CHAPTER VII

SHOCK WAVES STRUCTURE IN GASES

§ 1. Introduction

1. The Shock Front

§ 2. Viscous Shock Front

Since the compression shock in a shock front takes place over distances comparable with the gas kinetic molecular path, we should actually begin our study front structure on the bases of the kinetic theory of gases. As a first step in this direction, however, it is natural to consider the problem within the framework of the hydrodynamics of real fluid in which dissipative processes are taken into account, i.e., with viscosity and heat conduction. Here, in contrast to § 23, Chapter I, we do not impose limitations on the strength of the shock wave. To provide continuity of presentation we repeat here some conclusions and calculations of that section. In order not to complicate the presentation by unnecessary detail connected with the slow excitation of non-translational degree of freedom, we regard the gas as monatomic and neglect ionization.

The one-dimensional flow equations for a viscous and heat conducting gas flow , stationary in the system of coordinates moving with the wave front are

$$\left.\begin{aligned} &\frac{d}{dx}\rho u=0\\ &\rho u\frac{du}{dx}+\frac{dp}{dx}-\frac{d}{dx}\left(\frac{4}{3}\mu\frac{du}{dx}\right)=0\\ &\rho uT\frac{d\Sigma}{dx}=\frac{4}{3}\mu\left(\frac{du}{dx}\right)^2-\frac{dS}{dx}\end{aligned}\right\}. \qquad (7.1)$$

Here Σ is the specific entropy; μ is the coefficient of viscosity,*and S is non-hydrodynamic energy flow equal according to the ordinary heat conduction law to

$$S=-\kappa\frac{dT}{dx}, \qquad (7.2)$$

where κ is the coefficient of thermal conductivity.

To the system of equations (7.1) one should add the boundary conditions expressing the absence of any gradients "ahead" and "behind" the wave front, and also the fact that the flow variables must approach their initial (for $x=-\infty$) and final (for $x=+\infty$) values.

Rewriting the third equation in (7.1) with the help of the second law of thermodynamics

$$Td\Sigma = d\varepsilon + pdV = dh - \frac{1}{\rho}dp$$

and integrating each equations in (7.1), we obtain the first integrals of the system:

$$\left.\begin{aligned} \rho u &= \rho_0 D \\ p + \rho u^2 - \frac{4}{3}\mu\frac{du}{dx} &= p_0 + \rho_0 D^2 \\ h + \frac{u^2}{2} + \frac{1}{\rho_0 D}\left(S - \frac{4}{3}\mu u\frac{du}{dx}\right) &= h_0 + \frac{D^2}{2} \end{aligned}\right\}. \qquad (7.3)$$

The constants of integration here are expressed in terms of the initial values of the flow variables, distinguished by the subscript "0" and by the front velocity $D \equiv u_0$

If we refer the system of equations (7.3) to the final state (denoted by the subscript "1"), we obtain the well-known relations across a discontinuity, which we repeat here for convenience

$$\left.\begin{aligned} \rho_1 u_1 &= \rho_0 D \\ p_1 + \rho_1 {u_1}^2 &= p_0 + \rho_0 D^2 \\ h_1 + \frac{{u_1}^2}{2} &= h_0 + \frac{D^2}{2} \end{aligned}\right\}. \qquad (7.4)$$

It follows from these equations that the entropy jump across a shock wave $\Sigma_1 - \Sigma_0 = \Sigma(p_1, \rho_1) - \Sigma(p_0, \rho_0)$ is entirely independent of both the dissipative mechanisms involved , or in this case of the values of the coefficients of viscosity μ and thermal conductivity κ. The latter determine only the internal structure of the wave front and its thickness δ. The thickness δ of viscous shock front is proportional to the coefficients μ and κ, which, in turn, are proportional to the molecular mean free path *l*. In the limit $l \to 0$ the hydrodynamics of a real liquid becomes, in the continuous flow regions, the hydrodynamics of an ideal fluid. The shock front in the limit $l \to 0$ becomes a mathematical surface, since $\delta \sim l \to 0$. In this case, the gradients of all the flow variables across the front tend to infinity as $1/l$ but their jumps remain finite.

Specifying the coefficients of viscosity and thermal conductivity and also the thermodynamic relation $h(p, \rho)$ (in monatomic gas $h = c_p T = \frac{5}{2}\frac{p}{\rho}$), we can numerically integrate equations (7.3) and (7.2)

with the given boundary conditions. It is much more convenient, however, to have an **analytic solution,** since it illustrates graphically all the relationships governing the phenomenon. Unfortunately, it is not possible, in general, to find an analytic solution to the system. The equations can be integrated analytically if we limit ourselves to weak waves and expand the solution in a series with respect to the small change in one of the flow variables. This method was used in §23 of Chapter I for estimating the front thickness (the complete solution is given in the book by L. D. Landau and Ye. M. Lifshitz [1]). An exact **analytic solution for a wave of arbitrary strength** can be found in one **special case**. This solution, obtained for the first time by Becker [2] and later investigated by Morduchow and Libby [3], describes all the physical laws governing the structure of a shock front, and is both simple and graphic. Let us describe this solution in greater detail.

Usually the transport coefficients in gases, that is, the values of the kinematic viscosity $\nu = \frac{\mu}{\rho}$ and thermal diffusivity $\chi = \frac{\kappa}{c_p \rho}$, are close to one another and to the diffusion coefficient $\frac{l\bar{v}}{3}$. Let us se the dimensionless group $\Pr = \frac{\mu c_p}{\kappa} = \frac{\nu}{\chi}$, called the ***Prandtl number***, equal to $\frac{3}{4}$. In this case, the expression in parentheses in the last equations (7.3) becomes a total differential of the quantity $h + \frac{u^2}{2}$, and the equation becomes

$$\left(h + \frac{u^2}{2}\right) - \frac{4}{3}\frac{\mu}{\rho_0 D}\frac{d}{dx}\left(h + \frac{u^2}{2}\right) = h_0 + \frac{D^2}{2}.$$

In writing the integral of this linear equation, it is evident that $h + \frac{u^2}{2}$ can be finite at $x = +\infty$ only if it is independent of x,

$$h + \frac{u^2}{2} = h_0 + \frac{D^2}{2}. \qquad (7.5)[1]$$

Thus, for a **Prandtl number** Pr = 3/4, relationship (7.5) is satisfied not only behind the shock front (see (7.4)), but also at any intermediate point x.

Equation (7.5) gives a curve in the p, V plane along which the gas

[1] This equation is analogous to Bernoulli equation in steady flow theory.

changes from the initial to the final state. Noticing that for the **monatomic gas** under consideration $h = c_p T = \frac{\gamma}{\gamma - 1} pV = \frac{5}{2} pV$ ($\gamma = \frac{5}{3}$), and introducing the dimensionless velocity or specific volume

$$\eta = \frac{u}{D} = \frac{V}{V_0} = \frac{\rho_0}{\rho},$$

we find the equation of this curve

$$\frac{p}{p_0} = \frac{1 + \frac{1}{3} M^2 (1 - \eta^2)}{\eta} = \frac{4\eta_1 - \eta^2}{(4\eta_1 - 1)\eta}. \tag{7.6}$$

Here η_1 refers to the final state behind the shock front:

$$\eta_1 = \frac{1}{4} + \frac{5}{4} \frac{p_0}{\rho_0 D^2} = \frac{1}{4} + \frac{3}{4} \frac{1}{M^2}, \tag{7.7}$$

and M is the Mach number $M = \frac{D}{c_0}$, where c_0 is the speed of sound in the initial state ($c_0{}^2 = \gamma p_0 V_0 = \frac{5}{3} p_0 V_0$. In deriving (7.6) and (7.7) we have made use of the equation relating the variables on both sides of the shock front. The **Hugoniot equation** in terms of the variables p_1 / p_0 and η_1 is

$$\frac{p_1}{p_0} = \frac{4 - \eta_1}{4\eta_1 - 1}.$$

Figure 7.2 shows the Hugoniot curve and the curve along which the state of a particle changes through the wave (and also the characteristic straight line connecting initial and final states).

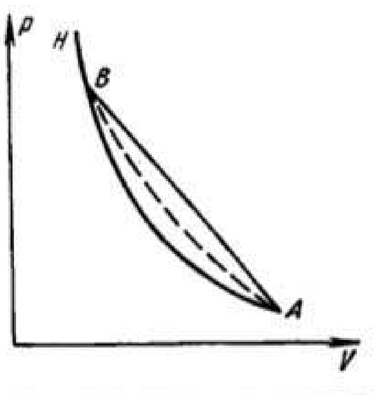

Fig. 7.2 Shock transition $A \to B$ on a p, V diagram. H is the Hugoniot curve. The point describing the state inside the wave front passes from A to B along the dashed line.

With the help of formula (7.6) and the first two equations of (7.3) we write a differential equation describing the velocity and specific volume distributions through the shock front $\eta(x)$

$$\frac{5}{3}\frac{\mu}{\rho_0 D}\eta\frac{d\eta}{dx}=-(1-\eta)(\eta-\eta_1)\,. \qquad (7.8)$$

We will, for simplicity, assume the **coefficient of viscosity to be independent of temperature** and equals to $\mu=\frac{\rho_0 l_0 \bar{v}_0}{3}$ (it is independent of density since $\mu \sim \rho l$ and $l \sim 1/\rho$). The integral of equation (7.8) contains an additive constant consistent with the arbitrariness in the choice of the coordinate origin. Placing the origin of the coordinate at the point of inflection of the velocity profile (in the "center" of the wave) and using (7.7), we find for $\eta(x)$ the expression

$$\frac{1-\eta}{(\eta-\eta_1)^{\eta_1}}=\frac{1-\sqrt{\eta_1}}{(\sqrt{\eta_1}-\eta_1)^{\eta_1}}\exp\left[a\frac{M^2-1}{M}\frac{x}{l_0}\right], \qquad (7.9)$$

$$\left(a=\frac{27}{40}\sqrt{\frac{5\pi}{6}}=1.1\right).$$

Knowing the velocity profile $u=D\eta$, it is easy to determine the distributions of other variables. Thus, for the temperature we have from (7.5) $\frac{T}{T_0}=1+\frac{1}{3}M^2(1-\eta^2)$; the pressure is defined in terms of η by (7.6) and the entropy is

$$\Sigma-\Sigma_0=c_p\ln\frac{T}{T_0}-R\ln\frac{p}{p_0} \quad \left(R=\frac{p}{\rho T}\right).$$

It is evident from (7.9) that as $x\to-\infty$, $\eta\to 1$ and as $x\to+\infty$, $\eta\to\eta_1$, with the initial and final values being approached asymptotically in an exponential manner. Flow variables – velocity, density, pressure and temperature – change monotonically across the wave from their initial to the final values, which are approached asymptotically as $x\to\mp\infty$.[2] The entropy, on the other hand, does not change monotonically, and has a maximum within the wave (this has been already shown in §23, Chapter I). We can easily satisfy ourselves that this is so by rewriting the entropy equation (the third of equation (7.1)) with the aid of the second law of the thermodynamics, "Bernoulli's equation" (7.5), and the second of equation (7.1). We find

[2] The inflection points for the various variables in the wave front do not coincide.

$$
\begin{aligned}
\rho u T \frac{d\Sigma}{dx} &= \rho u\left(\frac{dh}{dx} - V\frac{dp}{dx}\right) \\
&= \rho u\left(-u\frac{du}{dx} - V\frac{d}{dx}\frac{4}{3}\mu\frac{du}{dx} + V\rho u\frac{du}{dx}\right) \\
&= -u\frac{d}{dx}\frac{4}{3}\mu\frac{du}{dx} \\
&= -\frac{4}{3}\mu u\frac{d^2 u}{dx^2}
\end{aligned}.
$$

Hence it is clear that the entropy has an extremum at the inflection point in the velocity, i.e., at the "center" of the wave. The existence of the maximum of the entropy in the wave is connected with the presence of the heat conduction. One of the dissipative processes, viscosity, produces only the entropy increase, proportional to $\left(\frac{du}{dx}\right)^2$. Heat conduction, however, produces an irreversible transfer of heat from the hotter to colder gas layers. The increase in entropy of fluid particles through heat conduction in the colder layers (where $\frac{dS}{dx} \sim -\frac{d^2T}{dx^2} < 0$) is positive, and in the hotter layers (where $\frac{dS}{dx} \sim -\frac{d^2T}{dx^2} > 0$) is negative.

The entropy decrease in the more heated layers of gas does not in any way contradict the second law of thermodynamics. The entropy of the gas as a whole or of an individual particle increase across the whole shock discontinuity as a result of the process of shock compression. However, an individual layer of gas passing through the wave is no longer an isolated system. Its entropy increase at the beginning , when it is supplied heat through heat conduction and the work of the viscous forces, and then decreases when the heat loss due to heat conduction I the direction of the colder gas layers behind it exceeds the heat supplied by the work of the viscous forces.

The front thickness, as § 23, Chapter I, is given by

$$\delta = \frac{D - u_1}{\left(\frac{du}{dx}\right)_{\max}}.$$

It is evident from (7.9) that the order of magnitude of the front thickness is

$$\delta \sim l_0 \frac{M}{M^2 - 1}.$$

In a weak shock wave when $M-1<<1$, $\delta \sim \frac{l_0}{M-1}$, in agreement with the results presented in § 23, Chapter I. In this case, the front thickness can be equal to many molecular free paths. In the case when $M = 2$, shown in Fig. 7.3, the front thickness is approximately equal to three

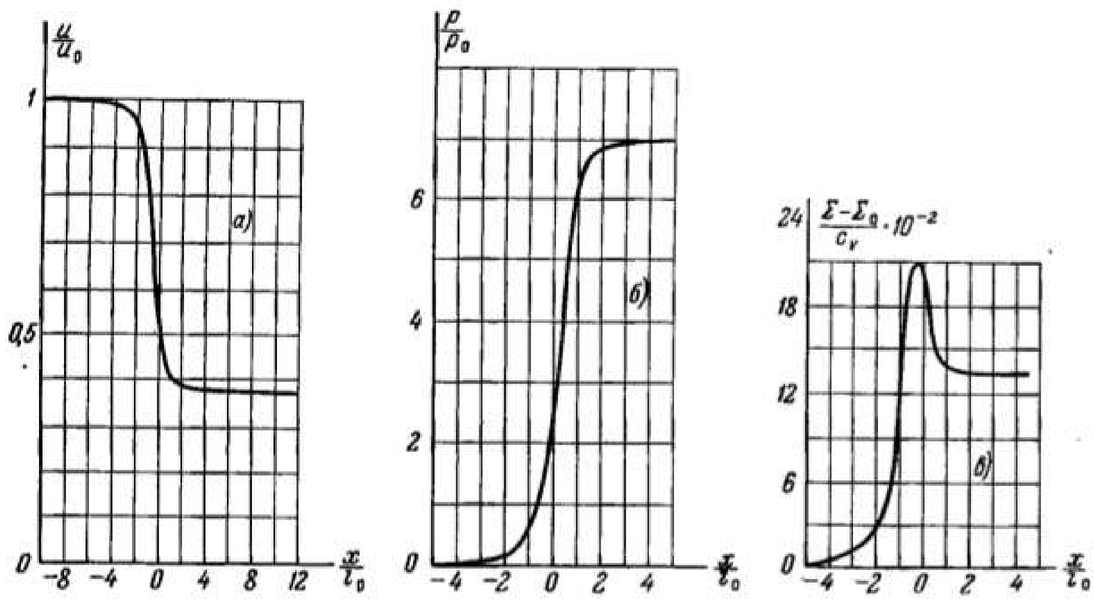

Fig. 7.3. Distributions of (a) velocity, (b) pressure, and (c) entropy through a viscous shock front with Mach number M = 2 in a gas with a specific heat ratio γ =7/5 and a temperature-independent viscosity coefficient. The abscissa is in units of molecular mean free path in the undisturbed gas (graphs taken from [3]).

molecular free paths. In the limiting case of a strong shock wave, when $M \to \infty$, $\delta \sim \frac{l_0}{M} \to 0$. The statement that the front thickness vanishes as the wave strength increases should, of course, not be taken literally. The fact is simply that when the front thickness becomes of the order of a mean free path, the hydrodynamic theory loses its meaning, since it is based on the assumption that the gradients are small, i.e., the mean free path is small as compared to the distance at which there occur considerable changes of hydrodynamic parameters. Therefore, to sufficiently strong waves the theory is simply not applicable. Physically, it is clear that the thickness of the shock wave in a wave of any strength cannot become less than a mean free path, since molecules of gas, incident on the discontinuity, must make at least several collisions so that directional momentum disperse and kinetic energy of directed motion be turned into kinetic energy of random motion (into heat). At the same time,

the thickness of the shock front in the case of a strong wave cannot include many mean free paths, since the molecules of the incident stream lose, on the average, a considerable fraction of their momentum during each collision.

The problem of the **structure of strong shock fronts** should be treated on the basis of the **kinetic theory of gases**, and therefore, numerous studies directed toward the improvement of the simple theory presented above, by taking the dependence of the transport coefficients on temperature into account, by calculating the effect of the Prandtl number on the front structure, and so forth [4-13], do not contribute anything new in principle beyond the particular case considered above, and at best are of interest for the case of weak wave only.[3]

I. Ye. Tamm and, independently, Mott-Smith [16], applied the Boltzmann kinetic equation to the problem of the structure of a shock front. An approximate solution to the Boltzmann equation in the neighborhood of the shock front is constructed as a superposition of two Maxwellian distributions corresponding to temperatures and macroscopic velocity in the initial and final states. The relative weight of each function varies over the width of the wave, from 0 to 1. The front thickness for an infinite strength shock wave tends to a finite limit. Sakurai [17], who has somewhat refined the Mott-Smith's method on the basis of a hard-sphere model for the interactions of molecules, obtained shock thickness in units of the mean free path based on the initial conditions, of[4] $\frac{\delta}{l_0} =$ 2.11, 1.68, 1.46 and 1.42 for Mach numbers equal to M = 2.5, 4, 10 and ∞, respectively. We note several other papers which have developed Mott-Smith's method and treated the shock front on the basis of the Boltzmann equation [52-55].

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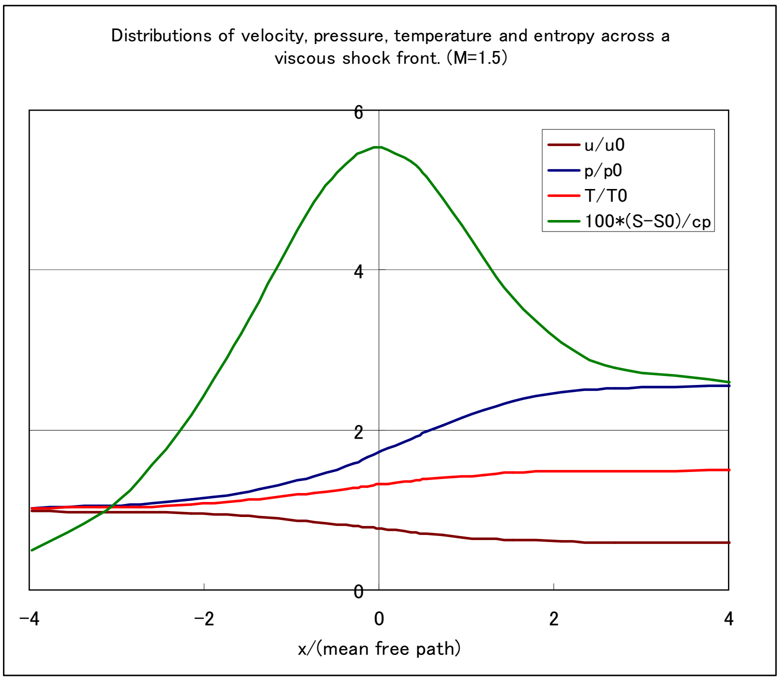
[3] An attempt to refine the hydrodynamic approach by taking into account second derivatives in the expressions for the transfer terms (the so-called Burnett approximation), undertaken by Zoller [14], somewhat improves the results for weak waves and, essentially, only indicates the limits of applicability of the hydrodynamic theory. For a wave strength $p_1 / p_0 = 1.5$, the thickness of the front, according to Zoller, is equal to 17 mean free paths, and for $p_1 / p_0 = 1.5$ is equal to 6 paths. The front thickness of weak shock waves in monatomic gases was treated by the method of light reflection in the works of Greene, Cowan, and Hornig, and others [15] (see § 5, Chapter IV), The thickness was found equal to 30, 19, and 13 mean free paths for Mach numbers M = 1.1, 1.5, and 2.5, respectively. Calculations of Zoller give good agreement with these results. See also [56].

[4] The front thickness δ is defined in the following way. If f_α and f_β are distribution functions of molecules in the initial and final states, then the distribution function at an intermediate point x in the wave is, according to the theory, $f = \nu(-x) f_\alpha + \nu(x) f_\beta$, , where $\nu(x) = \frac{1}{2}\left\{1 + \tanh \frac{2x}{\delta}\right\}$.

[Reader's exercise]

For Mach number M = 1.5 the distributions of velocity, pressure, temperature and entropy is calculated as shown in the following Figure.



§ 3. The role of viscosity and heat conduction in the formation of a shock front