Chapter 2

THE POINT SOURCE SOLUTION

by

John von Neumann

2.1 Introduction

The conventional picture of a blast wave is this: In a homogeneous atmosphere a certain sphere around the origin is suddenly replaced by homogeneous gas of much higher pressure. The high pressure area will immediately begin to expand against the surrounding low pressure atmosphere and send a pressure wave into it. As the high pressure area expands, its density decreases and with it the pressure; hence the effects it causes in the surrounding atmosphere weaken. As the pressure wave expands spherically through the atmosphere it is diluted over spherical shells of ever-increasing radii, and hence its intensity (the density of energy, and with it the overpressure) decreases continuously also. This pressure wave is known (both theoretically and experimentally) to consist at all times of a discontinuous shock wave at the head, and to weaken gradually as one goes backward from that head.

This description of the blast wave caused by an explosion is somewhat schematic, since the high pressure area caused by an explosion is not produced instantaneously, nor is its interior homogeneous, nor is it in general exactly spherical. Nevertheless, it seems to represent a reasonable approximation of reality.

Mathematically, however, this approximate description offers very great difficulties. To determine the details of the history of the blast, that is, of its decay, the following things must be computed: (I) The trajectory of the shock wave, that is, of the head of the blast wave, and (II) the continuous flow of air behind the shock (ahead of the shock the air is unperturbed and at rest). This requires the solution of a partial differential equation bounded by the unknown trajectory (I). Along this trajectory the theory of shocks imposes more boundary conditions than are appropriate for a differential equation of the type (II), and this overdetermination produces a linkage between (I) and (II) which should permit one to determine the trajectory of (I) and to solve (II). To this extent the problem is a so-called "free boundary" partial differential equation problem. However, the situation is further complicated by the fact that at each point (II) the local entropy is determined by the entropy change the corresponding gas underwent when it crossed the shock (I), that is, by the

shock strength at a certain point of (I). The latter depends on the shape of the trajectory (I), and the entropy in question influences the coefficients of the differential equation (II). Hence the differential equation (II) itself depends on the shape of the unknown trajectory (I). This dependence cannot be neglected as long as the entropy change caused by the shock is important, that is, as long as the shock is strong (in air a shock can be considered "strong" if the shock pressure exceeds 3 atm). Mathematically such problems are altogether inaccessible to our present analytical techniques. For this reason the general problem of the decay of blast has been treated only by approximate analytical methods, or numerically, or by combinations of these.

For very violent explosions a further simplification suggests itself, which changes the mathematical situation very radically. For such an explosion it may be justified to treat the original, central, high pressure area as a point. Clearly, the blast coming from a point, or rather from a negligible volume, can have appreciable effects in the outside atmosphere only if the original pressure is very high. One will expect that, as the original high pressure sphere shrinks to a point, the original pressure will have to rise to infinity. It is easy to see, indeed, how these two are connected. One will want the energy of the original high pressure area to have a fixed value E_0, and as the original volume containing E_0 shrinks to zero, the pressure in it will have to rise to infinity. It is clear that of all known phenomena nuclear explosions come nearest to realizing these conditions.

We will therefore investigate the laws of the decay of blast wave[1] due to a point explosion of energy E_0.

The essential simplification permitted by this model is the so-called "similarity property" of the solution. This property can be explained in the following manner:

Denote pressure, density, and temperature in the atmosphere by p, ρ, T. The significant data of the situation are these: The original (ball of fire) values of p, ρ, T in undisturbed air, p_0, ρ_0, T_0; the equation of state of the atmosphere, $p = c\rho T$; the caloric equation of state, $E_i = \frac{c}{\gamma - 1}T$;

[1] The main facts in the discussion which follows were presented by G. I. Taylor, British Report RC-210, June 27, 1941; and John von Neumann, NDRC, Div. B, Report AM-9, June 30, 1941. Important simplifications (in particular, the use of the variable θ of Eq. 2.44) are due to G. Y. Kynch, British Report BM-82, MS-69, Sept. 18, 1943. The results were generalized by J. H. Van Vleck, NDRC, Div. B, Report AM-11, Sept. 15, 1942. Compare also the later work of G. I. Taylor, Proc. Roy. Soc. (London), A201, 159 (1950).

and the original (explosive) release of energy, E_0. The mass and the characteristics of the point explosive are to be neglected, in the same sense in which a genuine point source is being assumed. Also, since the pressures which we propose to consider are to be very high, that is, very high compared to p_0, we will usually neglect p_0. (However, ρ_0 is not neglected!)

Put, accordingly, $p_0 = 0$ for the time being. Furthermore, let $t = 0$ be the time of the original energy release (explosion). Since the constant c is needed to connect the dimension of T to the "CGS" system, and since the constant γ is dimensionless, the only dimensioned quantities which appear among the data of the problem are $E_0 \sim ML^2T^{-2}$ and $\rho_0 \sim ML^{-3}$. Hence, the only combinations of the units of mass, length, and time (M, L, T) which can be significant in this problem are ML^{-3} and L^2T^{-2}.

Now let $t = ct'$, a lines change of time-scale. Then if our problem as. stated possesses a well-defined and unique solution, this solution must be unaffected by the above change in time-scale. This means

$$ML^2T^{-2} = M'\,L'^2\,T'^{-2} = ML^2c^{-2}T'^{-2},$$
$$ML^{-3} = M'\,L'^{-3}.$$

From this it follows that

$$L = L'c^{2/5};\quad M = M'c^{6/5}.$$

This will indeed be the case providing that

$$\text{lengths} \propto T^{2/5},$$
$$\text{mass} \propto T^{6/5}.$$

To put it more precisely, denote the distances from the site of the original energy release (explosion) by the letters x, X, Ξ. Let the trajectory of the shock wave (blast head) be

$$\Xi = \Xi(t). \tag{2.1}$$

If a gas element had originally (at $t = 0$) the (unperturbed) position x, then let its position at the time t be

$$X = X(x,t). \tag{2.2}$$

(x is the Lagrangian, X the Eulerian coordinate.) Now by the above Eq. 2.1 must have the form

$$\Xi = at^{2/5}, \tag{2.1'}$$

and Eq. 2.2 must have the form

$$\frac{X}{t^{2/5}} = f\left(\frac{x}{t^{2/5}}\right). \tag{2.2'}$$

It is evident that these relations will greatly simplify the entire problem.

Only a one-variable function, $f(z)$, is unknown; the partial differential equations must become ordinary ones and the unknown trajectory of the shock is replaced by one unknown parameter a. As will appear below, the situation is even more favorable. Everything can be determined by means of explicit t quadrature.

2.2 Analytical Solution of the Problem

We must now set up the equations controlling the two phenomena referred to in Section 2.1: (I) the trajectory of the shock wave; (II) the continuous airflow behind the shock. These are to be formulated with the help of Eqs. 2.1' and 2.2' of Section 2.1. We rewrite 2.1' unchanged: $\Xi = at^{2/5}$; but in 2.2' we replace $t^{2/5}$ by $at^{2/5}$

$$X = at^{2/5} F\left(\frac{x}{at^{2/5}}\right), \tag{2.2''}$$

and also introduce

$$z = \frac{x}{\Xi} = \frac{x}{at^{2/5}} . \tag{2.3}$$

Ahead of the shock lies the unperturbed atmosphere in the state $p_0 = 0$, (cf. Section 2.1) ρ_0, T_0 and with the mass velocity 0; behind the shock lies the shocked (compressed and heated) and then more or less re-expanded atmosphere in the state p, $\rho = \left[\dfrac{\partial(x^3)}{\partial(X^3)}\right]_t$, T, and with the mass velocity $u = \left(\dfrac{\partial X}{\partial t}\right)_X$. The shock itself has the velocity $U = \dfrac{d\Xi}{dt}$. Thus

$$\rho = \rho_0 \left[\frac{\partial(x^3)}{\partial(X^3)}\right]_t = \rho_0 \frac{x^2}{X^2} \frac{1}{\left(\dfrac{\partial X}{\partial x}\right)_t} = \rho_0 \frac{z^2}{F(z)^2} \frac{1}{F'(z)}, \tag{2.4}$$

$$u = \left(\frac{\partial X}{\partial t}\right)_X = \frac{2}{5} at^{-3/5} \left[F(z) - zF'(z)\right], \tag{2.5}$$

$$U = \frac{d\Xi}{dt} = \frac{2}{5} at^{-3/5} . \tag{2.6}$$

Let us now consider the conditions immediately behind the shock; that is, at $X = \Xi$ (precisely: $X = \Xi - 0$). Immediately before the shock got there, this gas was in its original state of rest, i.e., it had $X = x$. Since the shock causes no discontinuous changes in position (but only in pressure, density, mass velocity), hence $X = x$ remains true immediately behind the

shock. Thus, $X = x = \Xi$, i.e., $F(z) = z = 1$. In other words, the shock occurs at z = (I) (immediately behind it: z = 1- 0), and it imposes upon F the boundary condition

$$F(z) = 1 \quad \text{at } z = 1. \tag{2.7}$$

We note that for reasons of symmetry the origin can never be displaced; i.e., $x = 0$ goes at all times with $X = 0$. This gives for F the further boundary condition

$$F(z) = 0 \quad \text{at } z = 0. \tag{2.8}$$

Returning to the shock, Eqs. 2.4 to 2.6 above, with $z = 1$, give the conditions immediately behind it. The Hugoniot shock conditions express all that must be required at this point. They can be stated as follows:

$$\frac{\rho}{\rho_0} = \frac{(\gamma+1)p + (\gamma-1)p_0}{(\gamma-1)p + (\gamma+1)p_0}, \tag{2.9}$$

$$u = \frac{2(p - p_0)}{\sqrt{2\rho_0[(\gamma+1)p + (\gamma-1)p_0]}}, \tag{2.10}$$

$$U = \sqrt{\frac{(\gamma+1)p + (\gamma-1)p_0}{2\rho_0}}. \tag{2.11}$$

Considering $p_0 = 0$ (cf. above), these become

$$\rho = \frac{\gamma+1}{\gamma-1}\rho_0, \tag{2.9'}$$

$$u = \sqrt{\frac{2}{\gamma+1}\frac{p}{\rho_0}}, \tag{2.10'}$$

$$U = \sqrt{\frac{\gamma+1}{2}\frac{p}{\rho_0}}. \tag{2.11'}$$

We rewrite Eqs. 2.10’ and 2.11’ to express p and u in terms of U.

$$u = \frac{2}{\gamma+1}U, \tag{2.10''}$$

$$p = \frac{2}{\gamma+1}\rho_0 U^2. \tag{2.11''}$$

Equation 2.11” cannot be compared with 2.4 to 2.6, since it contains p, which does not occur there. Equations 2.9’ and 2.10’’ can be compared, putting $z = 1$ in 2.4 to 2.6 and using 2.7. Both give the same thing:

$$F'(z) = \frac{\gamma-1}{\gamma+1} \quad \text{at } z = 1. \tag{2.12}$$

Thus we have exhausted the discussion of the physical problem in Section 2.1, that is, essentially of the shock conditions. This turned out to be equivalent to the boundary conditions given by Eqs. 2.7 and 2.12 at z = 1; Eq. 2.8 at $z = 0$ is self evident.

The flow of the gas behind the shock is expected to be shock-free and hence adiabatic. That is, every particle x of the gas has the same entropy $p\rho^{-\gamma}$ at all times after it crossed the shock. We can therefore take for it the value of $p\rho^{-\gamma}$ immediately behind the shock, with the same x. Given z, t for a particle, and using Eqs. 2.1' and 2.3, its x is $at^{2/5}z$; hence the t' at which it crossed the shock is defined by $at'^{2/5}z = at^{2/5}z$, i.e., $t' = tz^{5/2}$. Hence by Eqs. 2.9' and 2.11' we have immediately behind the shock

$$\begin{aligned} p\rho^{-\gamma} &= \frac{2}{\gamma+1}\rho_0 U^2\left(\frac{\gamma+1}{\gamma-1}\rho_0\right)^{-\gamma} \\ &= \frac{2(\gamma-1)^{\gamma}}{(\gamma+1)^{\gamma+1}}\rho_0^{-(\gamma-1)}U^2 \end{aligned} .$$

Using Eqs. 2.4 and 2.6 this gives

$$p = \frac{8(\gamma-1)^{\gamma}}{25(\gamma+1)^{\gamma+1}}\rho_0 a^2 t'^{-6/5}\frac{z^{2\gamma}}{[F(z)]^{2\gamma}}\frac{1}{[F'(z)]^{\gamma}} .$$

that is, by the above

$$p = \frac{4}{25}\Phi\rho_0 a^2 t^{-6/5}\frac{z^{2\gamma-3}}{[F(z)\}^{2\gamma}}\frac{1}{[F'(z)]^{\gamma}} , \quad (2.13)$$

where

$$\Phi = \frac{2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma} . \quad (2.14)$$

We now pass to the consideration of (II) in Section 2.1, that is, of the continuous flow of air behind the shock. As we saw above, this region is defined by $0 < x < \Xi$, i.e., by $0 < z < 1$, and in it z, t, x, X are connected by Eqs. 2.2'' and 2.3, and p, ρ, u are given by Eqs. 2.13, 2.4 and 2.5.

With the help of these relations one can set up the equation of motion and thereby achieve a complete formulation of our problem. It turns out, however, that it is preferable to work with the energy principle instead. Since only one Lagrangian coordinate is involved (x), it is indeed adequate to consider the energy principle only. And by virtue of unusually favorable special circumstances, the energy principle leads to a differential equation of order 1, whereas the equation of motion would lead to one of order 2. A reduction of the order by another unit is possible in either case for reasons of symmetry, and therefore the former procedure permits the reduction of the entire problem to quadrature. This situation is mathematically of some interest and not at all trivial, but we do not propose to pursue this aspect here any further. At any rate we are going to use the energy principle, since it leads to an easier solution.

Consider the energy contained in the gas behind the shock. It is made up of the inner (thermic) energy $\frac{1}{\gamma-1}\frac{p}{\rho}$ and the kinetic energy $\frac{1}{2}u^2$ (both per unit mass); hence, the total energy per unit mass is

$$\varepsilon=\frac{1}{\gamma-1}\frac{p}{\rho}+\frac{1}{2}u^2 .$$

The amount of gas in the spherical shell reaching from the particles x to the particles $x + dx$ is the same for all t, and hence we may use its value for $t = 0$, which is clearly $4\pi\rho_0 x^2 dx$. Hence the total energy inside the sphere of the particles x is

$$\begin{aligned}\varepsilon_1(x)&=4\pi\rho_0\int_0^x \varepsilon x^2 dx\\ &=4\pi\rho_0\int_0^x\left(\frac{1}{\gamma-1}\frac{p}{\rho}+\frac{1}{2}u^2\right)x^2dx\end{aligned},$$

or upon introducing z, and using Eqs. 2.3 to 2.5 and 2.13

(1)

$$\varepsilon_2(z)=4\pi\rho_0\int_0^z\left\{\frac{1}{\gamma-1}\frac{\frac{4}{25}\Phi\rho_0 a^2 t^{-6/5}\frac{z^{2\gamma-3}}{[F(z)]^{2\gamma}}\frac{1}{[F'(z)]^{\gamma}}}{\rho_0\frac{z^2}{[F(z)]^2}\frac{1}{F'(z)}}+\frac{1}{2}\cdot\frac{4}{25}a^2t^{-6/5}[F(z)-zF'(z)]^2\right\}a^3t^{6/5}z^2dz$$

that is,

(2)

$$\varepsilon_2(z)=\frac{8\pi}{25}\rho_0 a^5\int_0^z\left\{\frac{2}{\gamma-1}\Phi\frac{z^{2(\gamma-1)-3}}{[F(z)]^{2(\gamma-1)}}\frac{1}{[F'(z)]^{\gamma-1}}+[F(z)-zF'(z)]^2\right\}t^2dz$$

(2.15)

From Eq. 2.15 we can draw two conclusions.

The first conclusion obtains by putting $z = 1$. Then Eq. 2.15 represents the entire energy within the shocked region. Outside the shocked region the energy of the gas is 0 (since $\rho_0 = 0$, $u = 0$), and at $t = 0$ this (energy = 0) would apply to the entire gas. Hence $\varepsilon_2(1)$ is the total energy acquired by the gas between $t = 0$ and present $t > 0$. This quantity is the same for all $t > 0$, and clearly positive. It is obvious that it must be identified with the explosion energy E_0 of Section 2.1. So we have

$$E_0=\frac{8\pi}{25}\rho_0 a^5\int_0^1\left\{\frac{2}{\gamma-1}\Phi\frac{z^{2(\gamma-1)-3}}{[F(z)]^{2(\gamma-1)}}\frac{1}{[F'(z)]^{\gamma-1}}+[F(z)-zF'(z)]^2\right\}z^2dz$$

(2.16)

The second conclusion obtains by considering a general z (> 0, < 1). It is clear from Eq. 2.15 that the energy within the z-sphere is constant. This was a physical necessity for $z = 1$, i.e., for the entire shock zone, but for general z it is a new fact with considerable consequences.

Indeed, let such a z (> 0, < 1) be given. This z-sphere contains the gas within the x-sphere, $x = at^{2/5}z$; i.e., its material content changes with t. The constancy of its energy amounts to stating that the energy flowing into it with the new material that enters is exactly compensated by the work which its original surface does by expanding against the surrounding pressure. It should be noted that in making this last statement we are stating the **energy principle**, that is, the **equivalent of the equation of motion**.

Let us therefore express the **two energy changes** referred to above and state their equality.

The energy of tie material entering the z-sphere, i.e., the x-sphere $x = at^{2/5}z$, in the time between t and $t + dt$ is

$$4\pi\rho_0 x^2 (dx)_t\, \varepsilon = 4\pi\rho_0\left(\frac{1}{\gamma-1}\frac{p}{\rho}+\frac{1}{2}u^2\right)x^2 (dx)_t\ ,$$

and using the form of the integrand in the first expression (1), $\varepsilon_2(z)$ of Eq. 2.15, the right hand side becomes

$$4\pi\rho_0\left\{\frac{1}{\gamma-1}\frac{\frac{4}{25}\Phi\rho_0 a^2 t^{-6/5}\frac{z^{2\gamma-3}}{[F(z)]^{2\gamma}}\frac{1}{[F'(z)]\gamma}}{\rho_0\frac{z^2}{[F(z)]^2}\frac{1}{F'(z)}}+\frac{1}{2}\cdot\frac{4}{25}a^2 t^{-6/5}[F(z)-zF'(z)]^2\right\}a^2 t^{4/5} z^2\,\frac{2}{5}at^{-3/5}dtz$$

that is,

$$\frac{16\pi}{125}\rho_0 a^5 t^{-1}\left\{\frac{2}{\gamma-1}\Phi\frac{z^{2(\gamma-1)-3}}{[F(z)]^{2(\gamma-1)}}\frac{1}{[F'z)]^{\gamma-1}}+[F(z)-zF'(z)]^2\right\}z^3 dt\ . \tag{2.17}$$

The work done by the original surface by expanding against the surrounding pressure is

$$4\pi p X^2 u dt\ ,$$

and using Eqs. 2.2'', 2.3, 2.5 and 2.13, this becomes

$$4\pi\frac{4}{25}\Phi\rho_0 a^2 t^{-6/5}\frac{z^{2\gamma-3}}{[F(z)]^{2\gamma}}\frac{1}{[F'(z)]^{\gamma}}a^2 t^{4/5}[F(z)]^2\frac{2}{5}at^{-3/5}[F(z)-zF'(z)]dt$$

that is,

$$\frac{32\pi}{125}\rho_0 a^5 t^{-1}\Phi\frac{z^{2\gamma-3}}{[F(z)]^{2(\gamma-1)}}\frac{1}{[F'(z)]^{\gamma}}[F(z)-zF'(z)]dt\ . \tag{2.18}$$

Equating 2.17 and 2.18 gives

$$\frac{2}{\gamma-1}\Phi\frac{z^{2(\gamma-1)}}{[F(z)]^{2(\gamma-1)}}\frac{1}{[F'(z)]^{\gamma-1}}+z^3[F(z)-zF'(z)]^2=2\Phi\frac{z^{2\gamma-3}}{[F(z)]^{2(\gamma-1)}[F'(z)]^{\gamma}}[F(z)-zF'(z)] \tag{2.19}$$

This equation is equivalent to the **equation of motion**, as pointed out earlier in this section. Together with the boundary conditions (Eqs. 2.7, 2.12, and 2.8) it contains the full statement of our problem while the connection with the given explosion energy E_0 is given by Eq. 2.16.

We now proceed to the integration of the differential equation, 2,19. Put

$$z=e^s\,, \tag{2.20}$$

$$F(z)=e^{\nu s}\Phi(s)\,, \tag{2.21}$$

the constant ν to be determined later. Then Eq. 2.19 becomes

$$\frac{2}{\gamma-1}\Phi\frac{e^{[2(\gamma-1)-2(\gamma-1)\nu-(\gamma-1)(\nu-1)]s}}{\Phi^{2(\gamma-1)}\left(\frac{d\Phi}{ds}+\nu\Phi\right)^{\gamma-1}}+e^{(3+2\nu)s}\left[\frac{d\Phi}{ds}+(\nu-1)\Phi\right]^2=-2\Phi e^{[2\gamma-3-2(\gamma-1)\nu-\gamma(\nu-1)+\nu]s}\frac{\frac{d\Phi}{ds}+(\nu-1)\Phi}{\Phi^{2(\gamma-1)}\left(\frac{d\Phi}{ds}+\nu\Phi\right)^{\gamma}}$$

Each of these three terms contains a factor e^{As}, the values of A being

(1) $2(\gamma-1)-2(\gamma-1)\nu-(\gamma-1)(\nu-1)$,

(2) $3+2\nu$,

(3) $2\gamma-3-2(\gamma-1)\nu-\gamma(\nu-1)+\nu$.

The first and the third are clearly equal, and they differ from the second by $(3\gamma-1)\nu-3(\gamma-2)$. Hence all three are equal, and thereby s no longer appears explicitly in the differential equation, if

$$\nu=\frac{3(\gamma-2)}{3\gamma-1}\,. \tag{2.22}$$

So we have

$$\frac{2}{\gamma-1}\Phi\frac{1}{\Phi^{2(\gamma-1)}\left(\frac{d\Phi}{ds}+\nu\Phi\right)^{\gamma-1}}+\left[\frac{d\Phi}{ds}+(\nu-1)\Phi\right]^2=-2\Phi\frac{\frac{d\Phi}{ds}+(\nu-1)\Phi}{\Phi^{2(\gamma-1)}\left(\frac{d\Phi}{ds}+\nu\Phi\right)^{\gamma}} \tag{2.23}$$

Now put

$$\Psi=\frac{d\Phi}{ds}+\nu\Phi\,, \tag{2.24}$$

that is,

$$zF'(z)=e^{\nu s}\Psi(s)\,. \tag{2.24'}$$

Then Eq. 2.23 becomes

$$\frac{2}{\gamma-1}\Phi\frac{1}{\Phi^{2(\gamma-1)}\Psi^{\gamma-1}}+[\psi-\Phi]^2=-2\Phi\frac{\Psi-\Phi}{\Phi^{2(\gamma-1)}\Psi^{\gamma}}$$

that is,

$$(\Psi-\Phi)^2+2\Phi\frac{\Psi-\Phi}{\Phi^{2(\gamma-1)}\Psi^{\gamma}}+\frac{2}{\gamma-1}\Phi\frac{1}{\Phi^{2(\gamma-1)}\Psi^{\gamma-1}}=0\,. \tag{2.25}$$

Thus Φ, Ψ are functions of each other by Eq. 2.25, and then Eqs. 2.22, 2.23 and 2.24' permit determination of z, $F(z)$, $F'(z)$ by one quadrature.

We first solve Eq. 2.25 explicitly by **parametrisation**. Recall

$$\Phi=\frac{2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}. \tag{2.14}$$

Put

$$D=\frac{\gamma-1}{\gamma+1}. \tag{2.26}$$

Then

$$\Phi=(1-D)D^{\gamma}, \tag{2.14'}$$

$$\frac{2}{\gamma-1}=\frac{1-D}{D}.$$

Now Eq. 2.25 may be written

$$\left(\frac{\Phi}{\Psi-1}\right)^2-2\Phi\frac{\frac{\Phi}{\Psi}-1}{\Phi^{2(\gamma-1)}\Psi^{\gamma+1}}+\frac{2}{\gamma-1}\Phi\frac{1}{\Phi^{2(\gamma-1)}\Psi^{\gamma+1}}=0\,,$$

that is,

$$\left(\frac{\Phi}{\Psi}-1\right)^2-2\frac{1-D}{D}\frac{\frac{\Phi}{\Psi}-1}{\Phi^{2(\gamma-1)}\left(\frac{\Psi}{D}\right)^{\gamma+1}}+\left(\frac{1-D}{D}\right)^2\frac{1}{\Phi^{2(\gamma-1)}\Psi^{\gamma+1}}=0$$

or equivalently,

$$\left(\frac{\frac{\Phi}{\Psi}-1}{\frac{1}{D}-1}\right)^2-2\frac{\frac{\Phi}{\Psi}-1}{\Phi^{2(\gamma-1)}\left(\frac{\Psi}{D}\right)^{\gamma+1}}\frac{\frac{\Phi}{\Psi}-1}{\frac{1}{D}-1}+\frac{1}{\Phi^{2(\gamma-1)}\left(\frac{\Psi}{D}\right)^{\gamma+1}}=0\,.$$

Now put

$$\xi=\frac{\frac{\Phi}{\Psi}-1}{\frac{1}{D}-1}, \tag{2.27}$$

$$\eta = \Phi^{2(\gamma-1)}\left(\frac{\Psi}{D}\right)^{\gamma+1} . \qquad (2.28)$$

Then the above equation becomes

$$\xi^2 - 2\frac{\xi}{\eta} + \frac{1}{\eta} = 0 ,$$

that is,

$$\eta = \frac{2\xi - 1}{\xi^2} .$$

It is convenient to define a new quantity θ by

$$\xi = \frac{1+\theta}{2} . \qquad (2.29)$$

We can now express s explicitly in terms of θ, and then z, $F(z)$, $F'(z)$ also. To do ms, we first note that according to Eq. 2.29

$$\eta = \frac{4\theta}{(1+\theta)^2} . \qquad (2.30)$$

Next Eq. 2.27 gives

$$\frac{\Phi}{\Psi / D} = D\frac{\Phi}{\Psi} = D\left[\left(\frac{1}{D} - 1\right)\xi + 1\right] = (1-D)\xi + D$$

and then this relation and Eq. 2.28 give

$$\Phi = \eta^{1/(3\gamma-1)}\left[(1-D)\xi + D\right]^{(\gamma+1)/(3\gamma-1)} ,$$
$$\Psi = D\eta^{1/(3\gamma-1)}\left[(1-D)\xi + D\right]^{-2(\gamma-1)/(3\gamma-1)} .$$

Substituting from Eqs. 2.2, 2.29 and 2.30, we get

$$\Phi = \theta^{1/(3\gamma-1)}\left(\frac{\theta+1}{2}\right)^{-2/(3\gamma-1)}\left(\frac{\theta+\gamma}{\gamma+1}\right)^{(\gamma+1)/(3\gamma-1)} , \qquad (2.31)$$

$$\Psi = \frac{\gamma-1}{\gamma+1}\theta^{1/(3\gamma-1)}\left(\frac{\theta+1}{2}\right)^{-2/(3\gamma-1)}\left(\frac{\theta+\gamma}{\gamma+1}\right)^{-2(\gamma-1)/(3\gamma-1)} .$$

(2.32)

Note that θ must be positive: Φ is intrinsically positive by Eqs. 2.21 and 2.2'' along with $F'(z)$ and X; Ψ is intrinsically positive by Eqs. 2.24' and 2.4 along with $F'(z)$ and ρ_1; the positivity of Φ and Ψ implies the positivity of η by Eq. 2.28 and the positivity of θ by Eq. 2.30. Thus we require

$$\theta > 0 . \qquad (2.33)$$

By the definition of Ψ in Eq. 2.24,

$$\frac{d\Phi}{ds} = \Psi - \nu\Phi .$$

Hence,

$$s=\int\frac{d\Phi}{\Psi-\nu\Phi}=\int\frac{\frac{d\Phi}{\Phi}}{\frac{\Psi}{\Phi}-\nu}.$$

The integrand is easily rewritten with the help of Eqs. 2.27, 2.29, and 2.31, yielding

$$s=\int\frac{\frac{1}{3\gamma-1}\frac{d\theta}{\theta}-\frac{2}{3\gamma-1}\frac{d\theta}{\theta+1}+\frac{\gamma+1}{3\gamma-1}\frac{d\theta}{\theta+\gamma}}{\frac{\gamma-1}{\theta+\gamma}+\frac{3(2-\gamma)}{3\gamma-1}}$$

$$=\int\frac{(\theta+\gamma)\left(\frac{d\theta}{\theta}-2\frac{d\theta}{\theta+1}\right)+(\gamma+1)d\theta}{3(2-\gamma)\theta+2\gamma+1}$$

$$=\int\frac{\theta+\gamma}{\theta[3(2-\gamma)\theta+2\gamma+1]}d\theta-\int\frac{2(\theta+\gamma)}{(\theta+1)[3(2-\gamma)\theta+2\gamma+1]}d\theta+\int\frac{\gamma+1}{3(2-\gamma)\theta+2\gamma+1}d\theta$$

Carrying out the integration we get

$$s=c_1+\frac{\gamma}{2\gamma+1}\ln\theta-\frac{2}{5}\ln(\theta+1)+\frac{13\gamma^2-7\gamma+12}{15(2-\gamma)(2\gamma+1)}\ln[3(2-\gamma)\theta+2\gamma+1] \tag{2.34}$$

Before we go further, let us express the **boundary conditions**, Eqs. 2.12, in the new parameters.

Equations 2.7 and 2.12 require that at z = 1, $F(z)$ = 1 and $F'(z)=\frac{\gamma-1}{\gamma+1}$. By Eqs. 2.20, 2.21 and 2.24' this means that at s = 0, we must have $\Phi=1$, and $\Psi=\frac{\gamma-1}{\gamma+1}$. By Eqs. 2.31 and 2.32 this means that at s = 0, we have $\theta=1$. [$\theta=1$ clearly implies $\Phi=1$, $\Psi=\frac{\gamma-1}{\gamma+1}$, and it is implied by them since $\frac{\Psi}{\Phi}=\frac{\gamma-1}{\theta+\gamma}$.]

Hence Eqs. 2.7 and 2.12 are just sufficient to determine the constant of integration c_1 in Eq. 2.34, and they are satisfied if we rewrite 2.34 in the following form:

$$s=\frac{\gamma}{2\gamma+1}\ln\theta-\frac{2}{5}\ln\frac{\theta+1}{2}+\frac{13\gamma^2-7\gamma+12}{15(2-\gamma)(2\gamma+1)}\ln\frac{3(2-\gamma)\theta+2\gamma+1}{7-\gamma}. \tag{2.34'}$$

Now we express the original similarity variable z in terms of θ:

$$z=e^s=\theta^{\gamma/(2\gamma+1)}\left(\frac{\theta+1}{2}\right)^{-2/5}\left[\frac{3(2-\gamma)\theta+2\gamma+1}{7-\gamma}\right]^{(13\gamma^2-7\gamma+12)/[15(2-\gamma)(2\gamma+1)]}$$

(2.34'')

Next, using Eqs. 2.31 and 2.22 we obtain

$$F(z) = e^{\nu s}\Phi(s)$$
$$= \theta^{(\gamma-1)/(2\gamma+1)}\left(\frac{\theta+1}{2}\right)^{-2/5}\left(\frac{\theta+\gamma}{\gamma+1}\right)^{(\gamma+1)/(3\gamma-1)}\left[\frac{3(2-\gamma)\theta+2\gamma+1}{7-\gamma}\right]^{-(13\gamma^2-7\gamma+12)/[5(2\gamma+1)(3\gamma-1)]}$$

(2.35)

These equations show that the boundary condition, Eq. 2.8 is automatically satisfied: Eq. 2,34'' (with Eq. 2.33) shows that $z \to 0$ corresponds to $\theta \to 0$, and Eq. 2.35 shows that this implies $F(z) \to 0$.

Hence Eqs. 2.34" and 2.35 contain the complete solution of our problem in parametric form [2.34'' (with 2.33)] and show that the interval

$$0 < z \le 1 \quad (0 < X \le \Xi) \qquad (2.36)$$

corresponds to the interval $0 < \theta \le 1$.

It is convenient to express $F'(z)$ and $F(z) - zF'(z)$, too, in terms of θ. We find using Eqs. 2.34", 2.32, and 2.22 that

$$F'(z) = e^{(\nu-1)s}\Psi$$
$$= \frac{\gamma-1}{\gamma+1}\theta^{-1/(2\gamma+1)}\left(\frac{\theta+\gamma}{\gamma+1}\right)^{-2(\gamma-1)/(3\gamma-1)}\left[\frac{3(2-\gamma)\theta+2\gamma+1}{7-\gamma}\right]^{-(13\gamma^2-7\gamma+12)/[3(2-\gamma)(2\gamma+1)(3\gamma-1)]}$$

On the other hand, Eqs. 2.31 and 2.32 ive

$$\frac{F(z)}{zF'(z)} = \frac{\Phi}{\Psi} = \frac{\gamma+1}{\gamma-1}\frac{\theta+\gamma}{\gamma+1} = \frac{\theta+\gamma}{\gamma-1},$$

$$\frac{F(z)}{zF'(z)} - 1 = \frac{\theta+1}{\gamma-1} = \frac{2}{\gamma-1}\frac{\theta+1}{2}.$$

Using these relations together with Eqs. 2.34'' and 2.37 we have

$$F(z) - zF'(z) = e^{\nu s}(\Phi - \Psi)$$
$$= \frac{2}{\gamma+1}\theta^{(\gamma-1)/(2\gamma+1)}\left(\frac{\theta+1}{2}\right)^{3/5}\left(\frac{\theta+\gamma}{\gamma+1}\right)^{-2(\gamma-1)/(3\gamma-1)}\left[\frac{3(2-\gamma)\theta+2\gamma+1}{7-\gamma}\right]^{-(13\gamma^2-7\gamma+12)/[5(2\gamma+1)(3\gamma-1)]}$$

(2.38)

We can now use Eqs. 2.3, 2.2'', 2.13, 2.4 and 2.5 to express x, X, p, ρ, u in terms of θ. The results are

$$x = at^{2/5}\cdot\theta^{\frac{\gamma}{2\gamma+1}}\left(\frac{\theta+1}{2}\right)^{-2/5}\left[\frac{3(2-\gamma)\theta+2\gamma+1}{7-\gamma}\right]^{\frac{13\gamma^2-7\gamma+12}{15(2-\gamma)(2\gamma+1)}},$$

(2.39)

$$X = at^{2/5}\cdot\theta^{\frac{\gamma-1}{2\gamma+1}}\left(\frac{\theta+1}{2}\right)^{-2/5}\left(\frac{\theta+\gamma}{\gamma+1}\right)^{\frac{\gamma+1}{3\gamma-1}}\left[\frac{3(2-\gamma)\theta+2\gamma+1}{7-\gamma}\right]^{-\frac{13\gamma^2-7\gamma+12}{5(2\gamma+1)(3\gamma-1)}}$$

(2.40)

$$\rho = \frac{\gamma+1}{\gamma-1}\rho_0 \cdot \theta^{\frac{3}{2\gamma+1}}\left(\frac{\theta+\gamma}{\gamma+1}\right)^{-\frac{4}{3\gamma-1}}\left[\frac{3(2-\gamma)\theta+2\gamma+1}{7-\gamma}\right]^{\frac{13\gamma^2-7\gamma+12}{(2-\gamma)(2\gamma+1)(3\gamma-1)}},$$

(2.41)

$$u = \frac{4}{5(\gamma+1)}at^{-3/5}\cdot\theta^{\frac{\gamma-1}{2\gamma+1}}\left(\frac{\theta+1}{2}\right)^{3/5}\left(\frac{\theta+\gamma}{\gamma+1}\right)^{-\frac{2(\gamma-1)}{3\gamma-1}}\left[\frac{3(2-\gamma)\theta+2\gamma+1}{7-\gamma}\right]^{-\frac{13\gamma^2-7\gamma+12}{5(2\gamma+1)(3\gamma-1)}}$$

(2.42)

$$p = \frac{8}{25(\gamma+1)}\rho_0 a^2 t^{-6/5}\cdot\left(\frac{\theta+1}{2}\right)^{6/5}\left(\frac{\theta+\gamma}{\gamma+1}\right)^{-\frac{4\gamma}{3\gamma-1}}\left[\frac{3(2-\gamma)\theta+2\gamma+1}{7-\gamma}\right]^{\frac{13\gamma^2-7\gamma+12}{5(2-\gamma)(3\gamma-1)}}$$

(2.43)

We express the **internal** (thermal) **energy** $\frac{1}{\gamma-1}\frac{p}{\rho}$ and the **kinetic energy** $\frac{1}{2}u^2$ per unit mass

$$\varepsilon_i = \frac{1}{\gamma-1}\frac{p}{\rho}$$

$$= \frac{8}{25(\gamma+1)^2}a^2 t^{-6/5}\cdot\theta^{-\frac{3}{2\gamma+1}}\left(\frac{\theta+1}{2}\right)^{6/5}\left(\frac{\theta+\gamma}{\gamma+1}\right)^{-\frac{4(\gamma-1)}{3\gamma-1}}\left[\frac{3(2-\gamma)\theta+2\gamma+1}{7-\gamma}\right]^{-\frac{2(13\gamma^2-7\gamma+12)}{5(2\gamma+1)(3\gamma-1)}}$$

(2.44)

$$\varepsilon_c = \frac{1}{2}u^2$$

$$= \frac{8}{25(\gamma+1)^2}a^2 t^{-6/5}\cdot\theta^{\frac{2(\gamma-1)}{2\gamma+1}}\left(\frac{\theta+1}{2}\right)^{6/5}\left(\frac{\theta+\gamma}{\gamma+1}\right)^{-\frac{4(\gamma-1)}{3\gamma-1}}\left[\frac{3(2-\gamma)\theta+2\gamma+1}{7-\gamma}\right]^{-\frac{2(13\gamma^2-7\gamma+12)}{5(2\gamma+1)(3\gamma-1)}}$$

(2.45)

Hence,

$$\frac{\varepsilon_c}{\varepsilon_i} = \theta \qquad (2.46)$$

giving an immediate physical interpretation of the parameter θ.

We need finally the expression for the **total energy** E_0. Instead of calculating it using Eq. 2.16, it is now preferable to use a different procedure.

We replace the inner and kinetic energies ε_i, ε_c per unit mass by those ε'_i, ε'_c per unit volume.

Equation 2.46 gives again

$$\frac{\varepsilon'_c}{\varepsilon'_i} = \theta ,$$

and now

$$\begin{aligned} E_0 &= \int_0^{\Xi} (\varepsilon'_i + \varepsilon'_c) 4\pi X^2 dX \\ &= 4\pi \int_0^{\Xi} (\theta+1)\varepsilon'_i \, X^2 dX \\ &= 8\pi \Xi^3 \int_0^1 \frac{\theta+1}{2} \varepsilon'_i \, [F(z)]^2 \, dF(z) \end{aligned} .$$

Now $\varepsilon'_i = \rho \varepsilon_i = \frac{1}{\gamma - 1} p$; hence Eq. 2.43 gives

$$E_0 = K \rho_0 a^5 , \qquad (2.47)$$

where

$$K = \frac{64\pi}{75(\gamma^2 - 1)} \int_0^1 \left(\frac{\theta+1}{2} \right)^{11/5} \left(\frac{\theta+\gamma}{\gamma+1} \right)^{-\frac{4\gamma}{3\gamma-1}} \left[\frac{3(2-\gamma)\theta + 2\gamma + 1}{7-\gamma} \right]^{\frac{13\gamma^2 - 7\gamma + 12}{5(2-\gamma)(3\gamma-1)}} d(F)^3$$

(2.47')

F^3 being obtainable from Eq. 2.35.

2.3 Evaluation and Interpretation of the Results

The formulae 2.39 to 2.46 give a complete description of the physical situation, while, 2.47 and 2.47' connect the necessary constant a with the physically given constants E_0, ρ_0. We will now formulate verbally some of the main qualitative features expressed by Eqs. 2.39 to 2.46.

The center is at $\theta = 0 : x = X = 0$. The shock is at $\theta = 1$: x = X = Ξ = $at^{2/5}$. The ratio kinetic energy/internal energy is θ; hence it varies from the value $\theta = 0$ at the center to the value $\theta = 1$ at the shock.

In all formulae 2.39 to 2.45 the θ-dependent terns are 1 for $\theta = 1$, that is, at the shock. In other words: the first factor gives the value of the corresponding quantity at the shock.

The formulae are valid[2] for $1 < \gamma < 2$.

These formulae are regular in the limit $\gamma \to 1$, $\theta \to 0$ except for the powers of θ, and the factor $\frac{\gamma+1}{\gamma-1}$ in ρ. It should be noted that the three other factors become all $\frac{\theta+1}{2}$, and hence can give rise to no singularities. We restate these formulae in their limiting form for $\gamma = 1$

[2] At this point it should be mentioned that one reason for developing the theory in the present form was to facilitate application of the small $\gamma - 1$ theory of Bethe in Chap. 4.

except that we conserve terms of order $(\gamma-1)$ [but not $(\gamma-1)^2$ and higher terms] in the θ exponent and the leading $\frac{1}{\gamma-1}$ term (but no other terms) in ρ. This gives:

$$\boxed{x = at^{2/5}\cdot\theta^{\frac{1}{3}}}$$

$$x = at^{2/5}\cdot\theta^{\frac{1}{3}+\frac{\gamma-1}{9}}, \tag{2.39'}$$

$$X = at^{2/5}\cdot\theta^{\frac{\gamma-1}{3}}, \tag{2.40'}$$

$$\rho = \frac{2}{\gamma-1}\rho_0\cdot\theta^{1-\frac{2(\gamma-1)}{3}}\cdot\frac{\theta+1}{2}, \tag{2.41'}$$

$$u = \frac{2}{5}at^{-3/5}\cdot\theta^{\frac{\gamma-1}{3}}, \tag{2.42'}$$

$$p = \frac{4}{25}\rho_0 a^2 t^{-6/5}\cdot\frac{\theta+1}{2}, \tag{2.43'}$$

$$\varepsilon_i = \frac{2}{25}a^2 t^{-6/5}\cdot\theta^{-1+\frac{2(\gamma-1)}{3}}, \tag{2.44'}$$

$$\varepsilon_c = \frac{2}{25}a^2 t^{-6/5}\cdot\theta^{\frac{2(\gamma-1)}{3}}. \tag{2.45'}$$

For $\gamma\to 2$ the last factor has to be considered separately, since its basis, $\frac{3(2-\gamma)\theta+2\gamma+1}{7-\gamma}$ becomes 1, while the exponent becomes infinite in some cases (x, ρ, p). Where the exponent stays finite $(X,u,\varepsilon_i,\varepsilon_c)$, this factor is simply 1, but for the others (x, ρ, p as above) it assumes the indefinite form 1^∞. These cases may be discussed on the basis of the expression

$$\left[\frac{3(2-\gamma)\theta+2\gamma+1}{7-\gamma}\right]^{1/(2-\gamma)}.$$

This can be written

$$\left[1-(2-\gamma)\frac{3}{7-\gamma}(1-\theta)\right]^{1/(2-\gamma)}.$$

This has the same $\gamma\to 2$ limit as

$$e^{-\frac{3}{7-\gamma}(1-\theta)},$$

that is

$$e^{-\frac{3}{5}(1-\theta)}.$$

Hence the last factors in Eqs. 2.39 to 2.45 become

$$e^{-\frac{2}{5}(1-\theta)}, \qquad (2.39'')$$

$$1, \qquad (2.40'')$$

$$e^{-\frac{6}{5}(1-\theta)}, \qquad (2.41'')$$

$$1, \qquad (2.42'')$$

$$e^{-\frac{6}{5}(1-\theta)} \qquad (2.43'')$$

$$1, \qquad (2.44'')$$

$$1. \qquad (2.45'')$$

respectively. The other factors offer no difficulties at all.

The formulae which have been derived so far permit us to make some general qualitative remarks about the nature of the point source solution. These are the following:

1. Equation 2.41 shows that the density vanishes at the center. Table 2.3 shows in more detail that the density increases from 0 to its maximum value as one moves from the center to the shock. Table 2.1, referred to spatial positions with the help of Table 2.2, shows even more: most material is situated near the shock, and as γ approaches 1 all material gets asymptotically into positions near the shock.

2. By Eqs. 2.39 and 2.40, $x \propto X^{\gamma/(\gamma-1)}$ for $X \to 0$; and by Eqs. 2.39' and 2.40' even $X \to 0$ can be omitted if $\gamma \to 1$. That is, the amount of material within the sphere of radius $X\left(\frac{4\pi}{3}x^3\right)$ decreases with a high power [$\frac{\gamma}{\gamma-1}$] of the volume of that sphere ($\frac{4\pi}{3}X^3$), and this tendency goes to complete degeneration as $\gamma \to 1$ [$\frac{\gamma}{\gamma-1} \to \infty$]. Indeed, for any fixed volume, except the total one (that is, whenever $\theta^{(\gamma-1)/(2\gamma+1)}$ fixed = ω_0 < 1), the mass in the sphere tends to 0 as $\gamma \to 1$ (that is, with the above assumption $\theta^{\gamma/((2\gamma+1)} = \omega_0{}^{\gamma/(\gamma-1)} \to 0$).

3. Near the center $\rho = 0$, as we saw in paragraphs 1 and 2 above, but $p \to p_0$ where $p < p_0 < \infty$. Indeed, Table 2.5 shows that p_0 / p_{shock} has very moderate values: As γ varies from 1 to 2, this ratio varies from 1/2 to about 1/4. Table 2.5, referred to spatial positions with the help of Table 2.2, and. to the quantities of matter affected with the help of Table 2.1, also shows that p varies mostly near the shock, and only little in the region which contains little material. It shows also that this tendency, too,

goes to complete degeneration as $\gamma \to 1$.

4. Since $\rho \to 0$ and $p \to p_0$, $0 < p_0 < \infty$ near the center, temperature $T \propto \varepsilon_i \propto \dfrac{p}{\rho} \to \infty$ near the center. This is also clear from Eq. 2.44. Equations 2.44 and 2.45 show, furthermore, that $\varepsilon_i \to \infty$, $\varepsilon_c \to 0$ near the center.

5. Already Eqs. 2.1', 2.6, and 2.11'' show that $p_{shock} \propto \Xi^{-3}$. Equation 2.43 (with $\theta = 1$), 2.47, and 2.47' show more specifically that

$$p_{shock} = \lambda \frac{E_0}{\Xi^3}, \tag{2.48}$$

where

$$\lambda = \frac{8}{25(\gamma+1)} \frac{1}{K} = \frac{3(\gamma-1)}{8\pi} \frac{1}{\int \text{as\_in\_2.47'}}. \tag{2.48'}$$

To sum up: The point source blows all material away from the center. The gradually emptying region around the center has ρ degenerating to 0, $T \propto \varepsilon_i$ degenerating to ∞, while p tends to constancy, with moderate values of p / p_{shock}. As $\gamma \to 1$, these tendencies accentuate more and more, they go finally to complete degeneracy, and all material concentrates in the immediate vicinity of the shock.

Tables 2.1 to 2.8 give numerical values of some relations discussed in this chapter. (Tables are converted by the reader to figures.)

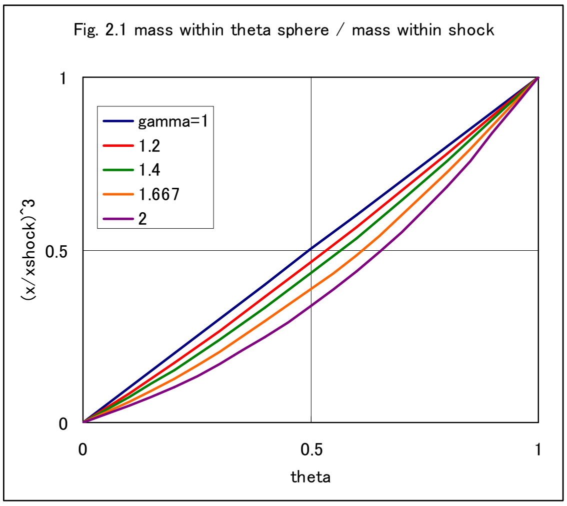


Fig. 2.2 volume within theta sphere / volume within shock

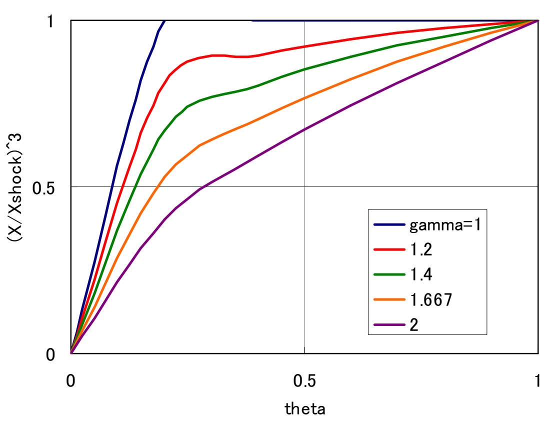


Fig. 2.3 row / row(shock)

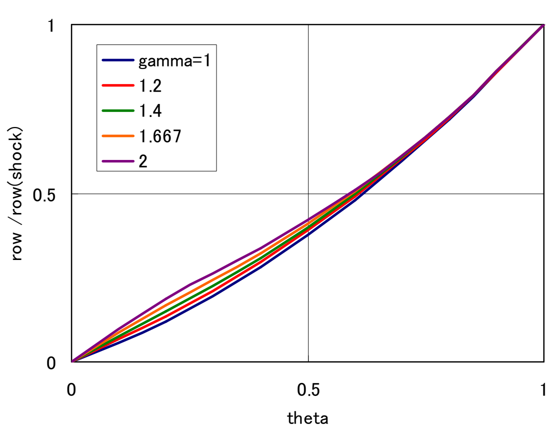
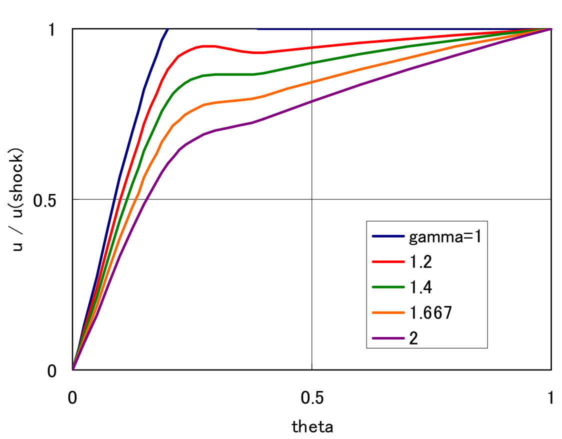
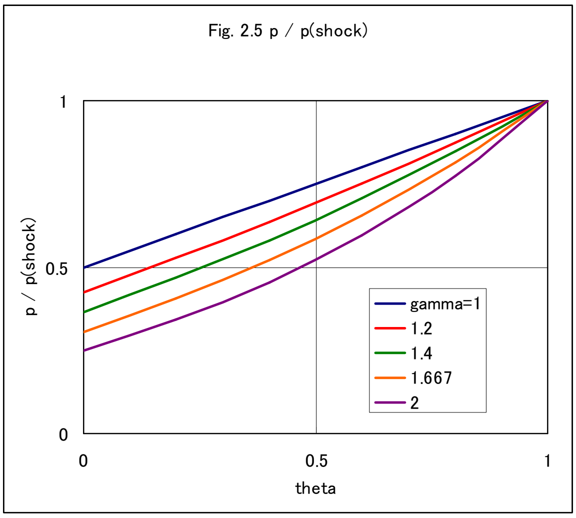
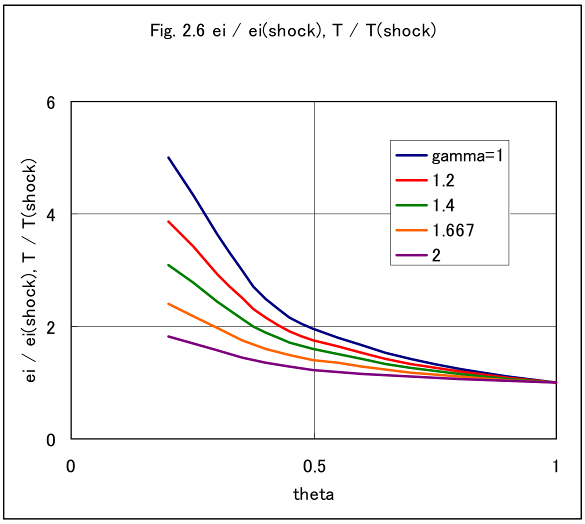
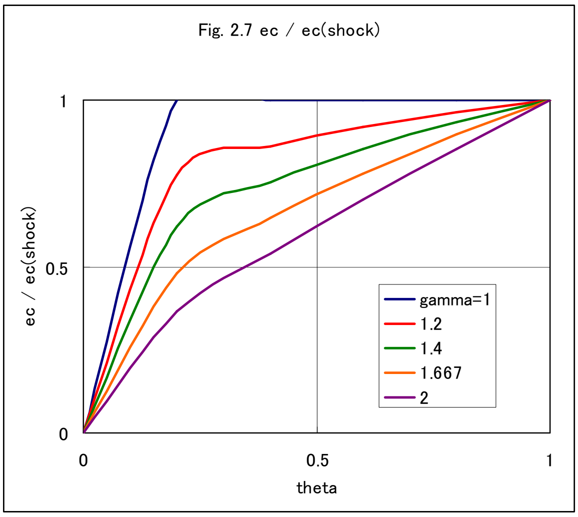


Fig. 2.4 u / u(shock)





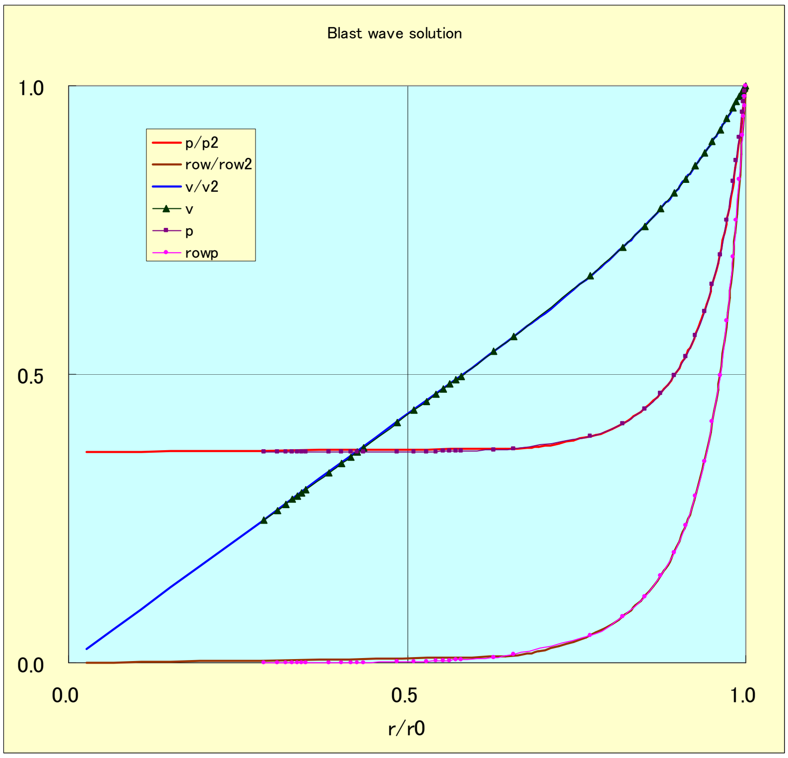




Reader's addition

L. I. Sedov solved the solution of this problem analytically. It is shown in *Similarity and Dimensional Methods in Mechanics*, Academic Press, 1959.

The typical quantities distribution for $\gamma = 1.4$ is as follows.



Blast wave solution
p/p2
row/row2
v/v2
v
p
rowp
1.0
0.5
0.0
0.0
0.5
1.0
r/r0