

命題論理 自然演繹 証明問題 解答 [樹形図方式]

問題 1

[1] $P \rightarrow Q, Q \rightarrow R, P \vdash R$

$$\frac{\frac{Q \rightarrow R \quad \frac{P \rightarrow Q \quad P}{Q} \text{MP}}{R} \text{MP}}$$

[2] $P, P \rightarrow Q, P \rightarrow (Q \rightarrow R) \vdash R$

$$\frac{\frac{\frac{P \quad P \rightarrow Q}{Q} \text{MP} \quad \frac{P \quad P \rightarrow (Q \rightarrow R)}{Q \rightarrow R} \text{MP}}{R} \text{MP}}$$

[3] $P \rightarrow (P \rightarrow Q), P \vdash Q$

$$\frac{P \quad \frac{P \rightarrow (P \rightarrow Q) \quad P}{P \rightarrow Q} \text{MP}}{Q} \text{MP}$$

[5] $P \rightarrow \sim Q, R \rightarrow Q, P \vdash \sim R$

$$\frac{R \rightarrow Q \quad \frac{P \rightarrow \sim Q \quad P}{\sim Q} \text{MP}}{\sim R} \text{MT}$$

[8] $P, P \rightarrow (Q \rightarrow R), P \rightarrow \sim R \vdash \sim Q$

$$\frac{\frac{P \quad P \rightarrow \sim R}{\sim R} \text{MP} \quad \frac{P \quad P \rightarrow (Q \rightarrow R)}{Q \rightarrow R} \text{MP}}{\sim Q} \text{MT}$$

問題 2

[1] $P \rightarrow \sim\sim Q, P \vdash Q$

$$\frac{P \rightarrow \sim\sim Q \quad P}{\sim\sim Q} \text{MP}$$

$$\frac{\sim\sim Q}{Q} \text{DNE}$$

[2] $\sim Q, \sim R \rightarrow Q, R \rightarrow P \vdash P$

$$\frac{\frac{\frac{1}{\sim Q} \quad \frac{2}{\sim R \rightarrow Q}}{\sim \sim R} \text{MT}}{\frac{\frac{3}{R \rightarrow P} \quad R}{P} \text{MP}} \text{DNE}$$

[6] $P, P \rightarrow Q, \sim(Q \rightarrow R) \rightarrow \sim P \vdash R$

$$\frac{\frac{\frac{1}{P} \quad \frac{3}{\sim(Q \rightarrow R) \rightarrow \sim P}}{\sim \sim(Q \rightarrow R)} \text{MT}}{\frac{Q \rightarrow R}{R} \text{DNE}} \text{DNI} \quad \frac{\frac{1}{P} \quad \frac{2}{P \rightarrow Q}}{Q} \text{MP} \text{MP}$$

問題 3

[1] $P \rightarrow Q \vdash \sim Q \rightarrow \sim P$

(i) $P \rightarrow Q \vdash \sim Q \rightarrow \sim P$

$$\frac{\frac{1}{P \rightarrow Q} \quad \frac{2}{\{\sim Q\}}}{\sim P} \text{MT} \quad \frac{\sim P}{\sim Q \rightarrow \sim P} \text{CP}_2$$

(ii) $\sim Q \rightarrow \sim P \vdash P \rightarrow Q$

$$\frac{\frac{1}{\sim Q \rightarrow \sim P} \quad \frac{2}{\{P\}}}{\sim \sim Q} \text{MT} \quad \frac{\sim \sim Q}{Q} \text{DNE} \quad \frac{Q}{P \rightarrow Q} \text{CP}_2$$

[3] $P \rightarrow Q, Q \rightarrow R \vdash P \rightarrow R$

$$\frac{\frac{2}{Q \rightarrow R} \quad \frac{\frac{1}{P \rightarrow Q} \quad \frac{3}{\{P\}}}{Q} \text{MP}}{P \rightarrow R} \text{CP}_3$$

[5] $P \rightarrow Q \vdash (Q \rightarrow R) \rightarrow (P \rightarrow R)$

$$\begin{array}{c}
 \frac{\frac{\frac{\overset{1}{P \rightarrow Q} \quad \overset{3}{\{P\}}}{\text{MP}}}{\frac{\overset{2}{\{Q \rightarrow R\}} \quad Q}{\text{MP}}} \text{MP} \\
 \frac{R}{P \rightarrow R} \text{CP}_3 \\
 \frac{P \rightarrow R}{(Q \rightarrow R) \rightarrow (P \rightarrow R)} \text{CP}_2
 \end{array}$$

[7] $\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$

$$\begin{array}{c}
 \frac{\frac{\overset{1}{\{P \rightarrow (Q \rightarrow R)\}} \quad \overset{3}{\{P\}}}{\text{MP}} \quad \frac{\frac{\overset{2}{\{P \rightarrow Q\}} \quad \overset{3}{\{P\}}}{\text{MP}}}{\frac{Q \rightarrow R \quad Q}{\text{MP}}} \text{MP} \\
 \frac{R}{P \rightarrow R} \text{CP}_3 \\
 \frac{P \rightarrow R}{(P \rightarrow Q) \rightarrow (P \rightarrow R)} \text{CP}_2 \\
 \frac{(P \rightarrow Q) \rightarrow (P \rightarrow R)}{(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))} \text{CP}_1
 \end{array}$$

[12] $P \vdash Q \rightarrow P$

$$\frac{\overset{1}{P}}{Q \rightarrow P} \text{CP}$$

問題 4

[1] $P \& Q \vdash Q \& P$

$$\frac{\frac{\overset{1}{P \& Q}}{Q} \&E \quad \frac{\overset{1}{P \& Q}}{P} \&E}{Q \& P} \&I$$

[4] $(P \& Q) \rightarrow R \vdash P \rightarrow (Q \rightarrow R)$

$$\begin{array}{c}
 \frac{\frac{\overset{1}{(P \& Q) \rightarrow R} \quad \frac{\frac{\overset{2}{\{P\}} \quad \overset{3}{\{Q\}}}{\&I}}{P \& Q} \&I}{R} \text{MP} \\
 \frac{R}{Q \rightarrow R} \text{CP}_3 \\
 \frac{Q \rightarrow R}{P \rightarrow (Q \rightarrow R)} \text{CP}_2
 \end{array}$$

[5] $P \rightarrow (Q \rightarrow R) \vdash (P \& Q) \rightarrow R$

$$\begin{array}{c}
 \begin{array}{c}
 \frac{P \rightarrow (Q \rightarrow R)}{Q \rightarrow R} \text{MP} \quad \frac{\frac{\frac{2}{\{P \& Q\}}}{P} \&E}{Q \rightarrow R} \text{MP} \\
 \frac{Q \rightarrow R}{R} \text{CP}_2 \\
 \frac{R}{(P \& Q) \rightarrow R} \text{CP}_2
 \end{array}
 \end{array}$$

[7] $P \& Q, (S \vee Q) \rightarrow R \vdash R$

$$\begin{array}{c}
 \frac{\frac{1}{P \& Q}}{Q} \&E \\
 \frac{(S \vee Q) \rightarrow R \quad \frac{Q}{S \vee Q} \vee I}{R} \text{MP}
 \end{array}$$

[8] $P \& (Q \& R) \vdash (P \& Q) \& R$

$$\begin{array}{c}
 \frac{\frac{1}{P \& (Q \& R)}}{P} \&E \quad \frac{\frac{1}{P \& (Q \& R)}}{Q \& R} \&E \quad \frac{\frac{1}{P \& (Q \& R)}}{Q \& R} \&E \\
 \frac{P \quad Q \& R}{P \& Q} \&I \quad \frac{Q \& R}{R} \&E \\
 \frac{P \& Q \quad R}{(P \& Q) \& R} \&I
 \end{array}$$

[10] $P \rightarrow Q, R \rightarrow S \vdash (P \& R) \rightarrow (Q \& S)$

$$\begin{array}{c}
 \frac{P \rightarrow Q}{Q} \text{MP} \quad \frac{\frac{3}{\{P \& R\}}}{P} \&E \quad \frac{R \rightarrow S}{S} \text{MP} \quad \frac{\frac{3}{\{P \& R\}}}{R} \&E \\
 \frac{Q \quad S}{Q \& S} \&I \\
 \frac{Q \& S}{(P \& R) \rightarrow (Q \& S)} \text{CP}_3
 \end{array}$$

問題 5

[1] $P \rightarrow (Q \vee R), P \& \sim Q \vdash R$

$$\begin{array}{c}
 \frac{P \rightarrow (Q \vee R)}{Q \vee R} \text{MP} \quad \frac{\frac{2}{P \& \sim Q}}{P} \&E \quad \frac{P \& \sim Q}{\sim Q} \&E \\
 \frac{Q \vee R \quad \sim Q}{R} \text{MTP}
 \end{array}$$

[3] $(P \vee Q) \vee R, \sim P \rightarrow \sim R \vdash \sim P \rightarrow Q$

$$\begin{array}{c}
 \frac{\frac{1}{(P \vee Q) \vee R} \quad \frac{\frac{2}{\sim P \rightarrow \sim R} \quad \frac{3}{\{\sim P\}}}{\sim R} \text{MP}}{P \vee Q} \text{MTP} \quad \frac{3}{\sim P} \text{MTP} \\
 \hline
 \frac{Q}{\sim P \rightarrow Q} \text{CP}_3
 \end{array}$$

問題 6

[3] $P \vee Q \vdash Q \vee P$

$$\begin{array}{c}
 \frac{1}{P \vee Q} \quad \frac{\frac{2}{\{P\}}}{Q \vee P} \text{vI} \quad \frac{\frac{3}{\{Q\}}}{Q \vee P} \text{vI} \\
 \hline
 Q \vee P \text{vE}_{2,3}
 \end{array}$$

[4] $P \vee Q, P \rightarrow R, Q \rightarrow S \vdash R \vee S$

$$\begin{array}{c}
 \frac{1}{P \vee Q} \quad \frac{\frac{2}{P \rightarrow R} \quad \frac{4}{\{P\}}}{R} \text{MP} \quad \frac{\frac{3}{Q \rightarrow S} \quad \frac{7}{\{Q\}}}{S} \text{MP} \\
 \frac{R}{R \vee S} \text{vI} \quad \frac{S}{R \vee S} \text{vI} \\
 \hline
 R \vee S \text{vE}_{4,7}
 \end{array}$$

[5] $P \vee (Q \vee R) \vdash (P \vee Q) \vee R$

$$\begin{array}{c}
 \frac{1}{P \vee (Q \vee R)} \quad \frac{\frac{2}{\{P\}}}{P \vee Q} \text{vI} \quad \frac{\frac{6}{\{Q\}}}{P \vee Q} \text{vI} \quad \frac{\frac{9}{\{R\}}}{(P \vee Q) \vee R} \text{vI} \\
 \frac{P \vee Q}{(P \vee Q) \vee R} \text{vI} \quad \frac{\frac{5}{\{Q \vee R\}} \quad \frac{P \vee Q}{(P \vee Q) \vee R}}{(P \vee Q) \vee R} \text{vE}_{6,9} \\
 \hline
 (P \vee Q) \vee R \text{vE}_{2,5}
 \end{array}$$

[6] $P \& (Q \vee R) \dashv\vdash (P \& Q) \vee (P \& R)$

(i) $P \& (Q \vee R) \vdash (P \& Q) \vee (P \& R)$

$$\begin{array}{c}
 \frac{1}{P \& (Q \vee R)} \quad \frac{\frac{1}{P \& (Q \vee R)}}{P} \&E \quad \frac{4}{\{Q\}} \&I \\
 \frac{P \& (Q \vee R)}{Q \vee R} \&E \quad \frac{P}{P \& Q} \&I \\
 \frac{P \& Q}{(P \& Q) \vee (P \& R)} \text{vI} \\
 \hline
 (P \& Q) \vee (P \& R) \text{vE}_{4,7}
 \end{array}$$

(ii) $(P \& Q) \vee (P \& R) \vdash P \& (Q \vee R)$

$$\begin{array}{c}
 \begin{array}{c}
 \frac{(P \& Q) \vee (P \& R)}{P \& (Q \vee R)} \vee E_{2,7} \\
 \frac{\frac{\frac{1}{(P \& Q) \vee (P \& R)}}{P} \& E \quad \frac{\frac{\frac{2}{\{P \& Q\}}}{Q} \& E \quad \frac{\frac{2}{\{P \& Q\}}}{Q \vee R} \vee I}{P \& (Q \vee R)} \& I
 \end{array}
 \end{array}$$

[7] $P \vee (Q \& R) \dashv\vdash (P \vee Q) \& (P \vee R)$

(i) $P \vee (Q \& R) \vdash (P \vee Q) \& (P \vee R)$

$$\begin{array}{c}
 \begin{array}{c}
 \frac{P \vee (Q \& R)}{(P \vee Q) \& (P \vee R)} \vee E_{2,6} \\
 \frac{\frac{\frac{1}{P \vee (Q \& R)}}{P \vee Q} \vee I \quad \frac{\frac{\frac{2}{\{P\}}}{P \vee R} \vee I}{(P \vee Q) \& (P \vee R)} \& I
 \end{array}
 \end{array}$$

(ii) $(P \vee Q) \& (P \vee R) \vdash P \vee (Q \& R)$

$$\begin{array}{c}
 \begin{array}{c}
 \frac{(P \vee Q) \& (P \vee R)}{P \vee (Q \& R)} \vee E_{3,5} \\
 \frac{\frac{\frac{1}{(P \vee Q) \& (P \vee R)}}{P \vee Q} \& E \quad \frac{\frac{\frac{3}{\{P\}}}{P \vee (Q \& R)} \& I \quad \frac{\frac{\frac{1}{(P \vee Q) \& (P \vee R)}}{P \vee R} \vee I \quad \frac{\frac{\frac{7}{\{P\}}}{P \vee (Q \& R)} \vee I \quad \frac{\frac{\frac{5}{\{Q\}} \quad \frac{9}{\{R\}}}{Q \& R} \& I \quad \frac{Q \& R}{P \vee (Q \& R)} \vee I}{P \vee (Q \& R)} \vee E_{7,9}
 \end{array}
 \end{array}$$

問題 7

[1] $P \rightarrow Q, P \rightarrow \sim Q \vdash \sim P$

$$\begin{array}{c}
 \begin{array}{c}
 \frac{P \rightarrow Q \quad \{P\}}{Q} \text{MP} \quad \frac{P \rightarrow \sim Q \quad \{P\}}{\sim Q} \text{MP} \\
 \frac{Q \quad \sim Q}{Q \& \sim Q} \& I \\
 \frac{Q \& \sim Q}{\sim P} \text{RAA}_3
 \end{array}
 \end{array}$$

[2] $P \rightarrow R, Q \rightarrow \sim R \vdash \sim(P \& Q)$

$$\begin{array}{c}
 \begin{array}{c}
 \frac{1}{P \rightarrow R} \quad \frac{\frac{3}{\{P \& Q\}}}{P} \&E \\
 \hline
 R \quad \text{MP}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{2}{Q \rightarrow \sim R} \quad \frac{\frac{3}{\{P \& Q\}}}{Q} \&E \\
 \hline
 \sim R \quad \text{MP}
 \end{array} \\
 \hline
 R \& \sim R \quad \&I \\
 \hline
 \sim(P \& Q) \quad \text{RAA}_3
 \end{array}$$

[4] $P \vee R, \sim Q \rightarrow (\sim P \& \sim R) \vdash Q$

$$\begin{array}{c}
 \begin{array}{c}
 \frac{1}{P \vee R} \\
 \hline
 R \quad \text{MTP}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{2}{\sim Q \rightarrow (\sim P \& \sim R)} \quad \frac{3}{\{\sim Q\}} \text{MP} \\
 \hline
 \sim P \& \sim R \quad \&E \\
 \hline
 \sim P \quad \&E \\
 \hline
 R \& \sim R \quad \&I \\
 \hline
 \sim \sim Q \quad \text{RAA}_3 \\
 \hline
 Q \quad \text{DNE}
 \end{array}
 \end{array}$$

[5] $\vdash P \vee \sim P$

$$\begin{array}{c}
 \frac{2}{\{P\}} \text{VI} \\
 \hline
 P \vee \sim P \quad \text{VI}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{1}{\{\sim(P \vee \sim P)\}} \\
 \hline
 (P \vee \sim P) \& \sim(P \vee \sim P) \quad \&I \\
 \hline
 \sim P \quad \text{RAA}_2 \\
 \hline
 \frac{\sim P}{P \vee \sim P} \text{VI} \quad \frac{1}{\{\sim(P \vee \sim P)\}} \\
 \hline
 (P \vee \sim P) \& \sim(P \vee \sim P) \quad \&I \\
 \hline
 \sim \sim(P \vee \sim P) \quad \text{RAA}_1 \\
 \hline
 P \vee \sim P \quad \text{DNE}
 \end{array}$$

[9] $P \rightarrow \sim R, Q \rightarrow R \vdash P \rightarrow \sim(Q \vee R)$

$$\begin{array}{c}
 \begin{array}{c}
 \frac{4}{\{Q \vee R\}} \quad \frac{1}{P \rightarrow \sim R} \quad \frac{3}{\{P\}} \text{MP} \\
 \hline
 Q \quad \sim R \quad \text{MTP}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{1}{P \rightarrow \sim R} \quad \frac{3}{\{P\}} \text{MP} \\
 \hline
 Q \rightarrow R \quad \sim R \quad \text{MT} \\
 \hline
 \sim Q \quad \text{MT}
 \end{array} \\
 \hline
 Q \& \sim Q \quad \&I \\
 \hline
 \sim(Q \vee R) \quad \text{RAA}_4 \\
 \hline
 P \rightarrow \sim(Q \vee R) \quad \text{CP}_3
 \end{array}$$

[12] $\sim(P \& Q), P \vdash \sim Q$

$$\begin{array}{c}
 \begin{array}{cc}
 \overset{2}{P} & \overset{3}{\{Q\}} \\
 \hline
 P \& Q
 \end{array}
 \quad \&I
 \quad
 \begin{array}{c}
 \overset{1}{\sim(P \& Q)} \\
 \hline
 \end{array}
 \quad \&I \\
 \hline
 (P \& Q) \& \sim(P \& Q) \\
 \hline
 \sim Q \quad \text{RAA}_3
 \end{array}$$