 and S_0 is proportional to $S_1 - S_0 \sim (p_1 - p_0)^3$, while the vertical distance between points A and B is $p_1 - p_0$. Therefore, the section of straight line AN on which the dashed line, in principle, could pass above the straight line, is small as compared to the main part of straight line AN .

all points inside the wave. If the x -axis is parallel to the flow, then $u > 0$, i.e., the gas in wave only decelerates and consequently is compressed monotonically. Thus, consideration of structure of shock front, taking into account of the viscosity, leads to the conclusion that with $\left(\frac{\partial^2 p}{\partial V^2}\right)_S > 0$, the gas in the shock wave can only be compressed. The density and velocity distributions through the wave have the form shown in Fig. 1.43.

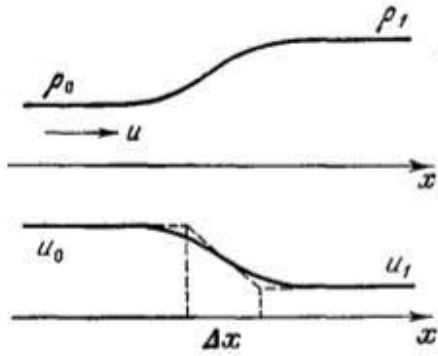


Fig. 1.43. Density and velocity profiles in a shock front. Δx is the effective front thickness.

We now define the effective shock front thickness by³

$$\frac{u_0 - u_1}{\Delta x} = \left| \frac{du}{dx} \right|_{\max}. \quad (1.108)$$

The geometrical meaning of the equation is clear. Maximum absolute value of the gradient $\left| \frac{du}{dx} \right|_{\max}$ is determined according to (1.107), by the maximum vertical deviation of the straight line AB from the dashed line, i.e., from the isentropes (Poisson adiabats) S_0 or S_1 . This deviation, as we already know, corresponds to the midpoint of the segment AB and is given by formula (1.102). Thus,

$$\frac{4}{3} \mu \left| \frac{du}{dx} \right|_{\max} = \frac{1}{8} \left(\frac{\partial^2 p}{\partial V^2} \right)_{S_A} (V_1 - V_0)^2.$$

Substituting this expression for $\left| \frac{du}{dx} \right|_{\max}$ into (1.108), and noting that $\mu = \rho_0 \nu \sim \rho_0 l \bar{v} \sim \rho_0 l c_0$ (ν is kinematic viscosity), and also that

$$u_0 - u_1 = [(p_1 - p_0)(V_0 - V_1)]^{1/2} \sim \left[(p_1 - p_0)^2 \left| \frac{\partial V}{\partial p} \right| \right]^{1/2} = \frac{p_1 - p_0}{p_0} c_0,$$

³ Δx is sometimes called *Prandtl front thickness*.

$$\left(\frac{\partial^2 p}{\partial V^2} \right)_S \sim \frac{p_0}{V_0^2},$$

we arrive again at formula (1.105) for the front thickness

$$\Delta x \sim l \frac{V_0}{V_0 - V_1} \approx l \frac{p_0}{p_1 - p_0}.$$

The front thickness can also be estimated from the entropy equation (1,106) by a procedure similar to that used for the first case:

$$\rho_0 u_0 T_0 \frac{S_1 - S_0}{\Delta x} \sim \mu \frac{(u_0 - u_1)^2}{\Delta x^2}.$$

Substituting here expression (1.89) for the entropy jump and rearranging, we arrive at the former formula for Δx .

During construction of continuous solution with only viscosity, no difficulties similar to those which appear during consideration of only thermal conduction appear. This circumstance, as already was noted, has a deep physical basis and indicates the fundamental role of viscosity in realization of the shock compression. Namely, **viscosity** is the mechanism due to which there occurs irreversible transformation of a part of kinetic energy of flow incident on shock into heat, i.e., transformation of energy of **ordered motion** of molecules of gas into energy of **random motion** due to the scattering of their momentum.

Heat conduction, in this sense, plays an indirect role in the conversion process, since it only participates in the transfer of energy of **random motion** of molecules from one point to another, but does not directly affect the **ordered motion**.

If we consider shock waves of moderate strength in an ordinary gas, where the transport coefficients – the kinematic viscosity ν and the thermal diffusivity κ – are approximately equal to each other and are determined by the same mean free path of molecules l ($\nu \approx \kappa \sim lc$), then we again obtain formula (1,105) for the front thickness. This is easily checked by considering the general entropy equation (1.98) which contains both the viscosity and the heat conduction.

Formula (1.105) shows that for a pressure jump in the wave of the order of magnitude of the actual pressure ahead of the front, the front thickness is of the order of the mean free path of molecules. With further increase of amplitude of wave, if we use the same formula, the thickness becomes smaller than the mean free path. This result, of course, does not have physical meaning. If gas dynamic quantities

change rapidly over distances of the order of the mean free path of molecules, then the hydrodynamic treatment of viscosity and heat conduction, which depends upon the assumption that the gradients are small, can no longer apply.

The thickness of an arbitrarily strong shock wave, of course, cannot become smaller than the mean free path of molecules, as indicated by studies based on the kinetic theory of gases (see Chapter VII).

Under certain conditions the front of a strong shock wave can become as thick as several mean free path, and it is then possible to divide it into regions of smooth and sharp changes of the flow variables. In particular, this occurs in a gas with delayed excitation of certain degrees of freedom of molecules or during the course of a reversible chemical reaction in the wave. These problems, as well as many others arising in the detailed study of internal structure of the shock fronts, will be considered in detail in Chapter VII.